Models of Set Theory II

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Abstract

Martin's Axiom and applications, iterated forcing, forcing Martin's axiom, adding various types of generic reals.

1 Introduction

2 Martin's axiom

2.1 The definition

We have produced several different models of set theory by the forcing method. Take a forcing partial order $(P, \leqslant, 1)$ in a ground model M. Then take an M-generic filter G on P. Infinitary combinatorics in the new model $M[G]$ is determined by the combinatorics of P in the ground model M. In particular it is important to control the collections of dense subsets and antichains in P.

Recall

Definition 1. *Let M be a ground model and $(P, \leqslant, 1_P) \in M$ be a forcing.*

a) *$D \subseteq P$ is* dense *in P iff $\forall p \in P \exists q \in D\, q \leqslant p$.*

b) *A filter G on P is M-*generic *iff $D \cap G \neq \emptyset$ for every $D \in M$ which is dense in P.*

If $M[G]$ is an extension of M by an M-generic filter we call $M[G]$ a generic extension.

We can define genericity for arbitrary collections of dense sets:

Definition 2. *Let $(P, \leqslant, 1_P)$ be a forcing and $\mathcal{D} \in X$ be any set. Then a filter G on P is $\mathcal{D}$-*generic *iff $D \cap G \neq \emptyset$ for every $D \in \mathcal{D}$ which is dense in P.*

For any countable $\mathcal{D}$ we obtain the existence of generic filters just like in the case of ground models.

Theorem 3. *Let $(P, \leqslant, 1_P)$ be a partial order, let $\mathcal{D}$ be countable, and let $p \in P$. Then there is a $\mathcal{D}$-generic filter G on P with $p \in G$.*

Proof. Take an enumeration $(D_n|n<\omega)$ of all $D\in\mathcal{D}$ which are dense in P. Define an ω-sequence $p=p_0\geqslant p_1\geqslant p_2\geqslant\ldots$ recursively, using the axiom of choice:

$$\text{choose } p_{n+1} \text{ such that } p_{n+1}\leqslant p_n \text{ and } p_{n+1}\in D_n\,.$$

Then $G=\{p\in P\,|\,\exists n<\omega\; p_n\leqslant p\}$ is as desired. □

For larger sets $\mathcal{D}$ there is in general no $\mathcal{D}$-generic filter. The arguments of the following counterexamples correspond to certain arguments in our forcing constructions of ¬CH und CH.

Example 4. Let $(P,\leqslant,1_P)$ with

$$P=\mathrm{Fn}(\omega,2,\aleph_0)=\{p\,|\,p\colon\mathrm{dom}(p)\to 2\wedge\mathrm{dom}(p)\subseteq\omega\wedge\mathrm{card}(\mathrm{dom}(p))<\aleph_0\}$$

be COHEN forcing partially ordered by reverse inclusion

$$p\leqslant q \text{ iff } p\supseteq q$$

and with weakest element $1_P=\emptyset$. Define $\mathcal{D}=\{D_x|x\in\mathbb{R}\}\cup\{D_n|n<\omega\}$, where

$$D_x=\{p\in P\,|\,p\nsubseteq x\} \text{ and } D_n=\{p\in P\,|\,n\in\mathrm{dom}(p)\}.$$

For us, the set of real numbers is $\mathbb{R}={}^{\omega}2$. We saw before that every D_x and D_n is dense in P.

Now assume that G were $\mathcal{D}$-generic. Define

$$c=\bigcup G.$$

The definition of the forcing relation and since every D_n is met by G imply that c behaves like a COHEN real, i.e., $c\colon\omega\to 2$.

But on the other hand we have that $G\cap D_c\neq\emptyset$. Take $p\in G\cap D_c$. This implies $p\subseteq c$ and $p\nsubseteq c$, a contradiction.

So we have a set $\mathcal{D}$ of size $2^{\aleph_0}$ such that there is no $\mathcal{D}$-generic filter on P.

Example 5. Let $(P,\leqslant,1_P)$ with

$$P=\mathrm{Fn}(\omega,\omega_1,\aleph_0)=\{p\,|\,p\colon\mathrm{dom}(p)\to\omega_1\wedge\mathrm{dom}(p)\subseteq\omega\wedge\mathrm{card}(\mathrm{dom}(p))<\aleph_0\}$$

the forcing for "making ω_1 countable". Again P is partially ordered by reverse inclusion

$$p\leqslant q \text{ iff } p\supseteq q$$

and with weakest element $1_P=\emptyset$. Define $\mathcal{D}=\{D_\alpha|\alpha<\omega_1\}$, where

$$D_\alpha=\{p\in P\,|\,\alpha\in\mathrm{ran}(p)\}.$$

Now assume that G were $\mathcal{D}$-generic. Define

$$f=\bigcup G.$$

The definition of the forcing relation imply that $f\colon\omega\rightharpoonup\omega_1$ is a partial function.

We show that f is surjective: Let $\alpha<\omega_1$. By genericity, $G\cap D_\alpha\neq\emptyset$. Take $p\in G\cap D_\alpha$. Then $\alpha\in\mathrm{ran}(p)\subseteq\mathrm{ran}(f)$.

But this is a contradiction since ω_1 cannot be a surjective image of some smaller ordinal.

So we have a set $\mathcal{D}$ of size $\aleph_1$ such that there is no $\mathcal{D}$-generic filter on P.

Exercise 1. Let M be a ground model with $2^{\aleph_0} = \aleph_2$. Define $P = \mathrm{Fn}(\omega, \omega_1, \aleph_0)^M$ and let $M[G]$ be a generic extension via P. Show that $M[G] \vDash 2^{\aleph_0} = \aleph_1$.

The second example shows that a forcing that collapses ω_1 cannot have generic sets for $\aleph_1$ many dense sets. We know from forcing $\neg$CH that forcings with the countable chain condition do not collapse ω_1. COHEN forcing satisfies the countable chain condition. The first example shows that COHEN forcing cannot have generic sets for $2^{\aleph_0}$ many dense sets. This analysis leaves open the possibility of ccc-forcings and collections of dense sets of size $< 2^{\aleph_0}$. Of course this only interesting in case that $2^{\aleph_0} > \aleph_1$:

Definition 6.

a) *Let κ be a cardinal. Then* MARTIN's axiom MA_κ *is the property: for every ccc partial order $(P, \leqslant, 1_P)$ and $\mathcal{D}$ with* $\mathrm{card}(\mathcal{D}) \leqslant \kappa$ *there is a $\mathcal{D}$-generic filter G on P.*

b) MARTIN's axiom MA *postulates that* MA_κ *holds for every $\kappa < 2^{\aleph_0}$.*

$\mathrm{MA}_{\aleph_0}$ holds by Theorem 3. Thus the continuum hypothesis $2^{\aleph_0} = \aleph_1$ trivially implies MA. We shall later see by a forcing construction that $2^{\aleph_0} = \aleph_2$ and MA are relatively consistent with ZFC.

2.2 Consequences of MA+¬CH

2.2.1 LEBESGUE measure

We shall not go into the details of LEBESGUE measure, since we shall only consider measure zero sets. We recall some notions and facts from before. For $s \in {}^{<\omega}2 = \{t | t\colon \mathrm{dom}(t) \to 2 \wedge \mathrm{dom}(t) \in \omega\}$ define the real *interval*

$$I_s = \{x \in \mathbb{R} | s \subseteq x\} \subseteq \mathbb{R}$$

with $\mathrm{length}(I_s) = 2^{-\mathrm{dom}(s)}$. Note that $I_s = I_{s \cup \{(\mathrm{dom}(s), 0)\}} \cup I_{s \cup \{(\mathrm{dom}(s), 1)\}}$, $\mathrm{length}(\mathbb{R}) = I_\emptyset = 2^{-0} = 1$, and $\mathrm{length}(I_{s \cup \{(\mathrm{dom}(s), 0)\}}) = \mathrm{length}(I_{s \cup \{(\mathrm{dom}(s), 1)\}}) = \frac{1}{2} \mathrm{length}(I_s)$.

Definition 7. *Let $\varepsilon > 0$. Then a set $X \subseteq \mathbb{R}$ has* measure $< \varepsilon$ *if there exists a sequence $(I_n | n < \omega)$ of intervals in $\mathbb{R}$ such that $X \subseteq \bigcup_{n<\omega} I_n$ and $\sum_{n<\omega} \mathrm{length}(I_n) \leqslant \varepsilon$. A set $X \subseteq \mathbb{R}$ has* measure zero *if it has measure $< \varepsilon$ for every $\varepsilon > 0$.*

Theorem 8. *Assume* MA_κ *and let $X \subseteq \mathbb{R}$ with* $\mathrm{card}(X) \leqslant \kappa$. *Then X has measure zero.*

Proof. Let $\varepsilon > 0$ be given. We want to cover X by a sequence $(I_n | n < \omega)$ of intervals as in the definition of measure zero sets. The idea is to define the intervals I_0, I_1, I_2, ... of lengths 2^{-i-1}, 2^{-i-2}, 2^{-i-3}, ... from some "COHEN generic" real c. Take $i < \omega$ such that $2^{-i} < \varepsilon$. For $n < \omega$ let $I_n = I_{s_n}$, where the finite sequence $s_n\colon i + n + 1 \to 2$ is given by

$$s_n(l) = c(n + l).$$

Then

$$\sum_{n<\omega} \text{length}(I_n) = \sum_{n<\omega} 2^{-i-n-1} = 2^{-i} < \varepsilon.$$

We shall apply MA_κ to COHEN forcing $P = \text{Fn}(\omega, 2, \aleph_0)$. Since P is countable it trivially satisfies the ccc. For every $x \in X$ let

$$D_x = \{p \in P | \exists n < \omega \, \forall l < i + n + 1 \, (n + l \in \text{dom}(p) \wedge p(n + l) = x(l))\}.$$

(1) D_x is dense in P.
Proof. Let $q \in P$. Take $n < \omega$ such that $\text{dom}(q) \subseteq n$. Set

$$p = q \cup \{(n + l, x(l)) | l < i + n + 1\}.$$

Then $p \leqslant q$ and $p \in D_x$. *qed*(1)

For $k < \omega$ let $D_k = \{p \in P | k \in \text{dom}(p)\}$. Set $\mathcal{D} = \{D_x | x \in X\} \cup \{D_k | k < \omega\}$. By MA_κ take a $\mathcal{D}$-generic filter G on P. As in example 4 $c = \bigcup G: \omega \to 2$ is a real number. Define $(I_n | n < \omega)$ from c as above. It suffices to show:

(2) $X \subseteq \bigcup_{n<\omega} I_n$.
Proof. Let $x \in X$. By the $\mathcal{D}$-genericity of G take $p \in G \cap D_x$. Take $n < \omega$ such that

$$\forall l < i + n + 1 \, (n + l \in \text{dom}(p) \wedge p(n + l) = x(l)).$$

Then

$$\forall l < i + n + 1 \; c(n + l) = x(l)$$

and

$$\forall l < i + n + 1 \; s_n(l) = x(l).$$

Hence $s_n \subseteq x$ and $x \in I_n \subseteq \bigcup_{n<\omega} I_n$. □

To strengthen this theorem we need some more topological and measure theoretic notions. The (standard) topology on $\mathbb{R}$ is generated by the basic open sets I_s for $s \in {}^{<\omega}2$. Hence every union $\bigcup_{n<\omega} I_n$ of basic open intervals is itself open. The basic open intervals I_s are also compact in the sense of the HEINE-BOREL theorem: every cover of I_s by open sets has a finite subcover.

