12. Problem set for "Models of set theory I", Summer 2011

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Problem 45 (Two-step iteration and κ-chain condition)**.** Let κ be an uncountable regular cardinal. A partial order $(\mathbb{P}, \leq)$ satisfies the κ-c.c. iff $|A| < \kappa$ for every antichain $A \subseteq \mathbb{P}$. Suppose $\mathbb{P}$ is a partial order and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name such that $1_{\mathbb{P}} \Vdash (\dot{\mathbb{Q}}$ is a partial order). Prove that if $\mathbb{P} * \dot{\mathbb{Q}}$ satisfies the κ-c.c., then $1_{\mathbb{P}} \Vdash (\dot{\mathbb{Q}}$ satisfies the κ-c.c.). You may prove the claim of this and the following problems inside a countable transitive model M of ZFC, if this is more convenient for you.

Problem 46 (Two-step iteration of κ-closed forcings)**.** Let κ be an uncountable regular cardinal. A partial order $(\mathbb{P}, \leq)$ is κ-closed iff for every $\lambda < \kappa$ and every sequence $p_0 \geq p_1 \geq ... \geq p_\alpha \geq ...$ for $\alpha < \lambda$ in $\mathbb{P}$, there is a condition $p \in \mathbb{P}$ with $p \leq p_\alpha$ for all $\alpha < \lambda$ (this should not be confused with another commonly used definition of κ-closure, where $\lambda < \kappa$ is replaced by $\lambda \leq \kappa$, as in Jech's book). Suppose $\mathbb{P}$ is a κ-closed partial order and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name such that $1_{\mathbb{P}} \Vdash (\dot{\mathbb{Q}}$ is a κ-closed partial order), then $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-closed.

Problem 47. Suppose $\mathbb{P}$ and $\mathbb{Q}$ are partial orders. Show that there is a dense embedding (see Problem 25) $f : \mathbb{P} \times \mathbb{Q} \to \mathbb{P} * \check{\mathbb{Q}}$.

Problem 48. Suppose $(T, \leq)$ is a pruned Suslin tree as in the proof of Theorem 9.4, i.e. T is a Suslin tree and every node of T has successors on every countable height. Let $(\mathbb{P}, \leq)$ denote T with the reversed partial order. We equip $\mathbb{P} \times \mathbb{P}$ with the product partial order, i.e. $(p, q) \leq (r, s)$ iff $p \leq r$ and $q \leq s$. Show that $(\mathbb{P} \times \mathbb{P}, \leq)$ does not satisfy the c.c.c.

Please hand in your solutions on Wednesday, 06 July before the lecture. This is the last problem set.