Higher Set Theory - Classical and Ordinal Computability

Exercise Sheet 5
due on Tuesday, 10 May 2011

11. Show, without using Kleene's T predicate, that for every partial recursive function $f : \omega^n \rightharpoonup \omega$ there is a primitive recursive relation R s.t.

$$f(x_0, x_1, \ldots, x_{n-1}) \simeq \mu g(P(x_0, x_1, \ldots, x_n, g)),$$

i.e., that all unbounded μ-operators can be replaced by a single one.

(6 points)

12. As known from any introductory lecture in logic, a formal proof can be expressed by finite a sequence of words ("formulas") $\phi_0, \phi_1, \ldots, \phi_{n-1}$ over the alphabet of first-order logic L_1, where ϕ_i is either an axiom from some axiom system Φ or is derived from some $\phi_{j_0}, \phi_{j_1}, \ldots, \phi_{j_{m-1}}$ $(j_0, j_1, \ldots, j_{m-1} < 1)$ by application of one of the finitely many deduction rules of first-order logic. Recall that $\Phi^{\models}$, the set of true formulas in the axiom system Φ, is not decidable.

    (a) Show, within reasonable bounds on attention to detail, that the predicate $\mathsf{Fml}(n)$ "n codes a well-formed formula of L_1" is primitive recursive *(Hint: Something similar may have been shown in your Logic class)*.

    (b) Show that for a given axiom system Φ the predicate $\vdash^\Phi (k, l)$ on $\omega \times \omega$ that states "k codes a sentence (a formula without free variables) and l codes a proof of ϕ" is primitive recursive.

    (c) Show that there is no recursive bound $F : \omega \to \omega$ s.t. $\forall k(\exists l \vdash^\Phi (k, l) \to \exists p \leq F(k) \vdash^\Phi (k, p))$. Standard coding codes more complex formulas by higher k and longer proofs by higher l, so this proves that there is no recursive bound on the length of a proof for a formula of given complexity. *You never (computably) know how long it will take to work on an exercise, even if you work flawlessly!*

(6 points)