Theorem 9. *Assume* MA_κ *and let* $(X_i | i < \kappa)$ *be a family of measure zero sets. Then* $X = \bigcup_{i<\kappa} X_i$ *has measure zero.*

Proof. Fix $\varepsilon > 0$. We show that $X = \bigcup_{i<\kappa} X_i$ has measure $< 2\varepsilon$. Let

$$\mathcal{I} = \{(a, b) | a, b \in \mathbb{Q}, a < b\}$$

the countable set of rational intervals $(a, b) = \{c \in \mathbb{R} | a < c < b\}$ in $\mathbb{R}$. The *length* of (a, b) is simply $\text{length}((a, b)) = b - a$. We shall apply MARTIN's axiom to the following forcing $P = (P, \supseteq, \emptyset)$ where

$$P = \{p \subseteq \mathcal{I} | \sum_{I \in p} \text{length}(I) < \varepsilon\}.$$

(1) P is ccc.
Proof. Let $\{p_i|i<\omega_1\}\subseteq P$. For every $i<\omega_1$ there is $n_i<\omega$ such that p_i has measure $<\varepsilon-\frac{1}{n_i}$. By a pigeonhole principle we may assume that all n_i are equal to a common value $n<\omega$. For every p_i we have

$$\sum_{I\in p_i}\text{length}(I)<\varepsilon-\frac{1}{n}.$$

For every $i<\omega_1$ take a finite set $\bar{p}_i\subseteq p_i$ such that

$$\sum_{I\in p_i\setminus\bar{p}_i}\text{length}(I)<\frac{1}{n}.$$

There are only countably many such set $\bar{p}_i$, and again by a pigeonhole argument we may assume that for all $i<\omega_1$

$$\bar{p}_i=\bar{p}$$

takes a fixed value. Now consider $i<j<\omega_1$. Then

$$\begin{aligned}\sum_{I\in p_i\cup p_j}\text{length}(I) &\leqslant \sum_{I\in p_i}\text{length}(I)+\sum_{I\in p_j\setminus\bar{p}}\text{length}(I)\\ &< \varepsilon-\frac{1}{n}+\frac{1}{n}\\ &= \varepsilon\end{aligned}$$

Hence $p_i\cup p_j\in P$ and $p_i\cup p_j\leqslant p_i,p_j$, and so $\{p_i|i<\omega_1\}$ is not an antichain in P. *qed*(1)

For $i<\kappa$ define

$$D_i=\{p\in P|X_i\subseteq\bigcup p\}.$$

(2) D_i is dense in P.
Proof. Let $q\in P$. Take $n<\omega$ such that

$$\sum_{I\in q}\text{length}(I)<\varepsilon-\frac{1}{n}.$$

Since X_i has measure zero, take $r\subseteq\mathcal{I}$ such that $X_i\subseteq\bigcup p$ and $\sum_{I\in r}\text{length}(I)\leqslant\frac{1}{n}$. Then

$$X_i\subseteq\bigcup(q\cup r)\ \text{ and }\ \sum_{I\in q\cup r}\text{length}(I)\leqslant\sum_{I\in q}\text{length}(I)+\sum_{I\in r}\text{length}(I)<\varepsilon-\frac{1}{n}+\frac{1}{n}=\varepsilon.$$

Hence $p=q\cup r\in P$, $p\supseteq q$, and $p\in D_i$. *qed*(2)

By MA_κ take a filter G on P which is $\{D_i|i<\kappa\}$-generic. Let $U=\bigcup G\subseteq\mathcal{I}$.

(3) $X=\bigcup_{i<\kappa}X_i\subseteq\bigcup_{I\in U}I$.
Proof. Let $i<\kappa$. By the generity of G take $p\in G\cap D_i$. Then

$$X_i\subseteq\bigcup p\subseteq\bigcup U$$

qed(3)

(4) $\sum_{I\in U}$ length$(I)\leqslant\varepsilon$.

Proof. Assume for a contradiction that $\sum_{I\in U}$ length$(I) > \varepsilon$. Then take a finite set $\bar{U}\subseteq U$ such that $\sum_{I\in\bar{U}}$ length$(I)>\varepsilon$. Let $\bar{B}=\{I_0,\ldots,I_{k-1}\}$. For every $I_j\in\bar{U}$ take $p_j\in G$ such that $I_j\in p_j$. Since all elements of G are compatible within G there is a condition $p\in G$ such that $p\supseteq p_0,\ldots,p_{k-1}$. Hence $\bar{U}\subseteq p$. But, since $p\in P$, we get a contradiction:

$$\varepsilon<\sum_{I\in\bar{U}}\text{length}(I)\leqslant\sum_{I\in p}\text{length}(I)<\varepsilon.$$

□

An easy corollary is:

Theorem 10. *Assume* MA. *Then* $2^{\aleph_0}$ *is regular.*

Proof. Assume instead that $\mathbb{R}=\bigcup_{i<\kappa}X_i$ for some $\kappa<2^{\aleph_0}$, where card$(X_i)<2^{\aleph_0}$ for every $i<\kappa$. Every singleton $\{r\}$ has measure zero. By Theorem 9, each X_i has measure zero. Again by Theorem, $\mathbb{R}=\bigcup_{i<\kappa}X_i$ has measure zero. But measure theory (and also intuition) shows that $\mathbb{R}$ does not have measure zero. □

2.2.2 Almost disjoint forcing

We intend to code subsets of κ by subsets of ω. If such a coding is possible then we shall have

$$2^{\aleph_0}\leqslant 2^{\kappa}\leqslant 2^{\aleph_0},\text{ i.e. }2^{\kappa}=2^{\aleph_0}.$$

We shall employ almost disjoint coding.

Definition 11. *A sequence* $(x_i|i\in I)$ *is* almost disjoint *if*

a) x_i *is infinite*

b) $i\neq j<\kappa$ *implies that* $x_i\cap x_j$ *is finite*

Lemma 12. *There is an almost disjoint sequence* $(x_i|i<2^{\aleph_0})$ *of subsets of* ω.

Proof. For $u\in{}^{\omega}2$ let $x_u=\{u\restriction m\,|\,m<\omega\}$. x_u is infinite. Consider $u\neq v$ from ${}^{\omega}2$. Let $n<\omega$ be minimal such that $u\restriction n\neq v\restriction n$. Then

$$x_u\cap x_v=\{u\restriction m\,|\,m<\omega\}\cap\{v\restriction m\,|\,m<\omega\}=\{u\restriction m\,|\,m<n\}$$

is finite. Thus $(x_u|u\in{}^{\omega}2)$ is almost disjoint. Using bijections $\omega\leftrightarrow{}^{<\omega}2$ and $2^{\aleph_0}\leftrightarrow{}^{\omega}2$ one can turn this into an almost disjoint sequence $(x_i|i<2^{\aleph_0})$ of subsets of ω. □

Theorem 13. *Assume* MA_{κ}. *Then* $2^{\kappa}=2^{\aleph_0}$.

Proof. By a previous example, $\kappa<2^{\aleph_0}$. By the lemma, fix an almost disjoint sequence $(x_i|i<\kappa)$ of subsets of ω. Define a map $c\colon\mathcal{P}(\omega)\to\mathcal{P}(\kappa)$ by

$$c(x)=\{i<\kappa|x\cap x_i\text{ is infinite}\}.$$

We say that x *codes* $c(x)$. We want to show that every subset of κ can be coded as some $c(x)$. We show this by proving that $c\colon \mathcal{P}(\omega) \rightarrow \mathcal{P}(\kappa)$ is surjective.

Let $A \subseteq \kappa$ be given. We use the following forcing $(P, \leqslant, 1)$ to code A:

$$P = \{(a, z) | a \subseteq \omega, z \subseteq \kappa, \mathrm{card}(a) < \aleph_0, \mathrm{card}(z) < \aleph_0\},$$

partially ordered by

$$(a', z') \leqslant (a, z) \text{ iff } a' \supseteq a, z' \supseteq z, i \in z \cap (\kappa \setminus A) \rightarrow a' \cap x_i = a \cap x_i .$$

The weakest element of P is $1 = (\emptyset, \emptyset)$.

The idea of the forcing is to keep the intersection of the first component with x_i fixed, provided $i \notin A$ has entered the second component. This will allow the almost disjoint coding of A by the finite/infinite method.

(1) $(P, \leqslant, 1)$ satisfies ccc.

Proof. Conditions $(a,\ y)$ and $(a,\ z)$ with equal first components are compatible, since $(a, y \cup z) \leqslant (a, y)$ and $(a, y \cup z) \leqslant (a, z)$. Incompatibel conditions have different first components. Since there are only countably many first components, an antichain in P can be at most countable. *qed*(1)

The outcome of a forcing construction results from an interplay between the partial order and some dense set arguments. We now define dense sets for our requirements.

For $i < \kappa$ let $D_i = \{(a, z) \in P | i \in z\}$. D_i is obviously dense in P. For $i \in A$ and $n \in \omega$ let $D_{i,n} = \{(a, z) \in P | \exists m > n\colon m \in a \cap x_i\}$.

(2) If $i \in A$ and $n \in \omega$ then $D_{i,n}$ is dense in P.

Proof. Consider $(a,\ z) \in P$. For $j \in z,\ j \neq i$ is the intersection $x_i \cap x_j$ finite. Take some $m \in x_i$, $m > n$ such that $m \notin x_i \cap x_j$ for $j \in z,\ j \neq i$. Then

$$(a \cup \{m\}, z) \leqslant (a, z) \text{ and } (a \cup \{m\}, z) \in D_{i,n} .$$

qed(2)

By MA_κ take a filter G on P which is generic for the dense sets in

$$\{D_i | i < \kappa\} \cup \{D_{i,n} | i \in A, n \in \omega\}.$$

Let

$$x = \bigcup \{a | (a, y) \in G\} \subseteq \omega.$$

(3) Let $i \in A$. Then $x \cap x_i$ is infinite.

Proof. Let $n < \omega$. By genericity take $(a, y) \in G \cap D_{i,n}$. By the definition of $D_{i,n}$ take $m > n$ such that $m \in a \cap x_i$. Then $m \in x \cap x_i$, and so $x \cap x_i$ is cofinal in ω. *qed*(3)

(4) Let $i \in \kappa \setminus A$. Then $x \cap x_i$ is finite.

Proof. By genericity take $(a, y) \in G \cap D_i$. Then $i \in y$. We show that $x \cap x_i \subseteq a \cap x_i$. Consider $n \in x \cap x_i$. Take $(b, z) \in G$ such that $n \in b$. By the filter properties of G take $(a',\ y') \in P$ such that $(a',\ y') \leqslant (a,\ y)$ and $(a',\ y') \leqslant (b,\ z)$. Then $n \in a'$, and by the definition of $\leqslant$, $a' \cap x_i = a \cap x_i$. Thus $n \in a \cap x_i$. *qed*(4)

So

$$c(x) = \{i < \kappa | x \cap x_i \text{ is infinite}\} = A \in \mathrm{range}(c).$$

□

2.2.3 Category

Lebesgue measure defines an ideal of "small" sets, namely the ideal of measure zero sets: arbitrary subsets of measure zero sets are measure zero, and, under MA, every union of less than $2^{\aleph_0}$ measure zero sets is again measure zero.

We now look at another ideal of small sets, namely the ideal of subsets X of $\mathbb{R}$ which are nowhere dense in $\mathbb{R}$: every nonempty open interval in $\mathbb{R}$ has a nonempty open subinterval which is disjoint from X. The union of all such subintervals is open, dense in $\mathbb{R}$, and disjoint from X.

