2. Problem set for "Models of set theory I", Summer 2011

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Problem 5. Let M be a transitive set. Show that $(M, \in)$ satisfies the Axiom of Extensionality. Find a set M such that $(M, \in)$ does not satisfy the Axiom of Extensionality.

Problem 6. Show that $\aleph_1$ is regular and $\aleph_\omega$ is singular.

Hint: If A is cofinal in $\aleph_1$, then $\aleph_1 = \bigcup A$ (why?). If $A \subseteq \aleph_1$ is countable, what is the size of $\bigcup A$? For the singularity of $\aleph_\omega$ use the fact that the union (supremum) of a set of cardinals is again a cardinal.

Problem 7. Let S be a (set-like) well-founded relation. Prove that there is a unique function $r_S : V \to Ord$ (the *rank function for* S) such that $r_S(y) = \sup_{(x,y)\in S}(r_S(x) + 1)$ for all $y \in V$. Prove that $r_\in(x)$ is the least ordinal α with $x \in V_{\alpha+1}$ (V_α is defined in Theorem 3.1).

Problem 8. Work in ZF without the Axiom of Foundation. An ordinal is a transitive set well-ordered (rather than just linearly ordered) by $\in$. Let $\varphi(\alpha)$ be the formula which states that V_α exists, i.e. α is an ordinal and there is a sequence $(W_\beta : \beta \leq \alpha)$ with

i. $W_0 = \emptyset$,
ii. $W_{\beta+1} = \mathcal{P}(W_\beta)$ for all $\beta < \alpha$, and
iii. $W_\gamma = \bigcup_{\beta<\gamma} W_\beta$ for all limit ordinals $\gamma \leq \alpha$.

Let $\psi(x, \alpha)$ be the formula which states that $x \in V_\alpha$, i.e. α is an ordinal and there is a sequence $(W_\beta : \beta \leq \alpha)$ as above with $x \in W_\alpha$. Suppose M is a transitive class and let V_α^M for $\alpha \in Ord^M$ denote the set of $x \in M$ with $\psi^M(x, \alpha)$. Prove that $V_\alpha \cap M \in M$ implies $V_{\alpha+1} \cap M = \mathcal{P}^M(V_\alpha \cap M)$. Prove that $\varphi^M(\alpha)$ implies $V_\alpha \cap M = V_\alpha^M$.

Please hand in your solutions on 20 April before the lecture.