9. Problem set for "Models of set theory I", Summer 2011

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Problem 33 (Chain condition)**.** Suppose κ is a cardinal and $(\mathbb{P}, \leq)$ is a partial order. $\mathbb{P}$ has the *κ-chain condition* (κ-c.c.) if every antichain in $\mathbb{P}$ has size less than κ. Let κ be least such that $\mathbb{P}$ has the κ-c.c.

a) Suppose $A \subseteq \mathbb{P}$ is a maximal antichain and every $p \in \mathbb{P}$ is an *atom*, i.e. there are no $q, r \leq p$ with $q \perp r$. Show that $\kappa = |A|$.

b) Show that $\kappa \neq \omega$. (To show this, begin with a maximal antichain and replace each nonatomic condition p with an antichain below p which is predense below p. Continue this process and find an infinite antichain.)

Problem 34 (Δ-systems)**.** A family $\mathcal{D}$ is a Δ-*system* if there is a finite set r (the *root*) such that $s \cap t = r$ for all $s \neq t$ in $\mathcal{D}$. Show:

a) Let $\mathcal{F}$ be a family of finite sets and $\kappa = |\mathcal{F}|$ a regular cardinal. Then there is a Δ-system $\mathcal{D} \subseteq \mathcal{F}$ of cardinality κ.

b) If κ is a singular cardinal, there is a family $\mathcal{F}$ of finite sets of cardinality κ with no Δ-system $\mathcal{D} \subseteq \mathcal{F}$ of cardinality κ.

Problem 35 (Nice names, cardinal arithmetic)**.**

a) Recall the definition of the sets a_α, $\alpha < \kappa$ mentioned in Lemma 8.1. Write down explicitly a nice name for each a_α. The name is allowed to mention Boolean values $[\![\ ]\!]^*$.

b) Suppose κ is an infinite cardinal and GCH holds. Show that $\kappa^\omega = \kappa$ if $cf(\kappa) > \omega$ and $\kappa^\omega = \kappa^+$ if $cf(\kappa) = \omega$.

Problem 36 (Random forcing is ω^ω-bounding)**.** Let $\mathbb{M}$ denote the collection of Borel subsets of $\mathbb{R}$ of positive Lebesgue measure μ and let $p \leq q$ iff $p \subseteq q$ for $p, q \in \mathbb{M}$ (*Random forcing*). Suppose N is a countable transitive model of ZFC and G is $\mathbb{M}^N$-generic over N.

a) Suppose $p \in \mathbb{M}^N$ forces over N that $\dot{f} : \omega \to \omega$. Let $\epsilon > 0$ and $n \in \omega$. Prove that there are $q \leq p$ in $\mathbb{M}^N$ and $g_n \in \omega$ such that $q \Vdash \dot{f}(n) \leq g_n$ and $\mu(p - q) < \epsilon$.

b) Prove that for any $f : \omega \to \omega$ in $N[G]$, there is a function $g : \omega \to \omega$ in M with $f \leq^* g$.

Please hand in your solutions on Wednesday, 08 June before the lecture.