The BAIRE category theorem says that the intersection of countably many dense open sets of reals in dense in $\mathbb{R}$. We can strengthen this to:

Theorem 14. *Assume* MA_κ. *Then the intersection of* κ *many dense open sets of reals is dense in* $\mathbb{R}$.

Proof. Consider a sequence $(O_i|i<\kappa)$ of dense open subsets of $\mathbb{R}$. We use standard COHEN forcing $P=\mathrm{Fn}(\omega,2,\aleph_0)$ for the density argument. Since P is countable it trivially has the ccc. For $i<\kappa$ define $D_i=\{p\in P|\forall x\in\mathbb{R}(x\supseteq p\rightarrow x\in O_i)\}$. This means that the interval determined by p lies within A_i. The density of D_i follows readily since O_i is open dense. For $n<\omega$ let $D_n=\{p\in P|n\in\mathrm{dom}(p)\}$. Obviously, D_n is also dense in P. By MA_κ let $G\subseteq P$ be $\{D_i|i<\kappa\}$-$\{D_n|n<\kappa\}$generic. Let $x=\bigcup G$. $p\in G\cap D_n$ implies that $n\in\mathrm{dom}(p)\subseteq\mathrm{dom}(x)$. So $x:\omega\rightarrow 2$ is a real number. □

Since $\mathrm{MA}_{\aleph_0}$ is always true in ZFC, we get the BAIRE category theorem:

Theorem 15. *The intersection of countably many dense open sets of reals is dense in* $\mathbb{R}$.

This says that dense open sets (of reals) have a largeness property, and correspondingly complements of dense open sets are small.

Definition 16. *A set* $A\subseteq\mathbb{R}$ *is* nowhere dense *if there is a dense open set* $O\subseteq\mathbb{R}$ *such that* $A\cap O=\emptyset$. *A set* $A\subseteq\mathbb{R}$ *is* meager *or* of 1st category *if it is a union of countably many nowhere dense sets.*

Proposition 17.

a) *A singleton set* $\{x\}\subseteq\mathbb{R}$ *is nowhere dense since* $\mathbb{R}\setminus\{x\}$ *is dense open in* $\mathbb{R}$.

b) *A countable set* C *is meager.*

c) *A set* $A\subseteq\mathbb{R}$ *is meager iff there are open dense sets* $(O_n|n<\omega)$ *such that* $A\cap\bigcap_{n<\omega}O_n=\emptyset$.

d) $\mathbb{R}$ *is not meager. Sets which are not meager are said to be of* 2nd category.

Proof. *c)* Let $A=\bigcup_{n<\omega}A_n$ be meager where each A_n is nowhere dense. For each n choose O_n dense open in $\mathbb{R}$ such that $A_n\cap O_n=\emptyset$. Then

$$(\bigcup_{n<\omega}A_n)\cap(\bigcap_{n<\omega}O_n)=A\cap(\bigcap_{n<\omega}O_n)=\emptyset.$$

Conversely assume that $A\cap(\bigcap_{n<\omega} O_n)=\emptyset$ where each O_n is dense open. $(A\setminus O_n)\cap O_n=\emptyset$, and so by definition, every $A_n=A\setminus O_n$ is nowhere dense. Obviously

$$\bigcup_{n<\omega} A_n\subseteq A.$$

For the converse consider $x\in A$. The property $A\cap(\bigcap_{n<\omega} O_n)=\emptyset$ implies that we may take $n<\omega$ such that $x\notin O_n$. Hence $x\in A\setminus O_n=A_n$. So $A=\bigcup_{n<\omega} A_n$ is meager.

d) If $\mathbb{R}$ were meager then there would be open dense sets $(O_n|n<\omega)$ such that $\mathbb{R}\cap\bigcap_{n<\omega} O_n=\emptyset$. But by Theorem 15,

$$\mathbb{R}\cap\bigcap_{n<\omega} O_n=\bigcap_{n<\omega} O_n\neq\emptyset,$$

contradiction. □

We would now like to show as in the case of measure that a union of $<2^{\aleph_0}$ small sets in the sense of category is again small if MARTIN's axiom holds.

Theorem 18. *Assume* MA_κ . *Let* $(A_i|i<\kappa)$ *be a family of meager sets. Then* $A=\bigcup_{i<\kappa} A_i$ *is meager.*

Proof. Obviously it suffices to consider the case where each A_i is nowhere dense. We shall use MA_κ to find dense open sets $(O_n|n<\omega)$ such that

$$(\bigcup_{i<\kappa} A_i)\cap(\bigcap_{n<\omega} O_n)=A\cap(\bigcap_{n<\omega} O_n)=\emptyset.$$

The forcing will consist of approximations to a family $(O_n|n<\omega)$ of open dense sets which makes this equality true.

The forcing conditions will consist of finitely many finite approximations to the O_n . Moreover there will be for every n a finite collection of $i<\kappa$ such that an approximation to the equation holds for those i. We shall see that by appropriate density considerations the full equality may be satisfied.

For ccc-reasons, much like in the argument of measure-zero sets, we only consider approximations to the O_n by finitely many *rational* intervals. Let

$$\mathcal{I}=\{(a,b)|a,b\in\mathbb{Q},a<b\}$$

the countable set of rational open intervals $(a,b)=\{c\in\mathbb{R}|a<c<b\}$ in $\mathbb{R}$. Now let

$$P=\{(r,s)|r\colon\omega\to[\mathcal{I}]^{<\omega}, s\colon\omega\to[\kappa]^{<\omega}, \{n<\omega|r(n)\neq\emptyset\}\text{ is finite}, \{n<\omega|s(n)\neq\emptyset\}\text{ is finite}, \forall n<\omega\forall i\in s(n)\, A_i\cap\bigcup r(n)=\emptyset\}.$$

Define

$$(r',s')\leqslant(r,s)\quad\text{iff}\quad\forall n<\omega\,(r'(n)\supseteq r(n)\wedge s'(n)\supseteq s(n)).$$

(1) $(P, \leqslant)$ satisfies the countable chain condition.
Proof. Consider (r, s) and (r, s') in P having the same first component. Then define s'': $\omega \to [\kappa]^{<\omega}$ by $s''(n) = s(n) \cup s'(n)$. It is easy to check that $(r, s'') \in P$, and also $(r, s'') \leqslant (r, s)$ and $(r, s'') \leqslant (r, s')$. So (r, s) and (r, s') are compatible in P.

An antichain in P must consist of conditions whose first components are pairwise distinct. Since there are only countably many first components, an antichain in P is at most countable. *qed*(1)

For each $n < \omega$ the following dense sets ensures the density of the O_n in $\mathbb{R}$: for $I \in \mathcal{I}$ let

$$D_{n,I} = \{(r', s') | \exists J \in r'(n)\ J \subseteq I\}.$$

(2) $D_{n,I}$ is dense in P.
Proof. Let $(r, s) \in P$. Let $s(n) = \{i_0, \ldots, i_{k-1}\}$. Since $A_{i_0}, \ldots, A_{i_{k-1}}$ are nowhere dense one can go find intervals $I \supseteq I_{i_0} \supseteq \ldots \supseteq I_{k-1} = J$ in $\mathcal{I}$ such that $A_{i_l} \cap I_{i_l} = \emptyset$. Define r': $\omega \to [\mathcal{I}]^{<\omega}$ by $r' \restriction (\omega \setminus \{n\}) = r \restriction (\omega \setminus \{n\})$ and $r'(n) = r(n) \cup \{J\}$. Then $(r', s) \in P$, $(r', s) \leqslant (r, s)$, and $(r', s) \in D_{n,I}$. *qed*(2)

We also need that every $i < \kappa$ is considered by some O_n. Define

$$D_i = \{(r', s') | \exists n < \omega\ i \in s'(n)\}.$$

(3) D_i is dense in P.
Proof. Let $(r, s) \in P$. Take $n < \omega$ such that $r(n) = \emptyset$. Define s': $\omega \to [\mathcal{I}]^{<\omega}$ by $s' \restriction (\omega \setminus \{n\}) = s \restriction (\omega \setminus \{n\})$ and $s'(n) = s(n) \cup \{i\}$. Then $(r, s') \in P$, $(r, s') \leqslant (r, s)$, and $(r, s') \in D_i$. *qed*(3)

By MA_κ we can take a filter G on P which is generic for

$$\{D_{n,I} | n < \omega, I \in \mathcal{I}\} \cup \{D_i | i < \kappa\}.$$

For $n < \omega$ define

$$O_n = \bigcup \bigcup \{r(n) | (r, s) \in G\}.$$

(4) O_n is open, since it is a union of open intervals.
(5) O_n is dense in $\mathbb{R}$.
Proof. Let $I \in \mathcal{I}$. By genericity take $(r', s') \in G \cap D_{n,I}$. Take $J \in r'(n)$ such that $J \subseteq I$. Then

$$\emptyset \neq J \subseteq \bigcup r'(n) \subseteq \bigcup \bigcup \{r(n) | (r, s) \in G\} = O_n.$$

qed(5)
(6) Let $i < \kappa$. Then $A_i \cap \bigcap_{n<\omega} O_n = \emptyset$.
Proof. By genericity take $(r', s') \in G \cap D_i$. Take $n < \omega$ such that $i \in s'(n)$. We show that $A_i \cap O_n = \emptyset$. Assume not, and let $x \in A_i \cap O_n$. Take $(r, s) \in G$ and $I \in r(n)$ such that $x \in I$. Since G is a filter, take $(r'', s'') \in P$ such that $(r'', s'') \leqslant (r, s)$ and $(r'', s'') \leqslant (r', s')$. Then $I \in r''(n)$, $i \in s''(n)$, and

$$x \in A_i \cap I \subseteq A_i \cap \bigcup r''(n) \neq \emptyset.$$

The last inequality contradicts the definition of P. *qed*(6)

By (6), $\bigcup_{i<\kappa} A_i \cap \bigcap_{n<\omega} O_n = \emptyset$, and so $\bigcup_{i<\kappa} A_i$ is meager. □

3 Iterated forcing

MARTIN's axiom postulates that for every ccc partial order $(P, \leqslant, 1_P)$ and $\mathcal{D}$ with $\mathrm{card}(\mathcal{D}) < 2^{\aleph_0}$ there is a $\mathcal{D}$-generic filter G on P. Syntactically this axiom has a $\forall\exists$-form: $\forall P \forall \mathcal{D} \exists G...$. $\forall\exists$-properties are often realised through chain constructions: build a chain

$$M = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_\alpha \subseteq \ldots \subseteq M_\beta \subseteq \ldots$$

of models such that for any $P, \mathcal{D} \in M_\alpha$ there is some $\beta \geqslant \alpha$ such that M_β contains a generic G as required. Then the "union" or limit of the chain should contain appropriate G's for all P's and $\mathcal{D}$'s.

Such chain constructions are wellknown from algebra. To satisfy closure under square roots ($\forall x \exists y$: $yy = x$) one can e.g. start with a countable field M_0 and along a chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$ adjoin square roots for all elements of M_n. Then $\bigcup_{n<\omega} M_n$ satisfies the closure property.

