Automatic Structures and Model Theory

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Automatic Structures

Regular Languages

Languages accepted by finite automaton.
Alternative definition: smallest class of languages which contains all subsets of a finite alphabet $\mathbf{\Sigma}$ and is closed under concatenation, union, intersection, star-operation and complementation.
Concatenation: $\mathbf{00} \cdot \mathbf{1} = \mathbf{001}$; $\mathbf{A} \cdot \mathbf{B} = \{\alpha \cdot \beta : \alpha \in \mathbf{A} \wedge \beta \in \mathbf{B}\}$;
$\mathbf{A}^* = \{\lambda\} \cup \mathbf{A} \cup (\mathbf{A} \cdot \mathbf{A}) \cup (\mathbf{A} \cdot \mathbf{A} \cdot \mathbf{A}) \cup \ldots$ (Kleene star).

Convolution

Given $\alpha, \beta \in \mathbf{\Sigma}^*$ and $\# \notin \mathbf{\Sigma}^*$, $\mathbf{conv}(\alpha, \beta)$ is the sequence all pairs made of $\alpha_\mathbf{n}$ and $\beta_\mathbf{n}$ for $\mathbf{n} = \mathbf{0}, \mathbf{1}, \ldots, \mathbf{max}\{|\alpha|, |\beta|\} - \mathbf{1}$ with $\alpha_\mathbf{n} = \#$ for $\mathbf{n} \geq |\alpha|$ and $\beta_\mathbf{n} = \#$ for $\mathbf{n} \geq |\beta|$.
Relation $\mathbf{R}$ is automatic iff $\{\mathbf{conv}(\alpha, \beta) : \mathbf{R}(\alpha, \beta)\}$ is regular.

Automatic Structures

Domain and all relations of the structure are automatic.

Examples

Functions

A function is automatic iff its graph is an automatic relation; all functions which are first-order definable in a given set of automatic relations are automatic.

Fibonacci Numbers and Addition [Tan 2008]

Domain $(\mathbf{0}^*\mathbf{01})^*$, addition $+$, comparison $<$; predicate $\mathbf{F}$. Here $\mathbf{a_1 a_2 \ldots a_n}$ represents $\mathbf{F_1 \cdot a_1 + F_2 \cdot a_2 + \ldots + F_n \cdot a_n}$ where $\mathbf{F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5}$.

Various Algebras

There are automatic presentations of the algebra of the eventually constant functions from $\mathbb{N}$ into a given finite field. Similarly for the Boolean algebra of finite and cofinite subsets of $\mathbb{N}$.

Rationals [Tsankov 2009]

The group $(\mathbb{Q}, +)$ has no automatic presentation.

Model Theory

Special models of a complete theory

Countable models: Either a theory has exactly one model which is finite or it has at least one countable model.
Prime model: Smallest countable model, is isomorphic to a substructure of every countable model.
Saturated model: Largest countable model, every countable model is isomorphic to a substructure of it.

Categoricity

A theory $\mathbf{T}$ is $\aleph_0$-categorical iff all countable models of $\mathbf{T}$ are isomorphic.
A theory $\mathbf{T}$ is $\aleph_1$-categorical iff for every cardinal $\kappa \geq \aleph_1$ all models of cardinality κ are isomorphic.

Easy Structures

The theory of structures consisting of infinitely many $\mathbf{4}$-circles is $\aleph_0$-categorical and $\aleph_1$-categorical; for each cardinal $\kappa \geq \aleph_0$ there is exactly one model consisting of κ many cycles.

```
o-o   o-o   o-o   o-o   o-o   o-o   o-o   ....
| |   | |   | |   | |   | |   | |   | |
o-o   o-o   o-o   o-o   o-o   o-o   o-o   ....
```

The theory of structures consisting of $\mathbb{Z}$-chains is $\aleph_1$-categorical but not $\aleph_0$-categorical:

```
.....  -o-o-o-o-o-o-o-o-o-o-o-  .......
.....  -o-o-o-o-o-o-o-o-o-o-o-  .......
.....  -o-o-o-o-o-o-o-o-o-o-o-  .......
```

For each $\mathbf{n} \in \{\mathbf{1}, \mathbf{2}, \ldots, \omega\}$ there is a countable model consisting of exactly $\mathbf{n}$ $\mathbb{Z}$-chains;
for $\kappa \geq \aleph_1$ there is exactly one model consisting of κ $\mathbb{Z}$-chains.

Circles and Chains

Let $\mathbf{T}$ be theory which has as model an infinite graph which is the union of disjoint circles of length $\mathbf{n = 2, 3, 4, \ldots}$, for each length $\mathbf{n}$ there is exactly one circle. There are the following models:

This model $\mathbf{A_0}$ is the prime model.

