Effective de Rham Cohomology

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Logik Oberseminar
Bonn, Mai 7, 2012

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Introduction

Algebraic Topology

Singular Homology or Counting Holes

We denote by

$$\Delta_k := \{(t_0, \dots, t_k) \in \mathbb{R}^{k+1} \mid t_i \geq 0, \sum_i t_i = 1\}$$

the standard n-simplex. Its i-th face is $F_i := \Delta_k \cap \{t_i = 0\} \simeq \Delta_{k-1}$. Let X be a topological space. A singular n-simplex in X is a continuous map $\sigma\colon \Delta_k \to X$. Let K be a field, and

$$S_k(X) := S_k(X, K) := K\text{-vector space with basis all } k\text{-simplices in } X.$$

The boundary map $\partial_k\colon S_k(X) \to S_{k-1}(X)$ is defined by

$$\partial_k(\sigma) := \sum_{i=0}^{n} (-1)^i \sigma|F_i \quad \text{for a singular simplex} \quad \sigma,$$

extended by linearity.

Singular Homology or Counting Holes

One checks that $\partial_k \circ \partial_{k+1} = 0$, hence we get a (chain) complex

$$\cdots \xrightarrow{\partial_{k+1}} S_k(X) \xrightarrow{\partial_k} S_{k-1}(X) \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \longrightarrow 0,$$

and the k-th singular homology of X (with coefficients in K) is defined as

$$H_k(X) := H_k(X, K) := \ker \partial_k / \mathrm{im}\, \partial_{k+1}, \quad i \in \mathbb{N}.$$

The k-th Betti number of X is

$$b_k(X) := \dim_{\mathbb{Q}} H_k(X, \mathbb{Q}).$$

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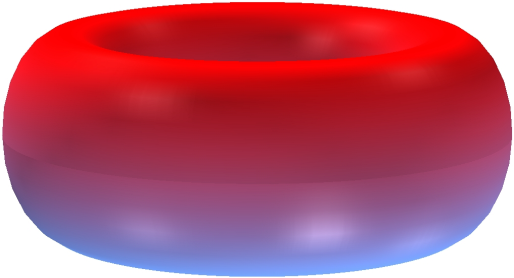
The k-th Betti number $b_k(X)$ counts the "k-dimensional holes" in X.

- $b_0(X) =$ number of (path) connected components of X
- For the n-dimensional sphere S^n, $n > 0$, we have

$$b_k(S^n) = \begin{cases} 1, & \text{if } i = 0 \text{ or } i = n \\ 0, & \text{otherwise} \end{cases}$$

Example 1



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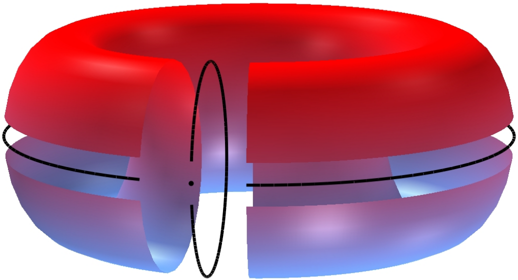

Introduction

Differential Topology

De Rham Cohomology

Consider the vector field

$$v\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2, \quad (x, y) \mapsto \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right).$$

Is v a gradient field, i.e., does there exists a smooth function $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ with $v = \operatorname{grad} f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$?

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Is v a gradient field, i.e., does there exists a smooth function $f\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ with $v = \operatorname{grad} f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$? A necessary condition is

$$\frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial x},$$

which is satisfied by v. On $\mathbb{R}^2$ this condition is also sufficient. However, our v is not a gradient field. In other words, the differential form

$$\omega = v_1 \mathrm{d}x + v_2 \mathrm{d}y$$

is closed, i.e., $\mathrm{d}\omega = \frac{\partial v_1}{\partial y}\mathrm{d}y \wedge \mathrm{d}x + \frac{\partial v_2}{\partial x}\mathrm{d}x \wedge \mathrm{d}y = (\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y})\mathrm{d}x \wedge \mathrm{d}y = 0$, but not exact, i.e., there is no f with $\omega = \mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$.