In set theory there is a difficulty that unions of models of set theory usually do not satisfy the theory ZF: assume that $M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$ is an ascending chain of transitive models of ZF such that $(M_{n+1} \setminus M_n) \cap \mathcal{P}(\omega) \neq \emptyset$ for all $n < \omega$. Let $M_\omega = \bigcup_{n<\omega} M_n$. Then $\mathcal{P}(\omega) \cap M_\omega \notin M_\omega$. Indeed, if one had $\mathcal{P}(\omega) \cap M_\omega \in M_\omega$ then $\mathcal{P}(\omega) \cap M_\omega \in M_n$ for some $n < \omega$ and $\mathcal{P}(\omega) \cap M_{n+1} \in M_n$ contradicts the initial assumption. So a "limit" model of models of ZF has to be more complicated, and it will itself be constructed by some limit forcing which is called iterated forcing.

Exercise 2. Check which axioms of set theory hold in $M_\omega = \bigcup_{n<\omega} M_n$ where $(M_n)_{n<\omega}$ is an ascending sequence of transitive models of ZF(C).

Since we want to obtain the limit by forcing over a ground model M the construction must be visible in the ground model. This means that the sequence of forcings to be employed to pass from M_α to $M_{\alpha+1}$ has to exist as a sequence $(\dot{Q}_\beta | \beta < \kappa)$ of names in the ground model. The initial sequence $(\dot{Q}_\beta | \beta < \alpha)$ already determines a forcing P_α and $\dot{Q}_\alpha$ is intended to be a P_α-name. If G_α is M-generic over P_α then furthermore $Q_\alpha = (\dot{Q}_\alpha)^{G_\alpha}$ is intended to be a forcing in the model $M_\alpha = M[G_\alpha]$, and $M_{\alpha+1}$ is a generic extension of M_α by forcing with Q_α. The following iteration theorem says that any sequence $(\dot{Q}_\beta | \beta < \kappa) \in M$ give rise to an iteration of forcing extensions. In applications the sequence has to be chosen carefully to ensure that some $\forall\exists$-property holds in the final model M_κ. Without loss of generality we only consider forcings Q_α whose maximal element is $\emptyset$.

Theorem 19. *Let M be a ground model, and let $((\dot{Q}_\beta, \dot{\leqslant}_\beta) | \beta < \kappa) \in M$ with the property that $\forall \beta < \kappa$: $\emptyset$. Then there is a sequence $((P_\alpha, \leqslant_\alpha, 1_\alpha) | \alpha \leqslant \kappa) \in M$ such that*

a) $(P_\alpha, \leqslant_\alpha, 1_\alpha)$ is a partial order which consists of α-sequences;

b) $P_0 = \{\emptyset\}$, $\leqslant_0 = \{(\emptyset, \emptyset)\}$, $1_0 = \emptyset$;

c) If $\lambda \leqslant \kappa$ is a limit ordinal then the forcing P_α is defined by:

$$\begin{aligned} P_\lambda &= \{p: \lambda \to V \,|\, (\forall \gamma < \lambda: p \restriction \gamma \in P_\gamma) \wedge \exists \gamma < \lambda \forall \beta \in [\gamma, \lambda)\, p(\beta) = \emptyset)\} \\ p \leqslant_\lambda q &\text{ iff } \forall \gamma < \lambda: p \restriction \gamma \leqslant_\gamma q \restriction \gamma \\ 1_\lambda &= (\emptyset | \gamma < \lambda) \end{aligned}$$

d) *If* $\alpha<\kappa$ *and* $1_\alpha\Vdash_{P_\alpha}(\dot{Q}_\alpha,\dot{\leqslant}_\alpha,\emptyset)$ *is a forcing, then the forcing* $P_{\alpha+1}$ *is defined by:*

$$\begin{aligned} P_{\alpha+1} &= \{p\colon \alpha+1\to V \mid p\restriction\alpha\in P_\alpha\wedge p(\alpha)\in\mathrm{dom}(\dot{Q}_\alpha)\wedge p\restriction\alpha\Vdash_{P_\alpha}p(\alpha)\in\dot{Q}_\alpha\}\\ p\leqslant_{\alpha+1}q &\text{ iff } p\restriction\alpha\leqslant_\alpha q\restriction\alpha\wedge p\restriction\alpha\Vdash_{P_\alpha}p(\alpha)\dot{\leqslant}_\alpha q(\alpha)\\ 1_{\alpha+1} &= (\emptyset|\gamma<\alpha+1)\end{aligned}$$

e) *If* $\alpha<\kappa$ *and* not $1_\alpha\Vdash_{P_\alpha}(\dot{Q}_\alpha,\dot{\leqslant}_\alpha,\emptyset)$ *is a forcing, then the forcing* $P_{\alpha+1}$ *is defined by:*

$$\begin{aligned} P_{\alpha+1} &= \{p\colon \alpha+1\to V \mid p\restriction\alpha\in P_\alpha\wedge p(\alpha)=\emptyset\}\\ p\leqslant_{\alpha+1}q &\text{ iff } p\restriction\alpha\leqslant_\alpha q\restriction\alpha\\ 1_{\alpha+1} &= (\emptyset|\gamma<\alpha+1)\end{aligned}$$

$((P_\alpha,\leqslant_\alpha,1_\alpha)|\alpha\leqslant\kappa)$, and in particular P_κ are called the (*finite support*) *iteration* of the sequence $((\dot{Q}_\beta,\dot{\leqslant}_\beta)|\beta<\kappa)$.

Proof. To justify the above recursive definition of the sequence $((P_\alpha,\leqslant_\alpha,1_\alpha)|\alpha\leqslant\kappa)$ it suffices to show recursively that every P_α is a forcing.

Obviously, P_0 is a trivial one-element forcing.

Consider a limit $\lambda\leqslant\kappa$ and assume that P_γ is a forcing for $\gamma<\alpha$. We have to show that the relation $\leqslant_\lambda$ is transitive with maximal element 1_λ. Consider $p\leqslant_\lambda q\leqslant_\lambda r$. Then $\forall\gamma<\lambda\colon p\restriction\gamma\leqslant_\gamma q\restriction\gamma$ and $\forall\gamma<\lambda\colon q\restriction\gamma\leqslant_\gamma r\restriction\gamma$. Since all $\leqslant_\gamma$ with $\gamma<\lambda$ are transitive relations, $\forall\gamma<\lambda\colon p\restriction\gamma\leqslant_\gamma r\restriction\gamma$ and so $p\leqslant_\lambda r$. Now consider $p\in P_\lambda$. Then $\forall\gamma<\lambda\colon p\restriction\gamma\in P_\gamma$. By the inductive assumption, $\forall\gamma<\lambda\colon p\restriction\gamma\leqslant_\gamma 1_\gamma=1_\lambda\restriction\gamma$ and so $p\leqslant_\lambda 1_\lambda$.

For the successor step assume that $\alpha<\kappa$ and that P_α is a forcing.

Case 1. $1_\alpha\Vdash_{P_\alpha}(\dot{Q}_\alpha,\dot{\leqslant}_\alpha,\emptyset)$ is a forcing.

For the transitivity of $\leqslant_{\alpha+1}$ consider $p\leqslant_{\alpha+1}q\leqslant_{\alpha+1}r$. Then $p\restriction\alpha\leqslant_\alpha q\restriction\alpha\wedge p\restriction\alpha\Vdash_{P_\alpha}p(\alpha)\dot{\leqslant}_\alpha q(\alpha)$ and $q\restriction\alpha\leqslant_\alpha r\restriction\alpha\wedge q\restriction\alpha\Vdash_{P_\alpha}q(\alpha)\dot{\leqslant}_\alpha r(\alpha)$. By the transitivity of $\leqslant_\alpha$: $p\restriction\alpha\leqslant_\alpha r\restriction\alpha$. Moreover $p\restriction\alpha\Vdash_{P_\alpha}p(\alpha)\dot{\leqslant}_\alpha q(\alpha)$, $p\restriction\alpha\Vdash_{P_\alpha}q(\alpha)\dot{\leqslant}_\alpha r(\alpha)$ and $p\restriction\alpha\Vdash_{P_\alpha}$"$\dot{\leqslant}_\alpha$ is transitive". This implies $p\restriction\alpha\Vdash_{P_\alpha}p(\alpha)\dot{\leqslant}_\alpha r(\alpha)$ and together that $p\leqslant_{\alpha+1}r$.

For the maximality of $1_{\alpha+1}$ consider $p\in P_{\alpha+1}$. Then $p\restriction\alpha\in P_\alpha\wedge p\restriction\alpha\Vdash_{P_\alpha}p(\alpha)\in\dot{Q}_\alpha$. Then $p\restriction\alpha\leqslant_\alpha 1_\alpha=1_{\alpha+1}\restriction\alpha$. Moreover $p\restriction\alpha\Vdash_{P_\alpha}$"$\emptyset$ is maximal in $\dot{\leqslant}_\alpha$ implies that $p\restriction\alpha\Vdash_{P_\alpha}p(\alpha)\dot{\leqslant}_\alpha\emptyset=1_{\alpha+1}(\alpha)$. Hence $p\leqslant_{\alpha+1}1_{\alpha+1}$.

Case 2. It is not the case that $1_\alpha\Vdash_{P_\alpha}(\dot{Q}_\alpha,\dot{\leqslant}_\alpha,\emptyset)$ is a forcing.

For the transitivity of $\leqslant_{\alpha+1}$ consider $p\leqslant_{\alpha+1}q\leqslant_{\alpha+1}r$. Then $p\restriction\alpha\leqslant_\alpha q\restriction\alpha$ and $q\restriction\alpha\leqslant_\alpha r\restriction\alpha$. By the transitivity of $\leqslant_\alpha$: $p\restriction\alpha\leqslant_\alpha r\restriction\alpha$ and so $p\leqslant_{\alpha+1}r$.

For the maximality of $1_{\alpha+1}$ consider $p\in P_{\alpha+1}$. Then $p\restriction\alpha\in P_\alpha$. By induction, $p\restriction\alpha\leqslant_\alpha 1_\alpha$ and so $p\leqslant_{\alpha+1}1_{\alpha+1}$. □

The term "finite support iteration" is justified by the following

Lemma 20. *In the above situation let* $p\in P_\kappa$. *Then*

$$\mathrm{supp}(p)=\{\alpha<\kappa|p(\alpha)\neq\emptyset\}$$

is finite.

Proof. Prove by induction on $\alpha \leqslant \kappa$ that $\mathrm{supp}(p)$ is finite for every $q \in P_\alpha$. The crucial property is the definition of P_λ at limit λ in the above iteration theorem. □

Let us fix a ground model M and the iteration $((\dot{Q}_\beta, \dot{\leqslant}_\beta)|\beta < \kappa) \in M$ and $((P_\alpha, \leqslant_\alpha, 1_\alpha)|\alpha \leqslant \kappa) \in M$ as above. Let G_κ be M-generic for P_κ. We analyse the generic extension $M_\kappa = M[G_\kappa]$ by an ascending chain

$$M = M_0 \subseteq M_1 = M[G_1] = M_0[H_0] \subseteq M_2 = M[G_2] = M_1[H_1] \subseteq \ldots \subseteq M_\alpha = M[G_\alpha] \subseteq \ldots \subseteq M_\kappa$$

of generic extensions.

Let us first note some relations within the tower $(P_\alpha)_{\alpha \leqslant \kappa}$ of forcings.

