3. Problem set for "Models of set theory I", Summer 2011

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Problem 9. Suppose M is a transitive class such that for every set $x \subseteq M$ there is a set $y \in M$ with $x \subseteq y$ and $(M, \in)$ satisfies the Separation Axiom for Δ_0 formulas. Prove that

1. L satisfies the assumptions on M,
2. if $x \in M$, then $\bigcup x \in M$, and
3. if $x \in M$, then there is some $y \in M$ with $(M, \in) \models$ 'y is the power set of x'.

Problem 10. Suppose $M = (X, R_i, f_i : i \in \omega)$ is a structure on a set $X \supseteq \aleph_1$ in a countable language. A set $C \subseteq \aleph_1$ is closed in $\aleph_1$ if $\alpha \in C$ for all $\alpha < \aleph_1$ with $C \cap \alpha$ unbounded in α. Prove that there is a closed unbounded (*club*) subset $C \subseteq \aleph_1$ such that for every $\alpha \in C$ there is a substructure $N = (\bar{X}, \bar{R}_i, \bar{f}_i : i \in \omega) \prec M$ with $X \cap \aleph_1 = \alpha$.

Hint: Use Skolem hulls (closure under Skolem functions, see the proof of Theorem 4.4) to construct a sequence $(X_\alpha : \alpha < \aleph_1)$ such that each $(X_\alpha, R_i \upharpoonright X_\alpha, f_i \upharpoonright X_\alpha : i \in \omega)$ is a countable elementary substructure of M and

i. $X_\alpha \subseteq X_\beta$ for all $\alpha < \beta < \aleph_1$,
ii. $X_\gamma = \bigcup_{\beta<\gamma} X_\beta$ for all limit ordinals $\gamma < \aleph_1$,
iii. $X_\alpha \cap \aleph_1 \in \aleph_1$ for each $\alpha < \aleph_1$,

and the set of $X_\alpha \cap \aleph_1$ forms a club subset of $\aleph_1$.

Problem 11. Let $\sqsubset$ be a well-ordering of a set X.

1. For all $a_1, ..., a_n, b_1, ..., b_n \in X$ let $\bar{a} = (a_1, ..., a_n) \sqsubset^n \bar{b} = (b_1, ..., b_n)$ iff $\bar{a} \neq \bar{b}$ and $a_i \sqsubset b_i$ for the minimal i with $a_i \neq b_i$. Prove that $\sqsubset^n$ is a well-ordering of X^n.
2. For all $\bar{a}, \bar{b} \in X^{<\omega} = \bigcup_{n\in\omega} X^n$ let $\bar{a} \sqsubset^{<\omega} \bar{b}$ if either $\bar{a}$ is a shorter finite sequence than $\bar{b}$ or for some $n \in \omega$, $\bar{a}, \bar{b} \in X^n$ and $\bar{a} \sqsubset^n \bar{b}$. Prove that $\sqsubset^{<\omega}$ is a well-ordering of $X^{<\omega}$.

Problem 12. Suppose $V = L$ and $\alpha < \aleph_1$. Give an explicit example of a subset a of ω such that $a \in L_{\aleph_1} - L_\alpha$.

Hint: See Exercise 5.15 in the lecture notes. You may use that the Reflection Principle holds for $(L_\alpha : \alpha < \aleph_1)$ (this is true since $\aleph_1$ has uncountable confinality). Note that $L_{\aleph_1}$ satisfies ZF without the Power set Axiom by Lemma 5.12. You may use that there is a finite fragment ZF* of ZF without the Power set Axiom which implies the following version of Theorem 4.5 (Mostowski collapse): If E is a binary relation on a set X and (X, E) is extensional and well-founded, then there are a unique transitive set Y and a unique isomorphism $\mu : (X, E) \to (Y, \in)$.

Please hand in your solutions on 27 April before the lecture.