8. Problem set for "Models of set theory I", Summer 2011

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Problem 29 (Strategic closure)**.** Suppose M is a countable transitive model of ZFC and $(\mathbb{P}, \leq)$ is a partial order. Consider the following two player game that lasts ω many rounds: Let $p_0 = q_0 = 1_{\mathbb{P}}$. In the n-th round the first player choses a condition $p_{n+1} \leq q_n$ and the second player replies by chosing a condition $q_{n+1} \leq p_{n+1}$. The second player wins the game iff there is a common extension $p \in \mathbb{P}$ of the p_n. Suppose that in M, the second player has a winning strategy for this game (i.e. $\mathbb{P}$ is ω-*strategically closed*). Show that for every G that is $\mathbb{P}$-generic over M, $M[G]$ does not contain any new function from ω into the ordinals.

Problem 30 (ω_1-closed forcings)**.** Suppose M is a transitive model of ZFC and $(\mathbb{P}, \leq)$ is a partial order in M. Suppose $\mathbb{P}$ is ω_1-*closed* in M, i.e. for every decreasing ω-sequence $p_0 \geq p_1 \geq ... \geq p_n \geq ...$ in M of conditions in $\mathbb{P}$, there is a condition p with $p \leq p_n$ for all $n \in \omega$. Suppose every ω-sequence of elements of M is an element of M. Prove similar to the Rasiowa-Sikorsky Theorem 6.4 that for every family $\mathcal{D} \in M$ of size ω_1, there is a $\mathcal{D}$-generic filter over M.

Problem 31 (Cohen subsets of ω_1)**.** Suppose M is a countable transitive model of ZFC. Let $\mathbb{Q} = \{p : A \to 2 : A \subseteq \omega_1 \text{ countable}\}^M$ and $p \leq q$ iff $p \supseteq q$. Suppose G is a $\mathbb{Q}$-generic filter over M. Show that CH holds in $M[G]$.

Problem 32 (Cohen forcing is not ω^ω-bounding)**.** Let $\mathbb{P} = \{p : n \to \omega : n \in \omega\}$ and $p \leq q :\Leftrightarrow p \supseteq q$ for $p, q \in \mathbb{P}$. Note that $\mathbb{P}$ is *equivalent* to Cohen forcing $Fn(\omega, 2)$ by Problem 26, i.e. every Cohen generic extension is a $\mathbb{P}$-generic extension and vice versa. Suppose M is a countable transitive model of ZFC. Let $f \leq^* g$ for $f, g : \omega \to \omega$ iff $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. Show that in every $\mathbb{P}$-generic extension of M, there is a function $f : \omega \to \omega$ such that $f \not\leq^* g$ for all $g \in M$.

Please hand in your solutions on Wednesday, 01 June before the lecture.