Lemma 21.

a) *Let* $\alpha \leqslant \kappa$, $r: \kappa \to V$, $\forall \gamma < \alpha (r(\gamma) \in \mathrm{dom}(\dot{Q}_\gamma) \vee r(\gamma) = \emptyset)$, *and let* $\mathrm{supp}(r)$ *be finite. Then* $r \in P_\alpha$ *iff* $\forall \gamma \in \mathrm{supp}(r): r \upharpoonright \gamma \Vdash_{P_\gamma} r(\gamma) \in \dot{Q}_\gamma$.

b) *Let* $\alpha \leqslant \kappa$ *and* $p, q \in P_\alpha$. *Then* $p \leqslant_\alpha q$ *iff* $\forall \gamma \in \mathrm{supp}(p) \cup \mathrm{supp}(q): p \upharpoonright \gamma \Vdash_{P_\gamma} p(\gamma) \dot{\leqslant}_\gamma q(\gamma)$.

c) *Let* $\alpha \leqslant \beta \leqslant \kappa$ *and* $p \in P_\beta$. *Then* $p \upharpoonright \alpha \in P_\alpha$.

d) *Let* $\alpha \leqslant \beta \leqslant \kappa$ *and* $p \leqslant_\beta q$. *Then* $p \upharpoonright \alpha \leqslant_\alpha q \upharpoonright \alpha$.

e) *Let* $\alpha \leqslant \beta \leqslant \kappa$, $q \in P_\beta$, $\bar{p} \leqslant_\alpha q \upharpoonright \alpha$. *Then* $\bar{p} \cup (q(\gamma)|\alpha \leqslant \gamma < \beta) \in P_\beta$ *and* $\bar{p} \cup (q(\gamma)|\alpha \leqslant \gamma < \beta) \leqslant_\beta q$.

Proof. $a), b)$ By a straightforward induction on $\alpha \leqslant \kappa$. Now $c-e)$ follow immediately. □

For $\alpha \leqslant \kappa$ define $G_\alpha = \{p \upharpoonright \alpha \,|\, p \in G_\kappa\}$.
(2) G_α is M-generic for P_α.
Proof. By (a), $G_\alpha \subseteq P_\alpha$. Consider $p \upharpoonright \alpha$, $q \upharpoonright \alpha \in G_\alpha$ with $p, q \in G_\kappa$. Take $r \in G_\kappa$ such that $r \leqslant_\kappa p, q$. By (b), $r \upharpoonright \alpha \leqslant_\alpha p \upharpoonright \alpha, q \upharpoonright \alpha$. Thus all elements of G_α are compatible in P_α.

Consider $p \upharpoonright \alpha \in G_\alpha$ with $p \in G_\kappa$ and $\bar{q} \in P_\alpha$ with $p \upharpoonright \alpha \leqslant_\alpha \bar{q}$. By (c),

$$q = \bar{q} \cup (\emptyset|\alpha \leqslant \gamma < \kappa)$$

is an element of P_κ XXX and $p \leqslant_\kappa q$ XXX. Since G_κ is a filter, $q \in G_\kappa$, and so $\bar{q} = q \upharpoonright \alpha \in G_\alpha$. Thus G_α is upwards closed.

For the genericity consider a set $\bar{D} \in M$ which is dense in P_α. We claim that the set

$$D = \{d \in P_\kappa \,|\, d \upharpoonright \alpha \in \bar{D}\} \in M$$

is dense in P_κ: let $p \in P_\kappa$. Then $p \upharpoonright \alpha \in P_\alpha$. Take $\bar{d} \in \bar{D}$ such that $\bar{d} \leqslant_\alpha p \upharpoonright \alpha$. By ( ),

$$d = \bar{d} \cup (p(\gamma)|\alpha \leqslant \gamma < \kappa) \in P_\kappa$$

and $d \leqslant_\kappa p$ XXX.

By the genericity of G_κ take $p \in D \cap G_\kappa$. Then $p \upharpoonright \alpha \in \bar{D} \cap G_\alpha \neq \emptyset$. *qed*(2)

So $M_\alpha = M[G_\alpha]$ is a welldefined generic extension of M by G_α.
(3) Let $\alpha < \beta \leqslant \kappa$. Then $G_\alpha \in M[G_\beta]$ and $M[G_\alpha] \subseteq M[G_\beta]$.
Proof. $G_\alpha = \{p \upharpoonright \alpha \mid p \in G_\kappa\} = \{(p \upharpoonright \beta) \upharpoonright \alpha \mid p \in G_\kappa\} = \{q \upharpoonright \alpha \mid q \in G_\beta\} \in M[G_\beta]$. *qed*(3)

For $\alpha < \kappa$ define

$$Q_\alpha = (Q_\alpha, \leqslant^{Q_\alpha}, \emptyset) = \begin{cases} (\dot{Q}_\alpha^{G_\alpha}, \dot{\leqslant}_\alpha^{G_\alpha}, \emptyset), \text{ if } 1_\alpha \Vdash_{P_\alpha} (\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset) \text{ is a forcing} \\ (\{\emptyset\}, \{(\emptyset, \emptyset)\}, \emptyset), \text{ else} \end{cases}$$

Then $Q_\alpha \in M_\alpha = M[G_\alpha]$ is a forcing. For $\alpha < \kappa$ define

$$H_\alpha = \{p(\alpha)^{G_\alpha} | p \in G_\kappa\}.$$

(4) H_α is M_α-generic for Q_α.
Proof. If it is not the case that $1_\alpha \Vdash_{P_\alpha} (\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset)$ is a forcing, then $(Q_\alpha, \leqslant^{Q_\alpha}, \emptyset) = (\{\emptyset\}, \{(\emptyset, \emptyset)\}, \emptyset)$ and $H_\alpha = \{\emptyset\}$ is trivially M_α-generic. So assume that $1_\alpha \Vdash_{P_\alpha} (\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset)$.
(a) $H_\alpha \subseteq Q_\alpha$. Let $p \in G_\kappa$. Then $p \upharpoonright \alpha + 1 \in P_{\alpha+1}$ and so $p \upharpoonright \alpha \Vdash_{P_\alpha} p(\alpha) \in \dot{Q}_\alpha$. Since $p \upharpoonright \alpha \in G_\alpha$ we have that $p(\alpha)^{G_\alpha} \in \dot{Q}_\alpha^{G_\alpha} = Q_\alpha$. *qed*(a)
(b) Let ....

(e) Let $D_\alpha \in M_\alpha$ be dense in Q_α. Then $D_\alpha \cap H_\alpha \neq \emptyset$.
Proof. Take $\dot{D}_\alpha \in M$ such that $D_\alpha = \dot{D}_\alpha^{G_\alpha}$. Take $p \in G_\kappa$ such that

$$p \upharpoonright \alpha \Vdash_{P_\alpha} \dot{D}_\alpha \text{ is dense in } \dot{Q}_\alpha.$$

Define

$$D = \{d \in P_\kappa \mid d \upharpoonright \alpha \Vdash d(\alpha) \in \dot{D}_\alpha\} \in M.$$

We show that D is dense in P_κ below p. Let $q \leqslant_\kappa p$. Then $q \upharpoonright \alpha \leqslant_\alpha p \upharpoonright \alpha$ and $q \upharpoonright \alpha \Vdash q(\alpha) \dot{\leqslant}_\alpha p(\alpha)$. Hence $q \upharpoonright \alpha \Vdash_{P_\alpha} \dot{D}_\alpha$ is dense in $\dot{Q}_\alpha$ and there is $\bar{d} \leqslant_\alpha q \upharpoonright \alpha$ and some $d(\alpha) \in \text{dom}(\dot{Q}_\alpha)$ such that

$$\bar{d} \Vdash_{P_\alpha} (d(\alpha) \dot{\leqslant}_\alpha q(\alpha) \wedge d(\alpha) \in \dot{D}_\alpha).$$

Define

$$d = \bar{d} \cup \{(\alpha, d(\alpha))\} \cup \{q(\gamma) | \alpha < \gamma < \kappa\}.$$

Then by a and b $d \in P_\kappa$, $d \leqslant_\kappa q$, and $d \in D$.
By the genericity of G_κ take $d \in D \cap G_\kappa$. Then $d(\alpha)^{G_\alpha} \in H_\alpha$, $d \upharpoonright \alpha \in G_\alpha$, and $d(\alpha)^{G_\alpha} \in (\dot{D}_\alpha)^{G_\alpha} = D_\alpha$. Thus $H_\alpha \cap D_\alpha \neq \emptyset$.

( ) $M_{\alpha+1} = M_\alpha[H_\alpha]$.
Proof. $\supseteq$ is straightforward. For the other direction, if suffices to show that $G_{\alpha+1} \in M_\alpha[H_\alpha]$, and indeed we show that

$$G_{\alpha+1} = \{q \in P_{\alpha+1} \mid q \upharpoonright \alpha \in G_\alpha \wedge q(\alpha)^{G_\alpha} \in H_\alpha\}.$$

Let $q\in G_{\alpha+1}$. Take $p\in G_\kappa$ such that $p\upharpoonright\alpha+1=q$. Then $q\upharpoonright\alpha=p\upharpoonright\alpha\in G_\alpha$ and $q(\alpha)^{G_\alpha}=p(\alpha)^{G_\alpha}\in H_\alpha$. For the converse consider $q\in P_{\alpha+1}$ such that $q\upharpoonright\alpha\in G_\alpha$ and $q(\alpha)^{G_\alpha}\in H_\alpha$. Take $p_1, p_2\in G_\kappa$ such that $q\upharpoonright\alpha=p_1\upharpoonright\alpha$ and $q(\alpha)^{G_\alpha}=p_2(\alpha)^{G_\alpha}$. Take $p\in G_\kappa$ such that $p\leqslant_\kappa p_1, p_2$. We also may assume that $p\upharpoonright\alpha\Vdash q(\alpha)=p_2(\alpha)$. $p\upharpoonright\alpha\leqslant_\alpha p_1\upharpoonright\alpha=q\upharpoonright\alpha$ and $p\upharpoonright\alpha\Vdash_{P_\alpha}p(\alpha)\dot{\leqslant}_\alpha p_2(\alpha)=q(\alpha)$. Hence $p\upharpoonright\alpha+1\leqslant_{\alpha+1}q$. Since $p\upharpoonright\alpha+1\in G_{\alpha+1}$ and since $G_{\alpha+1}$ is upward closed, we get $q\in G_{\alpha+1}$.

3.1 Two-step iterations

A two-step iteration is usually defined as follows: consider a forcing $(P,\leqslant_P,0)$ and names $\dot{Q}$, $\dot{\leqslant}$ such that

$$1_P\Vdash(\dot{Q},\dot{\leqslant},0)\text{ is a forcing.}$$

and $0\in\mathrm{dom}(\dot{Q})$. Then the two-step iteration $(P*\dot{Q},\preccurlyeq,1)$ is defined by:

$$\begin{aligned} P*\dot{Q} &= \{(p,\dot{q})|p\in P\wedge\dot{q}\in\mathrm{dom}(\dot{Q})\wedge p\Vdash_P\dot{q}\in\dot{Q}\}\\ (p',\dot{q}')\preccurlyeq(p,\dot{q}) &\text{ iff } p'\leqslant_P p\wedge p'\Vdash_P\dot{q}'\dot{\leqslant}\dot{q}'\\ 1 &= (0,0)\end{aligned}$$

Then this two-step iteration can be construed as a standard iteration as follows: set $\kappa=2$. Let ...