For each $\mathbf{n \in \{1, 2, 3, \ldots\}}$ there is exactly one model $\mathbf{A_n}$ consisting of the disjoint union of $\mathbf{A_0}$ and $\mathbf{n}$ $\mathbb{Z}$-chains; all these models are intermediate.

The model $\mathbf{A}_\omega$ consisting of the disjoint union of $\mathbf{A_0}$ and $\aleph_\mathbf{0}$ $\mathbb{Z}$-chains is the saturated model.

The theory $\mathbf{T}$ is $\aleph_\mathbf{1}$-categorical but not $\aleph_\mathbf{0}$-categorical.

The countable models of an $\aleph_\mathbf{1}$-categorical but not $\aleph_\mathbf{0}$-categorical theory always form this $\omega+\mathbf{1}$-chain $\mathbf{A_0, A_1, A_2, \ldots, A}_\omega$ with $\mathbf{A_i}$ being elementary embaddable into $\mathbf{A_j}$ iff $\mathbf{i \leq j}$.

Automatic Model Theory

In recursive model theory one discusses the property of structures which have presentations where all the operations are coded in a way that the domain is the set of natural numbers and every operation or relation is recursive.

In automatic model theory, one discusses the property of structures which have presentations where the domain is regular and every operation and relation on this domain is automatic. The central question is to find the parallels and differences between automatic and recursive model theory.

In particular structures with only finitely many countable models and structures which are $\aleph_1$-categorical but not $\aleph_0$-categorical are of special interest. Furthermore, structures with countably many countable models where the prime model or saturated model is automatic are also interesting.

Questions of Khoussainov and Nerode

Question 3.2. Let $n \geq 1$ be a natural number. Does there exist a theory with exactly n automatic models up to isomorphism? When $n = 1$, the theory should not be $\aleph_0$-categorical.

Question 3.3. Let T be a complete first-order decidable theory such that T has only countably many models. Is any of the following true?
1. If T has an automatic model then all countable models of T are automatic.
2. If T is $\aleph_1$-categorical and has an automatic model then all countable models of T are automatic.
3. If T has only finitely many countable models one of which is automatic then all of them are automatic.

Does the existence of an automatic saturated model of T imply that the prime model of T is also automatic?

Integers and Addition

The additive group $(\mathbb{Z}, +)$ of the integers has a theory which contains uncountably many countable models. The model $(\mathbb{Z}, +)$ is among these models and although it is in some sense simpler and smaller than the others, it is not a prime model. There are models of the theory of $(\mathbb{Z}, +)$ which do not contain $(\mathbb{Z}, +)$ as an elementary substructure.

The ordered additive group $(\mathbb{Z}, +, <)$ is the prime model of its theory. But also here there are uncountably many countable models.

The Prüfer Group

Let $\mathbb{Z}(\mathbf{p}^\infty)$ denote the Prüfer group

$$\{\mathbf{n} \cdot \mathbf{p}^{-\mathbf{m}} : \mathbf{m} \in \{\mathbf{0}, \mathbf{1}, \ldots\}, \mathbf{n} \in \{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{p}^{\mathbf{m}} - \mathbf{1}\}\}$$

where $\mathbf{x} + \mathbf{y}$ is identified with $\mathbf{x} + \mathbf{y} - \mathbf{1}$ whenever it is larger than $\mathbf{1}$. The theory of the Prüfer group is $\aleph_\mathbf{1}$-categorical but not $\aleph_\mathbf{0}$-categorical. The model $\mathbf{A_n}$ is the direct sum of $\mathbb{Z}(\mathbf{p}^\infty)$ and $\mathbf{n}$ copies of $(\mathbb{Q}, +)$.

Theorem (negative answer to Question 3.3.1–2)
The prime model $\mathbb{Z}(\mathbf{p}^\infty)$ is automatic [Nies and Semukhin 2009] while the other countable models are not.

Based on Tsankov's results, one can show that whenever a group $(\mathbf{E}, +)$ contains the rationals as a subgroup and satisfies that for each $\mathbf{e} \in \mathbf{E}$ and each prime $\mathbf{p}$ there are only finitely many $\mathbf{e}' \in \mathbf{E}$ with $\mathbf{e}' \cdot \mathbf{p} = \mathbf{e}$ then $(\mathbf{E}, +)$ is not automatic.

Two Models

Theorem

There is a theory $\mathbf{T}$ with exactly two countable automatic models plus uncountably many nonautomatic ones.