De Rham Cohomology

Let M be a smooth manifold of dimension m. Denote by $\mathcal{E}_M^p$ the vector bundle of smooth differential p-forms on M. Then there is a derivation $\mathrm{d}\colon \mathcal{E}_M^p \to \mathcal{E}_M^{p+1}$ called exterior derivative satisfying $\mathrm{d} \circ \mathrm{d} = 0$, hence we have a (cochain) complex

$$\mathcal{E}_M^\bullet \colon 0 \longrightarrow \mathcal{E}_M^0 \xrightarrow{\mathrm{d}} \mathcal{E}_M^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathcal{E}_M^p \xrightarrow{\mathrm{d}} \mathcal{E}_M^{p+1} \longrightarrow \cdots \mathcal{E}_M^m \longrightarrow 0.$$

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We define the p-th de Rham cohomology of M by

$$H^p(M) := H^p(\mathcal{E}_M^\bullet) = \ker(\mathcal{E}_M^p \xrightarrow{\mathrm{d}} \mathcal{E}_M^{p+1}) / \mathrm{im}\,(\mathcal{E}_M^{p-1} \xrightarrow{\mathrm{d}} \mathcal{E}_M^p).$$

The de Rham Theorem states that it is dual to singular homology:

$$H^p(M) = H_p(M, \mathbb{R})^*.$$

Mayer-Vietoris Sequence

Let M be covered by two open subsets $M = U \cup V$. Then there is a short exact sequence of differential forms

$$0 \to \mathcal{E}^p_M(M) \xrightarrow{\rho} \mathcal{E}^p_M(U) \oplus \mathcal{E}^p_M(V) \xrightarrow{\delta} \mathcal{E}^p_M(U \cap V) \to 0,$$

where $\rho(\omega) = (\omega|U, \omega|V)$ and $\delta(\alpha, \beta) = \beta|U \cap V - \alpha|U \cap V$.

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Every short exact sequence induces a long exact sequence in cohomology, hence we have an exact sequence

$$\cdots \to H^p(M) \xrightarrow{\rho} H^p(U) \oplus H^p(V) \xrightarrow{\delta} H^p(U \cap V) \to H^{p+1}(M) \to \cdots,$$

from which e.g. the cohomology of the sphere S^n can be computed.

Introduction

Algebraic Geometry

Algebraic Geometry

For a subset $S \subseteq \mathbb{C}[X_1, \ldots, X_n]$ we set

$$\mathcal{Z}(S) := \{x \in \mathbb{C}^n \mid \forall f \in S \colon f(x) = 0\}.$$

Any such $X = \mathcal{Z}(S) \subseteq \mathbb{C}^n$ is called a (complex affine) algebraic veriety.

Hilbert's Basis Theorem: We can choose S to be finite.

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Basic Notions:

- vanishing ideal $I(X) := \{f \in \mathbb{C}[X_1, \ldots, X_n] \mid \forall x \in X\colon f(x) = 0\}$.
- coordinate ring $\mathbb{C}[X] := \mathbb{C}[X_1, \ldots, X_n]/I(X)$.

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Hilbert's Nullstellensatz:

$$I(\mathcal{Z}(S)) = \sqrt{(S)}.$$

There is an inclusion-reversing one-to-one correspondence between varieties $X \subseteq \mathbb{C}^n$ and radical ideals $I \subseteq \mathbb{C}[X_1, \ldots, X_n]$ given by

$$X = \mathcal{Z}(I) \quad \leftrightarrow \quad I = I(X).$$

Kähler Differentials

Let Ω^1_X denote the module of Kähler differentials of $\mathbb{C}[X]$, i.e., the $\mathbb{C}[X]$-module generated by symbols $\mathrm{d}f$, $f \in \mathbb{C}[X]$, subject to the relations

$$\begin{aligned} \mathrm{d}(\lambda f + g) &= \lambda \mathrm{d}f + \mathrm{d}g, \quad \lambda \in \mathbb{C}, f, g \in \mathbb{C}[X] && \text{(linearity)},\\ \mathrm{d}(fg) &= f\mathrm{d}g + g\mathrm{d}f, \quad f, g \in \mathbb{C}[X] && \text{(Leibniz' rule)}. \end{aligned}$$

There is a natural derivation $\mathrm{d}\colon \mathbb{C}[X] \to \Omega^1_X$. Setting $\Omega^p_X := \bigwedge^p \Omega^1_X$, this extends to the algebraic de Rham complex

$$\Omega^\bullet_X\colon\ 0 \longrightarrow \mathbb{C}[X] \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^p_X \xrightarrow{\mathrm{d}} \Omega^{p+1}_X \xrightarrow{\mathrm{d}} \cdots .$$

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Example: For $X = \mathbb{C}^n$, the module $\Omega^1_{\mathbb{C}^n} = \bigoplus_i \mathbb{C}[X_1, \ldots, X_n]\mathrm{d}X_i$ is free, hence each p-form ω on $\mathbb{C}^n$ can be uniquely written as

$$\omega = \sum_{i_1 < \cdots < i_p} \omega_{i_1,\ldots,i_p} \mathrm{d}X_{i_1} \wedge \cdots \wedge \mathrm{d}X_{i_p}. \tag{1}$$

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A p-form on $X \subsetneq \mathbb{C}^n$ can still be written as in (1), but not uniquely.