3.2 Products of partial orders

A special case of a finite support iteration is a finite support product. So let M be a ground model, and let $((Q_\beta,\leqslant_\beta)|\beta<\kappa)\in M$ be a sequence of forcings such that $\emptyset$ is a maximal element of every Q_β. Define the *finite support product* $\prod_{\beta<\kappa}Q_\beta$ as the following forcing:

$$\begin{aligned}\prod_{\beta<\kappa}Q_\beta &= \{p:\kappa\to V|\forall\beta<\kappa: p(\beta)\in Q_\beta,\mathrm{supp}(p)\text{ is finite}\}\\ p\preccurlyeq q &\text{ iff } \forall\beta<\kappa: p(\beta)\leqslant_\beta q(\beta)\\ 1_\kappa &= (0|\beta<\kappa)\end{aligned}$$

We want to show that the product corresponds to a simple iteration. Define a sequence

$$((\check{Q}_\beta,\check{\leqslant}_\beta)|\beta<\kappa)\in M$$

where $\check{Q}_\beta$ is the canonical name for Q_β with respect to a forcing which has the β-sequence $1_\beta=(0|\gamma<\beta)$ as its maximal element. (Note that the definition of $\check{x}=\{(\check{y},1_\beta)|y\in x\}$ only depends on 1_β.) Let the sequence $((P_\alpha,\leqslant_\alpha,1_\alpha)|\alpha\leqslant\kappa)\in M$ be defined from the sequence $((\check{Q}_\beta,\check{\leqslant}_\beta)|\beta<\kappa)$ of names as in the iteration theorem.

Then there is a canonical isomorphism

$$\pi:\prod_{\beta<\kappa}Q_\beta\leftrightarrow P_\kappa$$

defined by: $p\mapsto p'$ where

$$p'(\beta)=\widecheck{p(\beta)}$$

with respect to a partial order with maximal element 1_β . It is straightforward to check that this defines an isomorphism.

3.3 Analysing a product of Cohen forcings

...

4 Ideals and cardinal coefficients

Ideals capture (some aspects of) the notion of *small sets*.

Definition 22. *A set* $\mathcal{I} \subseteq \mathcal{P}(R)$ *is an* ideal *on* R *if*

a) *if* $A, B \in \mathcal{I}$ *then* $A \cap B \in \mathcal{I}$

b) *if* $A \in \mathcal{I}$ *and* $B \subseteq A$ *then* $B \in \mathcal{I}$

c) *if* $r \in R$ *then* $\{r\} \in \mathcal{I}$

d) $R \notin \mathcal{I}$

An ideal is κ-complete *if for any family* $\mathcal{A} \subseteq \mathcal{I}$, $\mathrm{card}(\mathcal{A}) < \kappa$ *holds* $\bigcup \mathcal{A} \in \mathcal{I}$. *An ideal is* σ-complete *if it is* $\aleph_1$*-complete.*

We have already considered the following ideals on $\mathbb{R}$:

Definition 23. *Define the ideals* $\mathcal{N} = \{X \subseteq \mathbb{R} | X$ *has measure zero*$\}$ *(the ideal of* n*ullsets) and* $\mathcal{M} = \{X \subseteq \mathbb{R} | X$ *is meager*$\}$.

Both these ideals are σ-complete. They may have "more" completeness in certain models of set theory. We saw in Theorem 9 that under $\mathrm{MA}_{\aleph_1}$ the ideal $\mathcal{M}$ is $\aleph_2$-complete. On the other hand the continuum hypothesis CH implies that $\mathcal{M}$ is *not* $\aleph_2$-complete. So the value of the completeness of $\mathcal{M}$ is independent of the axioms of ZFC. To study such phenomena one introduces *cardinal characteristics* that capture properties of ideal and that may vary between different models of set theory. Sometimes these coefficients are misleadingsly called cardinal *invariants*.

Definition 24. *Let* $\mathcal{I}$ *be an ideal on* R. *Define the following cardinal characteristics:*

- $\mathrm{add}(\mathcal{I}) = \min \{\mathrm{card}(\mathcal{A}) | \mathcal{A} \subseteq \mathcal{I}, \bigcup \mathcal{A} \notin \mathcal{I}\}$ *is the* additivity (number) *of* $\mathcal{I}$
- $\mathrm{cov}(\mathcal{I}) = \min \{\mathrm{card}(\mathcal{A}) | \mathcal{A} \subseteq \mathcal{I}, \bigcup \mathcal{A} = R\}$ *is the* covering (number) *of* $\mathcal{I}$
- $\mathrm{non}(\mathcal{I}) = \min \{\mathrm{card}(X) | X \subseteq R, X \notin \mathcal{I}\}$
- $\mathrm{cof}(\mathcal{I}) = \min \{\mathrm{card}(\mathcal{A}) | \mathcal{A} \subseteq \mathcal{I}, \forall B \in \mathcal{I} \exists A \in \mathcal{A}: B \subseteq A\}$ *is the* cofinality *of* $\mathcal{I}$

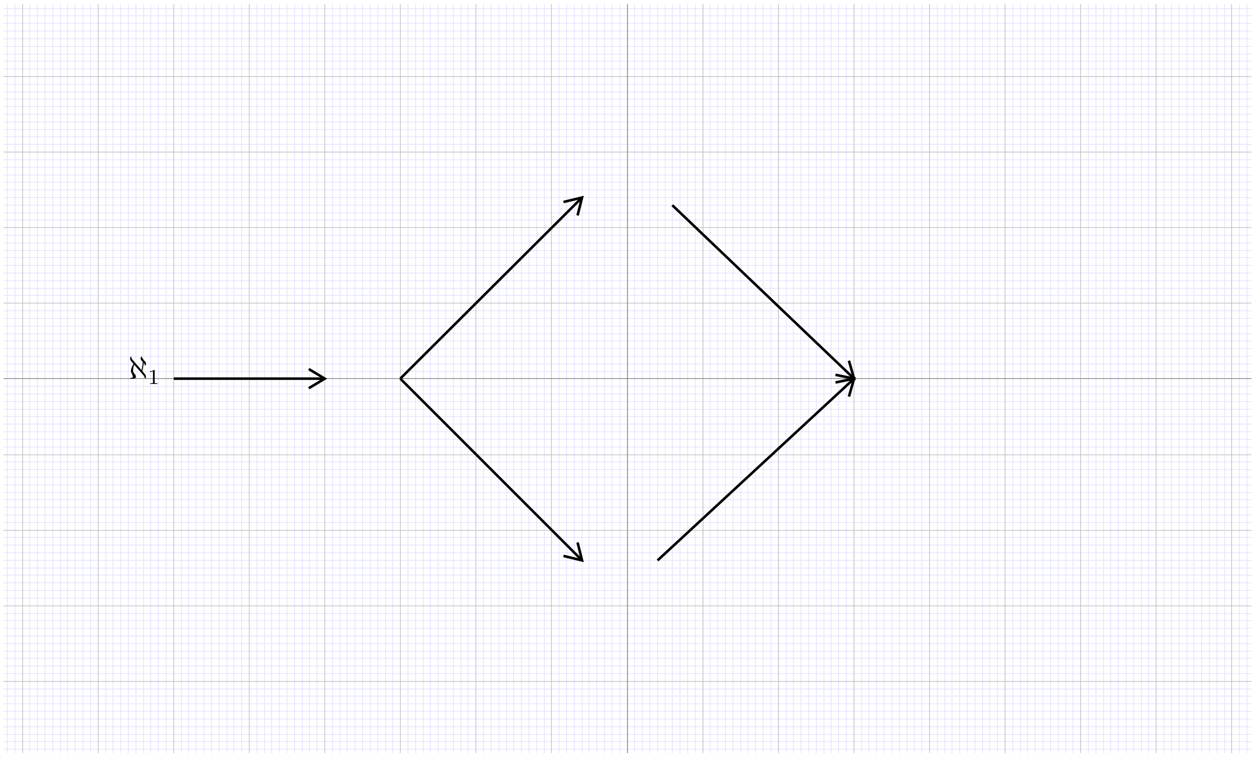
Proposition 25. *Let* $\mathcal{I}$ *be a* σ*-complete ideal on* $\mathbb{R}$. *Then*

$$\aleph_0 \leqslant \mathrm{add}(\mathcal{I}) \leqslant \mathrm{cov}(\mathcal{I}) \leqslant \mathrm{cof}(\mathcal{I}) \leqslant 2^{\aleph_0}$$

and

$$\text{add}(\mathcal{I}) \leqslant \text{non}(\mathcal{I}) \leqslant \text{cof}(\mathcal{I})$$

This can be pictured by the following diagram:



Proof. The inequalities

$$\aleph_0 \leqslant \text{add}(\mathcal{I}) \leqslant \text{cov}(\mathcal{I})$$

are trivial. To show that $\text{cov}(\mathcal{I}) \leqslant \text{cof}(\mathcal{I})$ consider a cofinal family $\mathcal{A} \subseteq \mathcal{I}$ with $\text{card}(\mathcal{A}) = \text{cof}(\mathcal{A})$. Then $\bigcup \mathcal{A} = R$ and so $\text{cov}(\mathcal{I}) \leqslant \text{card}(\mathcal{A}) = \text{cof}(\mathcal{I})$.

To show $\text{non}(\mathcal{I}) \leqslant \text{cof}(\mathcal{I})$ consider again a cofinal family $\mathcal{A} \subseteq \mathcal{I}$ with $\text{card}(\mathcal{A}) = \text{cof}(\mathcal{A})$. For each $B \in \mathcal{A}$ choose $x_B \in R \setminus B \neq \emptyset$. Then $X = \{x_B | B \in \mathcal{A}\}$ has cardinality $\leqslant \text{card}(\mathcal{A}) = \text{cof}(\mathcal{I})$. Assume for a contradiction that $X \in \mathcal{I}$. By cofinality take $B \in \mathcal{A}$ such that $X \subseteq B$. Then $x_B \in X \subseteq B$, contradiction. So $X \notin \mathcal{I}$ and

$$\text{non}(\mathcal{I}) \leqslant \text{card}(X) \leqslant \text{cof}(\mathcal{I}).$$

□

If the continuum hypothesis holds, then all these characteristics are equal to $\aleph_1 = 2^{\aleph_0}$. So it is interesting to study the characteristics in models of ZFC in which $\aleph_1 \neq 2^{\aleph_0}$. The obvious example that we can already study are the model for MA + $\aleph_1 \neq 2^{\aleph_0}$ and the COHEN model for $\aleph_1 \neq 2^{\aleph_0}$, and here one first looks at the ideals $\mathcal{M}$ and $\mathcal{N}$.

Theorem 26. *Assume* MA. *Then*

$$\text{add}(\mathcal{M}) = \text{cov}(\mathcal{M}) = \text{non}(\mathcal{M}) = \text{cof}(\mathcal{M}) = 2^{\aleph_0}$$

and

$$\operatorname{add}(\mathcal{N}) = \operatorname{cov}(\mathcal{N}) = \operatorname{non}(\mathcal{N}) = \operatorname{cof}(\mathcal{N}) = 2^{\aleph_0}$$

Proof. Because MA implies $\operatorname{add}(\mathcal{M}) = 2^{\aleph_0}$ (Theorem 9) and $\operatorname{add}(\mathcal{N}) = 2^{\aleph_0}$ (Theorem 18). □

Theorem 27. *Let M be a ground model of* ZFC + CH, *and let $M \vDash \kappa$ is a regular cardinal $> \aleph_1$. In M, let $(P, \leqslant, 1_P)$ be the forcing for adding κ* COHEN *reals:*

$$P = \operatorname{Fn}(\omega \times \kappa, 2, \aleph_0) = \{p | p: \operatorname{dom}(p) \to 2 \wedge \operatorname{dom}(p) \subseteq \omega \times \kappa \wedge \operatorname{card}(\operatorname{dom}(p)) < \aleph_0\},$$

partially ordered by reverse inclusion

$$p \leqslant q \text{ iff } p \supseteq q$$

and with weakest element $1_P = \emptyset$. Let $M[G]$ be a generic extension of M by P. Then in $M[G]$

$$\aleph_1 = \operatorname{add}(\mathcal{N}) = \operatorname{cov}(\mathcal{N}) < \operatorname{non}(\mathcal{N}) = \operatorname{cof}(\mathcal{N}) = 2^{\aleph_0}$$

and

$$\aleph_1 = \operatorname{add}(\mathcal{M}) = \operatorname{non}(\mathcal{M}) < \operatorname{cov}(\mathcal{M}) = \operatorname{cof}(\mathcal{M}) = 2^{\aleph_0}$$

Let us first prove some properties of the "COHEN model" $M[G]$.

