Mathematisches Institut
der Universität Bonn
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Winter 09/10
November 16, 2009
Exercise Sheet 5

Exercises for

Models of Set Theory II

Let $U \subseteq \mathfrak{P}(\omega)$ be an ultrafilter. Define the Mathias forcing

$$\mathbb{M}_U = \{(s, A) \mid s \in [\omega]^{<\omega},\ A \in U,\ max(s) < min(A)\}.$$

For $(s, A), (t, B) \in \mathbb{M}_U$ set $(s, A) \leq (t, B)$ iff $t \subseteq s$, $A \subseteq B$ and $s - t \subseteq B$.
If $A, B \in [\omega]^\omega$ and $A \cap B$ as well as $A - B$ are infinite, then A is said to be split by the set B.

17. Show that $\mathbb{M}_U$ satisfies ccc.

Let G be $\mathbb{M}_U$-generic over V and $X_G = \bigcup\{s \mid \exists A \in U\ (s, A) \in G\}$.

18. Show that X_G is not split by any $B \in \mathfrak{P}(\omega) \cap V$.

A filter F on ω is said to be rapid if for every function $f : \omega \to \omega$ there exists $A \in F$ such that $|A \cap f(n)| \leq n$ for all $n \in \omega$.

19. Show that if U is rapid and f_G is the increasing enumeration of X_G, then f_G eventually dominates every $g : \omega \to \omega$ from V.

20. Show that if $cov(\mathcal{M}) = 2^{\aleph_0}$ then there exists a rapid filter.
Hint: Let $\langle f_\alpha \mid \alpha < 2^{\aleph_0}\rangle$ enumerate all $f : \omega \to \omega$. Construct by induction sets $X_\alpha \subseteq \omega$ such that $X_\alpha \cap X_\xi$ is infinite and $|X_\alpha \cap f_\alpha(n)| \leq n$ for all $n \in \omega$. If $\langle X_\xi \mid \xi < \alpha\rangle$ is already constructed, set $g_\xi(n) = min(X_\xi \cap [f_\alpha(n), f_\alpha(n+1)[)$ and consider

$$G_\xi = \{x \in \omega^\omega \mid n \in dom(g_\xi)\ \wedge\ x(n) = g_\xi(n) \text{ for infinitely many } n\}.$$

Every problem will be graded with 8 points.

Please hand in your solutions during the lecture at November 23, 2009.