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Examples:

- $$H^p_{\mathrm{dR}}(\mathbb{C}^n) = \begin{cases} \mathbb{C} \cdot 1, & \text{if } p = 0 \\ 0, & \text{otherwise.} \end{cases}$$
- Let $f \in \mathbb{C}[X_1]$. For $X = \mathbb{C} \setminus \mathcal{Z}(f) = \mathbb{C} \setminus \{\zeta_1, \ldots, \zeta_d\}$ we have

$$H^p_{\mathrm{dR}}(X) = \begin{cases} \mathbb{C} \cdot 1, & \text{if } p = 0 \\ \bigoplus_i \mathbb{C} \cdot \mathrm{d}X_1/(X_1 - \zeta_i), & \text{if } p = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Introduction

Computational Algebraic Geometry

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Consider the following algorithmic problem:

$\#\textsc{Betti}_{\mathbb{R}}$

Given a first-order formula in the theory of ordered fields over $\mathbb{R}$ defining a semialgebraic set S in $\mathbb{R}^n$, compute the topological Betti-numbers $b_0(S), \ldots, b_n(S)$.

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The best known algorithms solving $\#\mathrm{BETTI}_{\mathbb{R}}$ run in double exponential time in the size of the formula (Collin's CAD, 1975).

Question

Can one solve $\#\mathrm{BETTI}_{\mathbb{R}}$ in single exponential time?

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State of the art: For fixed ℓ one can compute the Betti numbers $b_0(S), \ldots, b_\ell(S)$ in single exponential time (Basu, 2006).

Complex Varieties 1

What about the corresponding problem over $\mathbb{C}$? More specifically, we consider the following problem:

$\#\textsc{Betti}_{\mathbb{C}}$

Given polynomials $f_1, \ldots, f_r \in \mathbb{C}[X_1, \ldots, X_n]$ with common zero set $X = \mathcal{Z}(f_1, \ldots, f_r)$ in $\mathbb{C}^n$, compute the Betti-numbers $b_0(X), \ldots, b_{2n}(X)$.

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By considering $X \subseteq \mathbb{R}^{2n}$, one can apply the real algorithm and solve $\#\text{BETTI}_{\mathbb{C}}$ in double exponential time $d^{2^{n^{\mathcal{O}(1)}}}$, where d is a bound on the degrees of the f_i.

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However, of particular interest are algebraic algorithms using only the field structure of $\mathbb{C}$ (no "$<$"). These may be applicable to other fields (characteristic > 0, p-adic, ...).

Complex Varieties 2

Previous results:

- There are algorithms computing the cohomology of the complement $\mathbb{C}^n \setminus X$ of an affine variety X (Oaku/Takayama 1999, Walther 2000).
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These algorithms use Gröbner bases for algebraic D-modules. Unfortunately, Gröbner bases are inherently double exponential (Mayr/Meyer 1982).

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The computational model here is that of (uniform) families of algebraic circuits over $\mathbb{C}$. The parallel time corresponds to the depth, and the sequential time to the size of the circuits.

Effective Nullstellensätze

Hilbert's Nullstellensatz

Let k be a field.

Given $f_1, \dots, f_r \in k[X_1, \dots, X_n]$, how does one test if $\mathcal{Z}_{\overline{k}}(f_1, \dots, f_r) \neq \emptyset$?

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- Note that, if $\deg g_i f_i \leq D$, then (2) is a linear system of $\binom{D+n}{n}$ equations in $r\binom{D+n}{n}$ variables.
- In particular, any bound on the degrees of the g_i in terms of $\deg f_i$ and n proves that this problem is decidable (for $k = \mathbb{Q}$, say).
- Efficient algorithms for linear algebra immediately imply, that the problem can be solved in polynomial time in $r\binom{D+n}{n}$ (parallel time $\mathcal{O}(\log^2 r\binom{D+n}{n})$).