Lemma 28. *Let $X \subseteq \kappa$, $X \in M$. Then*

$$P = \operatorname{Fn}(\omega \times \kappa, 2, \aleph_0) \cong \operatorname{Fn}(\omega \times X, 2, \aleph_0) \times \operatorname{Fn}(\omega \times (\kappa \setminus X), 2, \aleph_0)$$

is isomorphic to a product forcing by the canonical isomorphism

$$p \mapsto (p \upharpoonright X, p \upharpoonright (\kappa \setminus X)).$$

Setting $G \upharpoonright X = \{p \upharpoonright X \mid p \in G\}$ and $G \upharpoonright (\kappa \setminus X) = \{p \upharpoonright (\kappa \setminus X) \mid p \in G\}$ we have:

a) *$G \upharpoonright X$ is M-generic for* $\operatorname{Fn}(\omega \times X, 2, \aleph_0)$

b) *$G \upharpoonright (\kappa \setminus X)$ is M-generic for* $\operatorname{Fn}(\omega \times (\kappa \setminus X), 2, \aleph_0)$

c) *$G \upharpoonright (\kappa \setminus X)$ is $M[G \upharpoonright X]$-generic for* $\operatorname{Fn}(\omega \times (\kappa \setminus X), 2, \aleph_0)$

d) *$G \upharpoonright X$ is $M[G \upharpoonright (\kappa \setminus X)]$-generic for* $\operatorname{Fn}(\omega \times X, 2, \aleph_0)$

Proof. These are standard results about product forcing. □

Lemma 29. *For every real $r \in M[G] \cap \mathcal{P}(\omega)$ there is some countable $X \subseteq \kappa$, $X \in M$ such that $r \in M[G \upharpoonright X]$. Moreover consider a set $S \in M[G]$, $S \subseteq \mathcal{P}(\omega)$ such that $M[G] \vDash \operatorname{card}(S) \leqslant \lambda$ where λ is an infinite cardinal in M (and in $M[G]$). Then there is some set $X \subseteq \kappa$, $X \in M$, $\operatorname{card}(X) \leqslant \lambda$ such that $S \in M[G \upharpoonright X]$.*

Proof. It suffices to prove the second statement. Let $\dot{S} \in M$ be a name for S, i.e., $\dot{S}^G = S$ and $p \Vdash \dot{S} \subseteq \mathcal{P}(\omega)$ where $p \in G$. ... □

We now prove the conclusions of the Theorem.

Lemma 30. $M[G] \vDash \mathrm{cov}(\mathcal{N}) = \aleph_1$.

Proof. Define in $M[G]$: for $\alpha < \kappa$ set $N_\alpha = M[G \restriction \kappa \setminus \{\alpha\}] \cap \mathcal{P}(\omega)$.
(1) $(N_\alpha | \alpha < \kappa) \in M[G]$.
Proof. In M there is a set of "canonical names" for reals, when forcing with $P \restriction ((\kappa \setminus \aleph_1) \cup \alpha)$. The interpretation function $\dot{x} \mapsto \dot{x}^G$ is definable in $M[G]$. So the above definition can be carried out in $M[G]$. *qed*(1)
(2) $\mathcal{P}(\omega) \cap M[G] = \bigcup_{\alpha < \aleph_1} N_\alpha$ follows directly from Lemma 29.
(3) N_α is a measure zero set in $M[G]$.
Proof. We already showed last term that in a generic extension by one COHEN real the set of ground model reals becomes a measure zero set:

$$M[G \restriction 1] \vDash M \cap \mathcal{P}(\omega) \text{ is a measure zero set.}$$

Let us indicate the argument. We may identify $\mathcal{P}(\omega)$ with the unit interval $[0,1] \subseteq \mathbb{R}$. Let $\varepsilon > 0$. Take $n \in \omega$ such that $\frac{1}{2^n} < \varepsilon$. Define intervals $(I_k | k \in \omega)$ from $G \restriction 1$. Define the COHEN real $c: \omega \to 2$ by

$$c(m) = (\bigcup G)(m, 0).$$

Then let

$$I_k = [\sum_{i=0}^{n+k+1} c(k+i) \cdot \frac{1}{2^i}, \frac{1}{2^{n+k+1}} + \sum_{i=0}^{n+k+1} c(k+i) \cdot \frac{1}{2^i}] \subseteq \mathbb{R}.$$

Then

$$\sum_{k<\omega} \mathrm{length}(I_k) = \sum_{k<\omega} \frac{1}{2^{n+k+1}} = \frac{1}{2^n} < \varepsilon.$$

We show by a standard density/genericity argument that $\bigcup_{k<\omega} I_k \supseteq M \cap \mathcal{P}(\omega)$.

Replacing M by $M[G \restriction \kappa \setminus \{\alpha\}]$ and $M[G \restriction 1]$ by $M[G \restriction \kappa \setminus \{\alpha\}][G \restriction \{\alpha\}]$ we obtain the claim. □

Lemma 31. $M[G] \vDash \mathrm{non}(\mathcal{N}) = 2^{\aleph_0}$.

Proof. Let $S \in M[G]$, $S \subseteq \mathcal{P}(\omega)$ such that $M[G] \vDash \mathrm{card}(S) < 2^{\aleph_0} = \kappa$. By Lemma 29 take $X \subseteq \kappa$, $X \in M$, $\mathrm{card}(X) < \kappa$ such that $S \in M[G \restriction X]$. Take $\alpha \in \kappa \setminus X$. Then

$$S \subseteq M[G \restriction X] \cap \mathcal{P}(\omega) \subseteq M[G \restriction \kappa \setminus \{\alpha\}] \cap \mathcal{P}(\omega) = N_\alpha$$

which is a measure zero set in $M[G]$. □

Concerning meager sets we have to make some preparations concerning "codes" of open sets in $\mathbb{R}$. In a transitive ZFC-model N consider an open set $A \subseteq \mathbb{R}$. A can be represented as

$$A = \bigcup c$$

where $c \in N$ is a set of rational open intervals $(r, s) \subseteq \mathbb{R}$, $r, s \in \mathbb{Q}$. We can view A as the interpretation of the *code* c within the model M and write $A = c^M$. If $N' \supseteq N$ is another transitive ZFC-model then $c \in N'$ and one can form

$$A' = c^{N'} = \bigcup c \in N'$$

within N'. Then $A \subseteq A'$ and if $\mathbb{R} \cap N \neq \mathbb{R} \cap N'$ it is possible that $A \neq A'$. Nevertheless we may view A and A' as the same open set, but interpreted in different models.

Definition 32. *A* G-code *is a countable set c of rational open intervals. The* interpretation *of c is the open set*

$$c^V = \bigcup c.$$

Lemma 33. *Let $c \in N \subseteq N'$ be a G-code. Then c^N is dense open in N if $c^{N'}$ is dense open in N'.*

Proof. Let c^N be dense open in N. Consider $r, s \in \mathbb{Q}$, $r < s$. By density take $x \in c^N \cap (r, s)$. Then $x \in c^{N'} \cap (r, s)$.

Conversely Let $c^{N'}$ be dense open in N'. Consider $r, s \in \mathbb{Q}$, $r < s$. By density, $c^{N'} \cap (r, s) \neq \emptyset$. Take a rational interval $(r_0, s_0) \in c$ such that $(r_0, s_0) \cap (r, s) \neq \emptyset$. Take $q \in (r_0, s_0) \cap (r, s) \cap \mathbb{Q}$. Then $q \in c^N \cap (r, s)$. □

Note that a set $X \subseteq \mathbb{R}$ is nowhere dense iff the complement of X contains a dense open set. A set $A \subseteq \mathbb{R}$ is meager iff the complement of A contains a countable intersection of dense open sets. Let us "code" countable intersections of open sets as follows.

Definition 34. *A* G_δ-code *is a countable set d of G-codes. The* interpretation *of d is the set*

$$d^V = \bigcap_{c \in d} c^V.$$

As an explanation of the notations G and G_δ note that in HAUSDORFF's times, open sets were called "Gebiet" with a "G" and countable intersections ("Durchschnitt" with a "d") were denoted by subscripts δ.

We show that COHEN reals "avoid" meager sets from the ground model.

Lemma 35. *Let M be a ground model and let $M[z] = M[H]$ be a generic extension of M by the standard* COHEN *forcing $P = \mathrm{Fn}(\omega, 2, \aleph_0)$: let H be M-generic for P and let $z = \bigcup H \in {}^\omega 2$ be the associated* COHEN *real. Consider a set $X \in M$ which is meager in the ground model and let $d \in M$ be a G_δ-code for a countable intersection of dense open sets such that $X \cap d^M = \emptyset$. Then $z \in d^{M[z]}$.*

Proof. Let us identify $\mathbb{R}$ with ${}^\omega 2$, linearly ordered lexicographically, and let us identify $\mathbb{Q}$ with the elements of $\mathbb{R}$ which are eventually 0. Consider $c \in d$. Define, in M,

$$D = \{p \in P | \exists (r, s) \in c \forall y \in \mathbb{R} (y \supseteq p \rightarrow y \in (r, s)\}.$$

(1) D is dense in P.

Proof. Let $q \in P$. Since c^M is dense, there exists a real $y_0 \supseteq q$ such that $y_0 \in c^M$. Take $(r, s) \in c$ such that $y_0 \in (r, s)$. Take $p \in P$, $p \supseteq q$ such that $\forall y \in \mathbb{R} (y \supseteq p \rightarrow y \in (r, s))$. Then $p \in D$ and D is dense. *qed*(1)

By genericity take $p \in D \cap H$. Then $z \supseteq p$ and by the definition of D there is $(r, s) \in c$ so that

$$z \in (r, s) \subseteq c^{M[z]}.$$

Since this holds for every $c \in d$:

$$z \in \bigcap_{c \in d} c^{M[z]} = d^{M[z]}.$$

□

We can now continue the proof of Theorem 27:

Lemma 36. $M[G] \vDash \mathrm{non}(\mathcal{M}) = \aleph_1$.

Proof. In $M[G]$ define the sequence $(z_i | i < \kappa)$ of COHEN reals $z_i: \omega \to 2$ by

$$z_i(n) = (\bigcup G)(n, i).$$

We claim that $A = \{z_i | i < \omega_1\} \notin \mathcal{M}^{M[G]}$. Assume not and let $d \in M[G]$ be a G_δ-code for a countable intersection of dense open sets so that

$$A \cap d^{M[G]} = \emptyset.$$

By previous lemmas take a countable $X \subseteq \kappa, X \in M$ such that $d \in M[G \upharpoonright X]$. Take $i \in \omega_1 \setminus X$. Then $d \in M[G \upharpoonright (\kappa \setminus \{i\})]$. We have

$$M[G] = M[G \upharpoonright (\kappa \setminus \{i\})][G \upharpoonright \{i\}] = M[G \upharpoonright (\kappa \setminus \{i\})][z_i]$$

where z_i is a COHEN real with respect to the model $M[G \upharpoonright (\kappa \setminus \{i\}]$. By the previous Lemma

$$z_i \in d^{M[G \upharpoonright (\kappa \setminus \{i\})][z_i]} = d^{M[G]}$$

contradicting that $A \cap d^{M[G]} = \emptyset$. □

Lemma 37. $M[G] \vDash \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}$.

