6. Problem set for "Models of set theory I", Summer 2011

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Problem 21 (Regular open sets)**.** Suppose $(\mathbb{P}, \leq)$ is a partial order.

a) Let $\mathcal{F} \subseteq ro(\mathbb{P})$ be a family of regular open sets. Show that $\bigcap \mathcal{F}$ is regular open.

b) Let $A \subseteq \mathbb{P}$. Show that $\neg A$ is a regular open subset of $\mathbb{P}$.

c) Suppose $\mathbb{P} = Fn(\omega, 2)$. Find regular open sets $A, B \subseteq \mathbb{P}$ such that $A \cup B$ is not regular open.

d) Suppose M is a countable transitive model of ZFC and $(\mathbb{P}, <) \in M$. Show that for every formula $\varphi(\tau_1, ..., \tau_n)$ in the forcing language, the truth value $[\![\varphi(\tau_1, ..., \tau_n)]\!]$ is a regular open subset of $\mathbb{P}$. Lemma 6.19 might be useful.

Problem 22 (Separative partial orders)**.** A partial order $(\mathbb{P}, <)$ is called *weakly separative* if for all $p, q \in \mathbb{P}$ we have

$$p = q \Leftrightarrow \forall r \in \mathbb{P}(r \perp p \Leftrightarrow r \perp q).$$

$\mathbb{P}$ is called *separative* if for all $p, q \in \mathbb{P}$ with $p \not\leq q$, there is $r \leq p$ with $r \perp q$. Suppose $\mathbb{B}$ is a Boolean algebra with smallest element 0. A set $A \subseteq \mathbb{B}$ is called dense in $\mathbb{B}$ if $A - \{0\}$ is dense in $\mathbb{B} - \{0\}$. Let $e : \mathbb{P} \to ro(\mathbb{P})$ be the map defined in section 6.3. Show:

a) Every separative partial order is weakly separative.

b) Give an example of a finite partial order that is weakly separative, but not separative. It is sufficient to find a partial order without a largest element.

c) The range of e is dense in $ro(\mathbb{P})$.

d) e is one-to-one iff $\mathbb{P}$ is weakly separative.

Problem 23 (Boolean algebras)**.** Suppose $\mathbb{B}$ is a complete Boolean algebra and $S \subseteq \mathbb{B} - \{0\}$. Let $p||q$ mean that p, q are *compatible* in the partial order $\mathbb{B} - \{0\}$, i.e. there is $r \leq p, q$ in $\mathbb{B} - \{0\}$. Show for all $p, q \in \mathbb{B} - \{0\}$:

a) $p \perp q$ iff $p \wedge q = 0$, and $p \leq q$ iff $p \wedge q = p$ iff $p \wedge \neg q = 0$.

b) $p \leq q$ implies $r \wedge p \leq r \wedge q$, and $(\mathbb{B}, \leq)$ is separative (see Problem 22).

c) $p \wedge \bigvee S = \bigvee_{s \in S}(p \wedge s)$ (to show this, write $s = (p \wedge s) \vee (\neg p \wedge s)$ for each $s \in S$).

d) S is predense below p iff $\forall q \leq p(q || \bigvee S)$ iff $p \leq \bigvee S$.

Problem 24 (Generic filters)**.** Suppose N is a countable transitive model of ZFC and $(\mathbb{P}, \leq) \in N$ is a partial order.

a) Suppose H is an arbitrary subset of $\mathbb{P}$. If σ, τ are $\mathbb{P}$-names, let $\sigma <_H \tau$ iff there is $p \in H$ with $(\sigma, p) \in \tau$. Show that $<_H$ is well-founded on $N^{\mathbb{P}}$. Find the function F that is used in the Recursion Theorem 2.4 to define τ_H for $\tau \in N^{\mathbb{P}}$ (as in Definition 6.8).

b) Suppose $H \subseteq \mathbb{P}$ and $H \in N$. Show by induction on $<_H$ that $N[H] \subseteq N$.

c) Let $\alpha := Ord^N$. Find a set $a \subseteq \omega$ such that $\alpha \in M$ for every model of ZFC with $a \in M$. You may use that there is a bijection $g : \omega \times \omega \to \omega$ which is an element of every transitive model of ZFC.

d) Let $\mathbb{P} = Fn(\omega, 2)$ and $\alpha := Ord^N$. Let $\chi_a : \omega \to 2$ be the characteristic function of the set a in c). Use Problem 13 to find a filter $H \subseteq \mathbb{P}$ such that $\alpha \in M$ for every transitive model M of ZFC with $H \in M$. Conclude that $H \notin N[G]$ for every generic extension $N[G]$ of N.

Please hand in your solutions on Wednesday, 18 May before the lecture.