Effective Nullstellensätze

Effective Nullstellensätze give bounds on the degrees of the g_i in terms of $\deg f_i =: d_i$. Set $d := \max_i\{d_i\}$ and $\mu := \min\{r, n\}$. Assume $n \geq 2$.

- Macaulay (?), Herman (1926), Masser/Wüstholz (1983): Double exponential bound in n.
- Brownawell (1987) For $k = \mathbb{C}$:

$$\deg g_i \leq n\mu d^\mu + \mu d.$$

- Kollár (1988): If $d_1 \geq \cdots \geq d_r \geq 3$, then

$$\deg g_i f_i \leq d_1 \cdots d_{\mu-1} d_r.$$

- Caniglia/Galligo/Heintz (1989), Fitchas/Galligo (1990), Sabia/Solerno (1993), Dubé (1993): Similar results with more elementary proofs.
- Sombra (1997), Krick/Sabia/Solerno (1997): Bound in terms of intrinsic degree.
- Sombra (1999): Sparse version.
- Ein/Lazarsfeld (1999): Geometric version.
- S. (2009): Completely elementary proof of Kollar's bound for $r = 2$.

Effective Nullstellensätze

Theorem (Jelonek 2005)

Let $X \subseteq \overline{k}^n$ be a subvariety of dimension $m \geq 1$ and degree D, and let $f_1, \ldots, f_r$ be polynomials of degrees $d_i \geq 1$ without common zeros in X. Then there exist $g_i \in k[X]$ such that $1 = \sum_i g_i f_i$ on X and

$$\deg(h_i g_i) \leq \begin{cases} Dd_1 \cdots d_r & \text{if} \quad r \leq m, \\ 2Dd_1 \cdots d_{m-1} d_r - 1 & \text{otherwise.} \end{cases}$$

Effective Nullstellensätze

Theorem (Jelonek 2005)

Let $X \subseteq \overline{k}^n$ be a subvariety of dimension $m \geq 1$ and degree D, and let $f_1, \ldots, f_r$ be polynomials of degrees $d_i \geq 1$ without common zeros in X. Then there exist $g_i \in k[X]$ such that $1 = \sum_i g_i f_i$ on X and

$$\deg(h_i g_i) \leq \begin{cases} D d_1 \cdots d_r & \text{if} \quad r \leq m, \\ 2D d_1 \cdots d_{m-1} d_r - 1 & \text{otherwise.} \end{cases}$$

Theorem (Kollár 1998)

Let $X_1, \ldots, X_r \subseteq \overline{k}^n$ be subvarieties with $X_1 \cap \cdots \cap X_r = \emptyset$. Then there exist $f_i \in I(X_i)$ such that

$$\sum_i f_i = 1 \quad \text{and} \quad \deg f_i \leq (n+1) \prod_i \deg X_i.$$

Counting Connected Components

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We will explain some ideas underlying the algorithm of

Theorem (Bürgisser/S. 2007)

Given polynomials $f_1, \ldots, f_r$ of degree $\leq d$ defining the variety $X \subseteq \mathbb{C}^n$, one can compute the number $b_0(X)$ of connected components of X in sequential time $d^{\mathcal{O}(n^4)}$ and parallel time $(n \log d)^{\mathcal{O}(1)}$.

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The algorithm uses the zeroth cohomology $H^0_{\mathrm{dR}}(X)$. We have

$$H^0_{\mathrm{dR}}(X) = \{f \in \mathbb{C}[X] \mid \mathrm{d}f = 0\} = \{f \in \mathbb{C}[X] \mid f \text{ locally constant}\}.$$

As a consequence, the following Theorem holds also in the singular case.

Proposition

An algebraic variety $X \subseteq \mathbb{C}^n$ has $\dim H^0_{\mathrm{dR}}(X)$ connected components.

Squarefree Regular Chains

Using an algorithm of Szántó (1999), which decomposes $I(X)$ into radical ideals described by squarefree regular chains, one can prove

Theorem

Let $f_1, \dots, f_r$ be polynomials of degree $\leq d$ defining the variety $X \subseteq \mathbb{C}^n$ of dimension m. Then for any $\delta \in \mathbb{N}$ the two vector spaces

$$\{f \in I(X) \mid \deg f \leq \delta\}$$

and

$$\{f \in \mathbb{C}[X] \mid \mathrm{d}f = 0 \text{ and } \deg f \leq \delta\}$$

are solution sets of linear systems of equations of size $\delta^{\mathcal{O}(n)} d^{\mathcal{O}(n^3(n-m))}$. Furthermore, given the f_i and δ, one can compute these systems in sequential time $\delta^{\mathcal{O}(n)} d^{\mathcal{O}(n^3(n-m))}$ and parallel time $(n \log(d\delta))^{\mathcal{O}(1)}$.

