5. Problem set for "Models of set theory I", Summer 2011

Stefan Geschke, Philipp Schlicht, Anne Fernengel, Allard van Veen

Suppose M is a countable transitive model of ZFC and $(\mathbb{P}, \leq) \in M$ is a partial order. $\mathbb{P}$ is called *nonatomic* if every $p \in \mathbb{P}$ has at least two incompatible extensions.

Problem 17. Show:

(1) If $\mathbb{P}$ is nonatomic, then no filter $G \subseteq \mathbb{P}$ that is $\mathbb{P}$-generic over M is actually an element of M.
(2) Every $\mathbb{P}$-generic filter G over M is a maximal filter.

Hint for (1): Given a filter $G \in M$, find a dense subset $D \in M$ that is disjoint from G.

Problem 18. Show: If $\mathbb{P}$ is nonatomic, then $|\{G \subseteq \mathbb{P} : \text{G is } \mathbb{P}\text{-generic over } M\}| = 2^\omega$.

Hint: Looking at the proof of Theorem 6.4 might be useful. Construct a family $(p_s : s \in 2^{<\omega})$ of conditions with $p_t \leq p_s$ for $s \subseteq t$ such that $G_x := \{p \in \mathbb{P} : \exists n(p_{x\restriction n} \leq p)\}$ is a generic filter for each $x \in 2^\omega$.

Problem 19. Given $\mathbb{P}$-names $\sigma, \tau \in M$, find $\mathbb{P}$-names $\mu, \nu, \pi, \rho \in M$ such that for every $\mathbb{P}$-generic filter G over M

(1) $\mu_G = (\sigma_G, \tau_G) = \{\{\sigma_G\}, \{\sigma_G, \tau_G\}\}$,
(2) $\nu_G = \sigma_G \cup \tau_G$,
(3) $\pi_G = \sigma_G \cap \tau_G$, and
(4) $\rho_G = \sigma_G - \tau_G$,

and prove that they have these properties, where you may assume $dom(\sigma) \cup dom(\tau) \subseteq \{\check{x} : x \in y\}$ for some $y \in M$ in (3) and (4).

Hint for (4): Let $\rho = \{(\check{x}, p) : x \in y \text{ and } \exists q \geq p((\check{x}, q) \in \sigma) \text{ and } \forall r || p((\check{x}, r) \notin \tau)\}$, where $r||p$ means that r and p are compatible, i.e. there is some $s \leq r, p$. If $x \in \sigma_G - \tau_G$, there is $q \in G$ with $(\check{x}, q) \in \sigma$ and $(\check{x}, p) \notin \tau$ for all $p \in G$. Show that $D := \{p \in \mathbb{P} : \forall r || p((\check{x}, r) \notin \tau) \text{ or } \exists r \geq p(\check{x}, r) \in \tau\}$ is dense.

Problem 20. Suppose M and $\mathbb{P}$ are as in problem 18 and G is a $\mathbb{P}$-generic filter over M. Assume that $M[G]$ satisfies ZF and show that in $M[G]$ every set can be well-ordered.

Hint: You may use that every name can be well-ordered in M, since M satisfies ZFC. Show that every surjective image of a well-orderable set can be well-ordered.

Please hand in your solutions on 11 May before the lecture.