Construction of models

Let $\mathbf{u_0}, \mathbf{u_1}, \ldots$ be nodes plus, perhaps, $\mathbf{u}_\omega$. For each unordered pair $\{\mathbf{u_i}, \mathbf{u_j}\}$ of two different members there is a copy $\mathbf{S_{i,j}}$ of the Prüfer group with one selected element $\mathbf{s_{i,j}}$ such that $\mathbf{s_{i,j}}$ has order $\mathbf{2}^{\min\{\mathbf{i},\mathbf{j}\}}$ in the Prüfer group. There is $\mathbf{R}$ such that $\mathbf{R}(\mathbf{u_k}, \mathbf{s_{i,j}})$ holds iff $\mathbf{k} \in \{\mathbf{i}, \mathbf{j}\}$. $\mathbf{s_{i,j}}$ is the unique element of $\mathbf{S_{i,j}}$ such that $\mathbf{R}(\mathbf{u_k}, \mathbf{s_{i,j}})$ holds for some $\mathbf{u_k}$.

$\mathbf{A_0}$ has only $\mathbf{u_0}, \mathbf{u_1}, \ldots$ and $\mathbf{s_{i,j}}, \mathbf{S_{i,j}}$ with $\mathbf{i} < \mathbf{j} < \omega$ and
$\mathbf{A_1}$ has besides those also $\mathbf{u}_\omega$ and $\mathbf{s}_{\mathbf{i},\omega}, \mathbf{S}_{\mathbf{i},\omega}$ for all $\mathbf{i} < \omega$.
Both models have the same theory $\mathbf{T}$.

Generalisation

Theorem

For every $\mathbf{n} \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \ldots, \omega\}$ there is a theory with uncountably many countable models and exactly $\mathbf{n}$ automatic models.

Proof

For $\mathbf{n} = \mathbf{0}$: Theory of $(\mathbb{Q}, +, <)$.
For $\mathbf{n} = \mathbf{1}$: Theory of $(\mathbb{Z}, +, <)$.
For $\mathbf{n} \in \{\mathbf{2}, \mathbf{3}, \ldots\}$: Theory of the disjoint union of $\mathbf{n} - \mathbf{1}$ copies of the structure from the previous slide. There are $\mathbf{n}$ automatic models; for $\mathbf{m} = \mathbf{0}, \mathbf{1}, \ldots, \mathbf{n} - \mathbf{1}$ there is an automatic model which is the disjoint union of $\mathbf{m}$ copies of $\mathbf{A_0}$ and $\mathbf{n} - \mathbf{m} - \mathbf{1}$ copies of $\mathbf{A_1}$.
For $\mathbf{n} = \omega$: Theory of disjoint union of $\mathbf{n}$ copies of $\mathbf{A_0}$ and $\mathbf{m}$ copies of $\mathbf{A_1}$ with $\mathbf{n} + \mathbf{m} \geq \aleph_\mathbf{0}$. Model is automatic iff $\mathbf{n} + \mathbf{m} = \aleph_\mathbf{0}$ and every copy is either $\mathbf{A_0}$ or $\mathbf{A_1}$.

Three Countable Models

For $\mathbf{n = 3}$, Question 3.2 is solved by adjusting known methods.

Theorem
There is a theory with exactly three countable models, all of them automatic.

Construction
Let $\mathbf{D}$ be the positive dyadic rationals and define the models modulo a linear preorder $\leq$ such that

$$
\begin{aligned}
\mathbf{A_0}/\leq \quad &\equiv \quad \{\mathbf{q} \in \mathbf{D} : \mathbf{q} \geq \mathbf{1}\} \\
\mathbf{A_1}/\leq \quad &\equiv \quad \{\mathbf{q} \in \mathbf{D} : \mathbf{q} \geq \mathbf{1}\} \cup \{\omega + \mathbf{q} : \mathbf{q} \in \mathbf{D}\} \\
\mathbf{A_2}/\leq \quad &\equiv \quad \{\mathbf{q} \in \mathbf{D} : \mathbf{q} \geq \mathbf{1}\} \cup \{\omega\} \cup \{\omega + \mathbf{q} : \mathbf{q} \in \mathbf{D}\}
\end{aligned}
$$

where every $\mathbf{q}$ with $\mathbf{n} \leq \mathbf{q} < \mathbf{n} + \mathbf{1}$ is made to be an equivalence class of $\mathbf{n}$ elements and ω and $\omega + \mathbf{q}$ with $\mathbf{q} \in \mathbf{D}$ are made to be equivalence classes with $\aleph_\mathbf{0}$ elements.

Four, Five, ... Countable Models

Theorem

Let $\mathbf{n} \in \{\mathbf{4}, \mathbf{5}, \ldots\}$. There is a theory with exactly $\mathbf{n}$ countable models, each of them being automatic.