Effective Bounds for the Zeroth Cohomology

Let $\deg(H^0_{\mathrm{dR}}(X))$ be the minimal $\delta \geq 0$ such that each $f \in H^0_{\mathrm{dR}}(X)$ has $\deg f \leq \delta$.

Proposition

Let $X \subseteq \mathbb{C}^n$ be a variety of dimension m and degree D. Then

$$\deg(H^0_{\mathrm{dR}}(X)) \leq D^{m+1}.$$

The proof uses Jelonek's effective Nullstellensatz

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The proof uses Jelonek's effective Nullstellensatz very similarly as in the proof of

Proposition

Let $X \subseteq \mathbb{C}^n$ be a variety of dimension m and degree D. Then

$$\deg(H^0_{\mathrm{dR}}(X)) \leq \frac{n+1}{4} D^2.$$

Remark. In the case $m = n - 1$ we can drop the factor $n + 1$.

Effective Bounds for the Zeroth Cohomology

Proof of the last bound. Let $X = Z_1 \cup \cdots \cup Z_t$, Z_i the connected components, and $D_i := \deg Z_i$. Since $I(X) = \bigcap_i I(Z_i)$, the Chinese Remainder Theorem implies

$$\mathbb{C}[X] = \mathbb{C}[X_1, \ldots, X_n]/I(X) \simeq \prod_{i=1}^{t} \mathbb{C}[Z_i].$$

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$$\mathbb{C}[X] = \mathbb{C}[X_1, \ldots, X_n]/I(X) \simeq \prod_{i=1}^{t} \mathbb{C}[Z_i].$$

We have a corresponding mcsoi $e_1, \ldots, e_t \in \mathbb{C}[X]$, i.e.,

- $e_i^2 = e_i$, $e_i e_j = 0$ for $i \neq j$,
- $1 = e_1 + \cdots + e_t$,
- $\forall i$: $e_i = e + f$ with orthogonal idempotents implies $e = 0$ or $f = 0$.

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Here,

$$e_i = \begin{cases} 1 & \text{on} \quad Z_i, \\ 0 & \text{on} \quad X \setminus Z_i. \end{cases}$$

Then: $e_1, \ldots, e_t$ is a basis of $H^0_{\mathrm{dR}}(X)$.

Effective Bounds for the Zeroth Cohomology

Let $i < j$. Since $Z_i \cap Z_j = \emptyset$, from Kollár's effective NS for arbitrary ideals we obtain $\varphi_{ij} \in I(Z_i), \varphi_{ji} \in I(Z_j)$ such that

$$\deg(\varphi_{ij}),\ \deg(\varphi_{ji}) \leq (n+1)D_iD_j \quad \text{and} \quad \varphi_{ij} + \varphi_{ji} = 1.$$

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Their degrees satisfy

$$\deg e_j \leq (n+1) D_j \sum_{i \neq j} D_i = (n+1) D_j (D - D_j) \leq (n+1) \left(D/2\right)^2 . \ \square$$

Example (derived from Masser, Philippon, and Brownawell)

The last bound is tight up to the factor of $n+1$. Consider

$$Z_1 := \mathcal{Z}(X_1, X_2X_3^{d-1} - 1),\ Z_2 := \mathcal{Z}(X_1X_3^{d-1} - X_2^d), \quad X := Z_1 \cup Z_2 \subseteq \mathbb{C}^3.$$

The Z_i are the connected components of X, and $D = \deg X = 2d$. Now let $e_1, e_2 \in \mathbb{C}[X_1, X_2, X_3]$ be the mcsoi and $\delta := \max\{\deg e_1, \deg e_2\}$.

