Combinatorial characterizations of supercompactness and their applications to forcing axioms

Christoph Weiß

My Ph.D. thesis dealt with a combinatorial principle that captures supercompactness for small cardinals like ω_2. It is the analog of the tree property, only for supercompactness.

Utilizing this principle, Matteo Viale and I were able to gain some new insights into the necessity of large cardinals for the forcing axioms PFA and MM. Suppose κ is an inaccessible cardinal and $\langle \mathbb{P}_\alpha \mid \alpha < \kappa \rangle$ is a forcing iteration such that $|\mathbb{P}_\alpha| < \kappa$ for all $\alpha < \kappa$ and such that direct limits are taken stationarily often. We conjecture that the following holds: If $V^{\mathbb{P}_\kappa} \models \mathsf{PFA}$, then κ is supercompact in V. We can prove the conjecture if "supercompact" is replaced by "strongly compact," and Menachem Magidor showed us a trick how to make our proposed proof work under the stronger assumption of $V^{\mathbb{P}_\kappa} \models \mathsf{PFA}^{++}$ and $\mathbb{P}_\kappa$ being proper. "Half of the trick" works if we use MM instead of PFA^{++}, so there might be some hope to get the result from MM alone without requiring the iteration to be proper.

The concept of these proofs splits into three parts that might be interesting on their own. Firstly, any such iteration satisfies both the κ-cc and the κ-approximation property. Secondly, PFA implies the principle for ω_2 which corresponds to supercompactness. Finally, if two models of set theory $V \subset W$ satisfy κ-covering and κ-approximation, κ is inaccessible in V, and the principle for supercompactness holds in W, then it also holds in V, making κ actually supercompact as it is inaccessible. The first two steps are correct, but unfortunately the third step introduces some problems we cannot yet surmount, which is why we either can only prove the weaker statement that κ is strongly compact or have to use the stronger assumptions mentioned above. The third step would also work under the stronger assumption that we have the τ-approximation property for some $\tau < \kappa$, however this would no longer be the case for the forcings we are actually interested in, that is the forcings used to create models of PFA or MM.

It should also be noted the principle itself can hopefully work as a framework to derive large cardinal consequences for ω_2. It implies the failure of weak versions of square, the tree property, the failure of the approachability property, and, due to Viale, together with $\mathsf{MA}(\omega_1)$ also SCH.