Proof

Models $\mathbf{A_0}, \mathbf{A_1}, \mathbf{A_2}$ from the previous slide with one additional structure: a predicate $\mathbf{P}$ with $\mathbf{n}-\mathbf{2}$ different values taking the same value on each equivalence class and between two equivalence classes $\mathbf{x}, \mathbf{y}$ with $\mathbf{x} < \mathbf{y}$ there is for each possible value of $\mathbf{P}$ a equivalence class $\mathbf{z}$ on which $\mathbf{P}$ takes that value. $\mathbf{P}$ takes on each equivalence class of $\mathbf{n} \in \{\mathbf{1}, \mathbf{2}, \ldots\}$ a fixed default value; in the case of the model $\mathbf{A_2}$, $\mathbf{P}$ can take on the equivalence class ω each of the $\mathbf{n}-\mathbf{2}$ values which gives $\mathbf{n}-\mathbf{2}$ different structures, each of them containing the equivalence class ω.

Prime Model not Automatic

Theorem

If $\mathbf{P} = \mathbf{LOGSPACE}$ then there is a $\aleph_1$-categorical but not $\aleph_0$-categorical structure such that $\mathbf{A_0}$ is not automatic and $\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A}_\omega$ are all automatic.

Construction

$\mathbf{A_0}$ consist of cycles of length $\mathbf{a_n}$ (two cycles each) and $\mathbf{A_n}$ consists of the disjoint union of $\mathbf{A_0}$ and $\mathbf{n}$ $\mathbb{Z}$-chains.

Here $\mathbf{b_0} = \mathbf{1}$, $\mathbf{b_{m+1}} = \mathbf{2^{2^{2^{b_m}}}}$ and $\mathbf{a_m} = \min\{\mathbf{n} \geq \mathbf{b_m} : \mathbf{L}(\mathbf{n}) \geq \mathbf{b_m}\}$ where $\mathbf{L}(\mathbf{n})$ is the size of the shortest string $\mathbf{x}$ such that a fixed universal machine with input $\mathbf{x}$ outputs $\mathbf{n}$ using space $\mathbf{n}$.

Kolmogorov complexity arguments show then that the cycles cover the set of strings not contiguously and that in an automatic model of $\mathbf{A_0}$ there would be large gaps without any string in contradiction to the pumping lemma.

Further Construction

Theorem

There is a theory with countably many countable models such that the saturated model exists and is automatic and the prime model exists and is not automatic.

Construction

The saturated model consists of chains where nodes are represented by equivalence classes of various cardinalities which are linked by edges. The chains (first of length $\mathbf{m+1}$) are

$\mathbf{2 - 1 - 1 - \ldots - 1 - 2^{2^{m} \cdot (m+1)}}$	once for each $\mathbf{m} \in \{\mathbf{2}, \mathbf{3}, \ldots\}$
$\mathbf{\ldots - 1 - 1 - 1 - 1 - \ldots}$	infinitely often
$\mathbf{n - 1 - 1 - 1 - 1 - \ldots}$	infinitely often for each $\mathbf{n} > \mathbf{1}$

The prime model has no chains of second type and has no chains of third type which start with $\mathbf{2}$ or ω. The nodes representing $\mathbf{2}$ are too sparse for a regular domain.

Summary

Working on questions by Khoussainov and Nerode, the following results were obtained:

1. The Prüfer group is an example of a $\aleph_1$-categorical but not $\aleph_0$-categorical theory where only the prime model is automatic.
2. If P = LOGSPACE then there is a $\aleph_1$-categorical but not $\aleph_0$-categorical theory where all countable models except the prime model are automatic.
3. There is a theory with countably many countable models where the saturated model is automatic and the prime model not.
4. For every $n \in \{0, 1, 2, \ldots, \omega\}$ there is a theory with uncountably many countable models and n automatic models.
5. For every $n \in \{1\} \cup \{3, 4, 5, \ldots, \omega\}$ there is a theory with n countable models, all of them are automatic.

Questions

1. Is there a theory with at most countably many countable models such that exactly two of them are automatic? Note that the number of models must be at least three.

2. The theory of the Prüfer group has an uncountable ω-automatic model; namely $\mathbb{Z}(\mathbf{p}^\infty) \oplus \mathbb{R}$. The theory of $(\mathbb{Q}, +)$ has no automatic model but the uncountable ω-automatic model $(\mathbb{R}, +)$. Some theories have neither a countable automatic nor an uncountable ω-automatic model. What about the fourth possibility?
If a theory has a countable automatic model, does it also have an uncountable ω-automatic model?