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$$E_1 + E_2 = X_0^\delta \text{ on } \overline{X}, \quad E_1 = 0 \text{ on } \overline{Z}_2, \quad E_2 = 0 \text{ on } \overline{Z}_1,$$

Hence $X_0^\delta \in I(\overline{Z}_1) + I(\overline{Z}_2) = (F_1, F_2, F_3)$, where

$$F_1 = X_1, \quad F_2 := X_2X_3^{d-1} - X_0^d, \quad F_3 = X_1X_3^{d-1} - X_2^d.$$

$$\Rightarrow \quad X_0^\delta = 0 \quad \text{in} \quad \mathbb{C}[X_0, \dots, X_3]/(F_1, F_2, F_3, X_3 - 1) \simeq \mathbb{C}[X_0]/(X_0^{d^2}),$$

$$\Rightarrow \quad \delta \geq d^2 = D^2/4.$$

Higher Betti Numbers

Smooth Projective Case

Smooth Projective Case

Theorem (S. 2009)

Given homogeneous $f_1, \ldots, f_r \in \mathbb{C}[X_0, \ldots, X_n]$ with $\deg f_i \leq d$, one can test whether $X := \mathcal{Z}(f_1, \ldots, f_r) \subseteq \mathbb{P}^n$ is smooth and if so, compute the Betti numbers $b_0(X), \ldots, b_{2n}(X)$ in sequential time $d^{\mathcal{O}(n^4)}$ and parallel time $(n \log d)^{\mathcal{O}(1)}$.

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How to define $H^{\bullet}_{\mathrm{dR}}(X)$ for a projective variety $X \subseteq \mathbb{P}^n$?

Problem: The are not enough globally defined differential forms, e.g., every globally defined regular function on X is constant!

Hypercohomology

Idea: cover X with open affine subsets $X = \bigcup_{i=0}^{m} U_i$ and do patchwork.

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Example: $m = \dim X = 1$, i.e., $X = U_0 \cup U_1$. The Mayer-Vietoris sequence reads

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\begin{aligned}
0 \to H^0(X) \to H^0(U_0) \oplus H^0(U_1) \overset{\delta^0}{\to} H^0(U_0 \cap U_1) \to \\
\to H^1(X) \overset{\rho}{\to} H^1(U_0) \oplus H^1(U_1) \overset{\delta^1}{\to} H^1(U_0 \cap U_1) \to \\
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\to H^2(X) \to 0.&
\end{aligned}
$$

Then we have

$$
\begin{aligned}
H^0(X) &\simeq \ker \delta^0, \\
H^1(X) &\simeq \ker \rho \oplus \operatorname{im} \rho \simeq \operatorname{coker} \delta^0 \oplus \ker \delta^1, \\
H^2(X) &\simeq \operatorname{coker} \delta^1.
\end{aligned}
$$

Recall that $\delta^i(\alpha, \beta) = \beta | U_0 \cap U_1 - \alpha | U_0 \cap U_1$.

Definition of Hypercohomology

Denote by Ω_X^p the sheaf of algebraic differential p-forms on X, i.e.,

$$\Omega_X^p(U) = \Omega^p_{\mathbb{C}[U]/\mathbb{C}} \quad \text{for an open affine } U \subset X.$$

Let $\mathcal{U} := \{U_i \,|\, 0 \leq i \leq s\}$ an open cover of X and for $0 \leq i_0 < \cdots i_q \leq s$ set $U_{i_0 \cdots i_q} := U_{i_0} \cap \cdots \cap U_{i_q}$. Define

$$C^{p,q} := C^{p,q}(\mathcal{U}, \Omega_X^\bullet) := \bigoplus_{i_0 < \cdots < i_q} \Omega_X^p(U_{i_0 \cdots i_q}) \quad \text{for all} \quad p, q \geq 0.$$

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We have the exterior differential $d \colon C^{p,q} \to C^{p+1,q}$ and the Čech differential $\delta \colon C^{p,q} \to C^{p,q+1}$ defined by

$$(\delta(\omega))_{i_0 \cdots i_{q+1}} := \sum_{\nu=0}^{q+1} (-1)^\nu \omega_{i_0 \cdots \widehat{i_\nu} \cdots i_{q+1}} | U_{i_0 \cdots i_{q+1}} \quad \text{for} \quad \omega = (\omega_{i_0 \cdots i_q}) \in C^q.$$

Then $\mathrm{d} \circ \mathrm{d} = 0$, $\delta \circ \delta = 0$, and $\mathrm{d} \circ \delta = \delta \circ \mathrm{d}$, i.e., $C^{\bullet,\bullet}$ is a double complex.

