10. Problem set for "Models of set theory I", Summer 2011

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Problem 37 (κ-chain condition)**.** Suppose κ is a regular cardinal. Adapt Lemmas 8.3 and 8.6 and their proofs to κ-c.c. forcings (see Problem 33).

Problem 38 (Ultrafilters)**.** Let A be a Boolean algebra.

a) A set $S \subseteq A$ has the *finite intersection property* (f.i.p.) if for all n and all $a_1, ..., a_n \in A$, $a_1 \wedge ... \wedge a_n \neq 0$. By Zorn's Lemma, every set with the f.i.p. is contained in a maximal set with the f.i.p. Show that a maximal set $S \subseteq A$ with the f.i.p. is an ultrafilter on A (see Definition 9.7).

b) For each $a \in A$ let $D_a = \{b \in A : b \leq a \vee b \perp a\}$. Then each D_a is a dense subset of A. If $F \subseteq A$ is a $(D_a)_{a \in A}$-generic filter, then F is an ultrafilter.

Problem 39 (Coherent map)**.** Construct a sequence $(f_\alpha : \alpha < \omega_1)$ such that for all $\alpha < \beta < \omega_1$,

1. $f_\alpha : \alpha \to \omega$ is injective,
2. $f_\alpha(\gamma) = f_\beta(\gamma)$ holds for all but finitely many $\gamma < \alpha$, and
3. $\omega - range(f_\alpha)$ is infinite.

Problem 40 (Aronszajn tree)**.** Let $(f_\alpha : \alpha < \omega_1)$ be a sequence as in Problem 39. Let $T := \{f_\alpha \upharpoonright \beta : \alpha, \beta < \omega_1\}$ and $pred_T(t) := \{s \in T : s \subsetneq t\}$. Let $Lev_\alpha(T) := \{t \in T :$ the order type of $pred_T(t)$ is $\alpha\}$ denote the α^{th} level of T. Show:

a) There is no branch of length ω_1 in T.

b) $0 < |Lev_\alpha(T)| < \omega_1$ for every $\alpha < \omega_1$.

Such a tree is called an *Aronszajn tree.*

Please hand in your solutions on Wednesday, 15 June before the lecture.