Definition of Hypercohomology 2

Define the total complex

$$T^k := \mathrm{tot}^k(C^{\bullet,\bullet}) = \bigoplus_{p+q=k} C^{p,q}$$

with the differential

$$d^{\mathrm{tot}}\colon T^k \to T^{k+1},\ (\omega_{p,q})_{p+q=k} \mapsto \left(\mathrm{d}\omega_{p-1,q} + (-1)^p \delta\omega_{p,q-1}\right)_{p+q=k+1}.$$

The sign is needed to ensure $d^{\mathrm{tot}} \circ d^{\mathrm{tot}} = 0$.

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The sign is needed to ensure $d^{\mathrm{tot}} \circ d^{\mathrm{tot}} = 0$.

The algebraic de Rham cohomology of X is defined as the cohomology

$$H^{\bullet}_{\mathrm{dR}}(X) := H^{\bullet}(T^{\bullet}, d^{\mathrm{tot}}).$$

Definition of Hypercohomology 3

Back to the case $m = 1$. Then the double complex looks like

$$\begin{tikzcd}
\mathbb{C}[U_{01}] = C^{0,1} \arrow[r, "\mathrm{d}"] & C^{1,1} = \Omega_{U_{01}} \\
\mathbb{C}[U_0] \oplus \mathbb{C}[U_1] = C^{0,0} \arrow[u, "\delta"] \arrow[r, "\mathrm{d}"] & C^{1,0} = \Omega_{U_0} \oplus \Omega_{U_1} \arrow[u, "\delta"']
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\end{tikzcd}$$

Hence

$$H^0_{\mathrm{dR}}(X) = \{(f, g) \mid f = g \text{ on } U_{01},\ \mathrm{d}f = \mathrm{d}g = 0\},$$
$$H^1_{\mathrm{dR}}(X) = \{(f, \alpha, \beta) \mid \mathrm{d}f = \beta - \alpha \text{ on } U_{01}\}/(\mathbb{C}[U_0] + \mathbb{C}[U_1]) \oplus \mathrm{im}\,\mathrm{d},$$
$$H^2_{\mathrm{dR}}(X) = \Omega_{U_{01}}/\mathrm{d}\mathbb{C}[U_{01}] + \Omega_{U_0} + \Omega_{U_1}.$$

Castelnuovo-Mumford Regularity

Let $X \subseteq \mathbb{P}^n$ be smooth, and let $\mathcal{O}_X(1)$ be the very ample line bundle of the embedding. For a coherent sheaf $\mathcal{F}$ on X put $\mathcal{F}(k) := \mathcal{F} \otimes \mathcal{O}_X(k)$.

The coherent sheaf $\mathcal{F}$ on X is called *k-regular* iff

$$H^i(X, \mathcal{F}(k-i)) = 0 \quad \text{for all} \quad i > 0.$$

Castelnuovo-Mumford regularity $\mathrm{reg}\,(\mathcal{F}) := \inf\{k \in \mathbb{Z} \mid \mathcal{F}\ k\text{-regular}\}$.

The required degree bounds are provided by the following theorem.

Theorem (S. 2009)

Let $X \subset \mathbb{P}^n$ be a smooth irreducible projective variety of dimension m and degree D. Let $e := n - m$ be the codimension of X. Then

$$\begin{aligned} \mathrm{reg}\,(\Omega_X^p) &\leq p(em+1)D \quad \textit{for} \quad p > 0, \\ \mathrm{reg}\,(\mathcal{O}_X) &\leq e(D-1). \end{aligned}$$

Higher Betti Numbers

Smooth Affine Case

Smooth Affine Case

Notation. $\deg(f\,\mathrm{d}X_{i_1}\wedge\cdots\wedge\mathrm{d}X_{i_p}) = \deg f + p, \quad f\in\mathbb{C}[X_1,\ldots,X_n]$,
$\deg(H^p_{\mathrm{dR}}(X)) := \min\{\delta \mid$ each class in $H^p_{\mathrm{dR}}(X)$ can be represented by a p-form of degree $\leq \delta\}$.

Theorem (S. 2011/2012)

Let $X\subseteq\mathbb{C}^n$ be a smooth m-dimensional variety of degree D.

(i) *Then*

$$\begin{aligned}\deg(H^p_{\mathrm{dR}}(X)) &\leq 2^{2pm+6m+2}p^{2pm+6m+1}D^{4pm+10m+1}+D^{m+1}\\ &= (pD)^{\mathcal{O}(pm)}.\end{aligned}$$

(ii) *If $m=n-1$ and $D\geq 3$, then*

$$\deg(H^p_{\mathrm{dR}}(X)) \leq (p+1)(D+1)(2D^n+D)^{p+1} = D^{\mathcal{O}(pn)}.$$

Hypersurface Complement

Our proof uses the following previously known special case. Let $f \in \mathbb{C}[X_1, \ldots, X_n]$ and $U := \mathbb{C}^n \setminus \mathcal{Z}(f)$, which is a smooth affine variety. Its coordinate ring is the localization

$$\mathbb{C}[X_1, \ldots, X_n]_f = \left\{ \frac{g}{f^s} \,|\, g \in \mathbb{C}[X_1, \ldots, X_n], s \in \mathbb{N} \right\}.$$

Theorem (Dimca/Deligne 1990)

Each class in $H^p_{\mathrm{dR}}(U)$ can be represented by a differential p-form

$$\alpha / f^p \quad \textit{with} \quad \deg \alpha = p(\deg f + 1).$$

Gysin Sequence

The main tool in our proof for the hypersurface case is the Gysin sequence.

Lemma

Let Y be an irreducible smooth variety and $X \subseteq Y$ a smooth hypersurface. Then there is an exact sequence

$$\cdots \to H^p_{\mathrm{dR}}(Y) \to H^p_{\mathrm{dR}}(Y \setminus X) \stackrel{\mathrm{Res}}{\to} H^{p-1}_{\mathrm{dR}}(X) \to H^{p+1}_{\mathrm{dR}}(Y) \to \cdots$$

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Corollary

For a smooth hypersurface $X \subseteq \mathbb{C}^n$ the residue map

$$\mathrm{Res}\colon H^p_{\mathrm{dR}}(\mathbb{C}^n \setminus X) \xrightarrow{\simeq} H^{p-1}_{\mathrm{dR}}(X)$$

is an isomorphism for all $p > 0$.

Effective Gysin Sequence

Let $A := \mathbb{C}[X_1, \dots, X_n]$ and $X := \mathcal{Z}(f)$ smooth, where $f \in A$ is squarefree. Then the relevant coordinate rings are

$$\mathbb{C}[\mathbb{C}^n \setminus X] = A_f \quad \text{and} \quad B := \mathbb{C}[X] = A/(f).$$

Furthermore, $D := \deg X = \deg f$.

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Furthermore, $D := \deg X = \deg f$.

Theorem

Let $D \geq 3, p > 0$. The residue map

$$\mathrm{Res}\colon H^p_{\mathrm{dR}}(A_f) \to H^{p-1}_{\mathrm{dR}}(B)$$

is induced by a map $\Omega^p_{A_f} \to \Omega^{p-1}_B$ which takes a p-form $\omega = \frac{\alpha}{f^s}$ to a $(p-1)$-form of degree at most

$$(2D^n + D)^s(\deg \omega + sD).$$

Example

Consider the case $n = 1$. Then $f = \prod_i^d (X - \zeta_i) \in \mathbb{C}[X]$.
Partial fraction decomposition implies that $H^1_{\mathrm{dR}}(\mathbb{C} \setminus \mathcal{Z}(f))$ has the basis

$$\frac{\mathrm{d}X}{X - \zeta_i}, \quad 1 \le i \le d.$$

The mcsoi

$$e_i := \prod_{j \neq i} (X - \zeta_j) / \prod_{j \neq i} (\zeta_i - \zeta_j), 1 \le i \le d$$

is a basis of $H^0_{\mathrm{dR}}(\mathcal{Z}(f))$. We have

$$\mathrm{Res}\colon H^1_{\mathrm{dR}}(\mathbb{C} \setminus \mathcal{Z}(f)) \to H^0_{\mathrm{dR}}(\mathcal{Z}(f)), \quad \frac{g}{f^s}\mathrm{d}X \mapsto \sum_{i=1}^d \mathrm{Res}_{\zeta_i}\left(\frac{g}{f^s}\right) e_i,$$

where Res_{ζ_i} denotes the classical *residue* at ζ_i of a meromorphic function, i.e., the coefficient of $(X - \zeta_i)^{-1}$ in the Laurent expansion around ζ_i.

Proof for Higher Codimension

The proof of the bound for arbitrary smooth affine varieties also proves an effective Gysing sequence for the case of a codimension 2 complete intersection of a very special type.

Furthermore, we make Čech cohomology, hypercohomology, and local cohomology effective. . .

Why not make the whole of algebraic geometry effective. . . ?

Thank you!

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