Proof. Assume for a contradiction that $(A_\xi | \xi < \lambda)$, $\lambda < \kappa$ is a sequence of meager sets such that $\mathbb{R} = \bigcup_{\xi < \lambda} A_\xi$. For each $\xi < \lambda$ choose a G_δ-code d_ξ such that $A_\xi \cap d_\xi^{M[G]} = \emptyset$. By Lemma 29 take $X \subseteq \kappa$, $\mathrm{card}(X) = \mathrm{card}(\lambda) + \aleph_0$ such that

$$\forall \xi < \lambda: d_\xi \in M[G \upharpoonright X].$$

Take $i \in \kappa \setminus X$. Then

$$\forall \xi < \lambda: d_\xi \in M[G \upharpoonright (\kappa \setminus \{i\})].$$

As above

$$z_i \in d_\xi^{M[G \upharpoonright (\kappa \setminus \{i\})][z_i]} = d_\xi^{M[G]}$$

for all $\xi < \lambda$. Hence

$$z_i \notin \bigcup_{\xi < \lambda} A_\xi = \mathbb{R},$$

contradiction. □

5 The Cichon diagram

We want to relate cardinal characteristics of the ideals $\mathcal{N}$ and $\mathcal{M}$ in a joint diagram called the Cichon diagram. We first have to define two more characteristics.

Definition 38.

a) *Define the partial ordering* $\leqslant^*$ *of* eventual domination *on* ω^ω *by*

$$f \leqslant^* g \text{ iff } \exists m < \omega \forall n \in [m, \omega): f(n) \leqslant g(n).$$

b) *The* bounding number *is*

$$\mathfrak{b} = \min \{\text{card}(F) | F \subseteq {}^\omega\omega, \forall g \in {}^\omega\omega \exists f \in F: f \nleqslant^* g\},$$

i.e., the smallest cardinality of an unbounded family in $\leqslant^*$.

c) *The* dominating number *is*

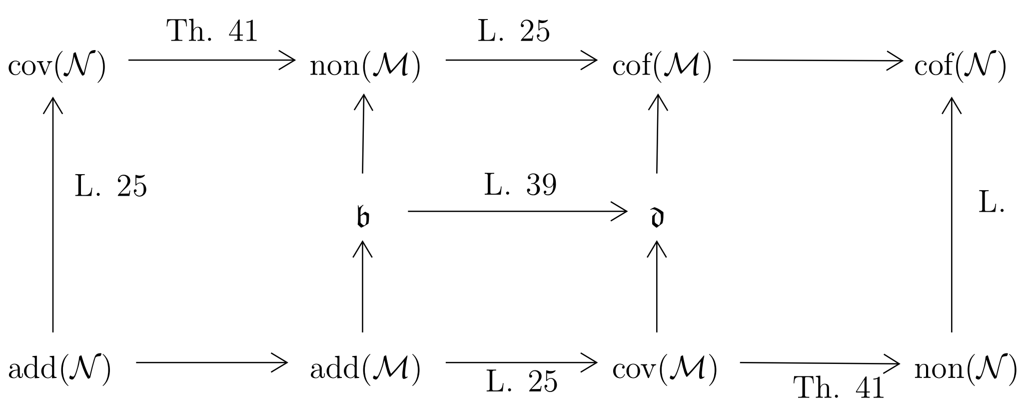
$$\mathfrak{d} = \min \{\text{card}(F) | F \subseteq {}^\omega\omega, \forall g \in {}^\omega\omega \exists f \in F: f \leqslant^* g\},$$

i.e., the smallest cardinality of a cofinal (or dominating) family in $\leqslant^*$.

Lemma 39. $\mathfrak{b} \leqslant \mathfrak{d}$.

Proof. Every cofinal family is unbounded. □

The following diagram records provable relations between the cardinal characteristics introduced so far. An arrow $\longrightarrow$ stands for the $\leqslant$-relation between cardinals. Some inequalities have already been proved:



It is remarkable that there are inequalities connecting the ideals $\mathcal{N}$ and $\mathcal{M}$.

Lemma 40. *There are sets* $A \in \mathcal{N}$ *and* $B \in \mathcal{M}$ *such that* $A \cup B = \mathbb{R}$, *i.e.,* $\mathbb{R}$ *is the (disjoint) union of two sets which are both "small".*

Proof. We work with the standard reals $\mathbb{R}$. Let $(q_n|n<\omega)$ enumerate the rational numbers. For $m<\omega$ let

$$U_m = \bigcup_{n>m} (q_n - \frac{1}{2^n}, q_n + \frac{1}{2^n}).$$

U_m is dense open in $\mathbb{R}$ and

$$\sum_{n>m} \text{length}((q_n - \frac{1}{2^n}, q_n + \frac{1}{2^n})) = \sum_{n>m} \frac{2}{2^n} = \frac{2}{2^m}.$$

Let $A = \bigcap_{m\in\omega} U_m$. By the calculation of the sum of interval lengths, A is a measure zero set, i.e., $A \in \mathcal{N}$.

$\mathbb{R}\setminus U_m$ is nowhere dense. Then $B = \bigcup_{m\in\omega} (\mathbb{R}\setminus U_m)$ is meager, i.e., $B\in\mathcal{M}$. Moreover

$$z\notin A \leftrightarrow \exists m<\omega: z\notin U_m \leftrightarrow \exists m<\omega: z\in(\mathbb{R}\setminus U_m) \leftrightarrow z\in B.$$

□

Theorem 41. (ROTHBERGER, 1938) $\text{cov}(\mathcal{M}) \leqslant \text{non}(\mathcal{N})$ *and* $\text{cov}(\mathcal{N}) \leqslant \text{non}(\mathcal{M})$.

Proof. Let $A\in\mathcal{N}$ and $B\in\mathcal{M}$ such that $A\cup B=\mathbb{R}$ as in the preceding Lemma.
(1) Let $X\notin\mathcal{M}$. Then $X+A=\{x+a|x\in X, a\in A\}=\mathbb{R}$.
Proof. Let $z\in\mathbb{R}$. Then $z-X\nsubseteq B$. Take $x\in X$ such that $z-x\in A$. Then $z\in x+A\in X+A$. *qed*(1)

Now take $X\notin\mathcal{M}$ with $\text{card}(X)=\text{non}(\mathcal{M})$. Then

$$\mathbb{R} = X + A = \bigcup_{x\in X} (x+A).$$

The right hand side is a covering of $\mathbb{R}$ by $\leqslant \text{card}(X)$ many sets in $\mathcal{N}$. So $\text{cov}(\mathcal{N}) \leqslant \text{card}(X) = \text{non}(\mathcal{M})$.

The proof of the other inequality proceeds in the same way, with $\mathcal{M}$ and $\mathcal{N}$ interchanged. □

Before we prove further inequalities in the CICHON diagram let us check the values in the diagram in the models of set theory considered so far.

If we assume MA or CH then we know already that all entries except possible $\mathfrak{b}$ or $\mathfrak{d}$ are equal to $2^{\aleph_0}$.

Lemma 42. *Assume* MA. *Then* $\mathfrak{b}=2^{\aleph_0}$ *(and so* $\mathfrak{d}=2^{\aleph_0}$*).*

Proof. Let $F\subseteq {}^\omega\omega$ and $\text{card}(F)<2^{\aleph_0}$. It suffices to show that F is bounded in the structure $({}^\omega\omega, \leqslant^*)$. Define HECHLER *forcing* by

$$P = \{(a, A)|a\in {}^{<\omega}\omega, A\subseteq {}^\omega\omega, \text{card}(A)<\aleph_0\}$$

with

$$(a', A') \leqslant (a, A) \text{ iff } a'\supseteq a, A'\supseteq A, \text{ and } \forall n\in\text{dom}(a')\setminus\text{dom}(a) \forall f\in A: a'(n) > f(n)$$

and $1_P = (\emptyset, \emptyset)$.
(1) HECHLER forcing has the ccc.
Proof. If $(a, A), (a, B)\in P$ with the same "stem" a, then they are compatible:

$$(a, A\cup B) \leqslant (a, A), (a, B).$$

So if $\mathcal{C}$ is an antichain in P, then the map $(a, A) \mapsto a$ is injective on $\mathcal{C}$. Since there are only countably many possible stems a, $\mathrm{card}(\mathcal{C}) \leqslant \aleph_0$. *qed*(1)

For every $f \in {}^{\omega}\omega$ set

$$D_f = \{(a, A) \in P \,|\, f \in A\}.$$

(2) D_f is dense in P.

Proof. Since $(a, A \cup \{f\}) \leqslant (a, A)$ and $(a, A \cup \{f\}) \in D_f$. *qed*(2)

For every $n < \omega$ set

$$D_n = \{(a, A) \in P \,|\, n \in \mathrm{dom}(a)\}.$$

(3) D_n is dense in P.

Proof. Let $(b, B) \in P$. Define $a: n + 1 \to \omega$ by

$$a(i) = \begin{cases} b(i), \text{ if } i \in \mathrm{dom}(b) \\ \max\{f(i) | f \in B\} + 1 \end{cases}$$

Then $(a, B) \leqslant (b, B)$ and $(a, A) \in D_n$. *qed*(3)

By MA take a $\{D_f | f \in F\} \cup \{D_n\}$-generic filter G on P. Let

$$h = \bigcup \{a | (a, A) \in G\}.$$

Then $h: \omega \to \omega$, since G meets every D_n.

(4) $\forall f \in F$: $f \leqslant^* h$, i.e., F is bounded.

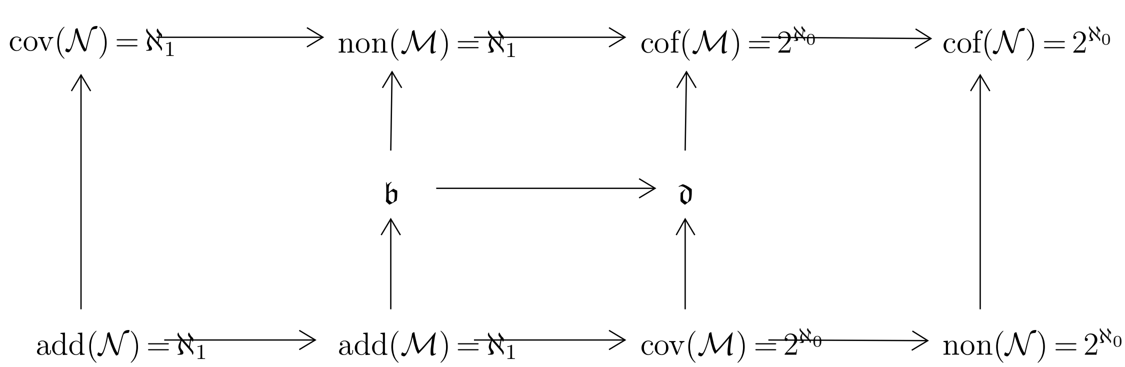
Proof. Let $f \in F$. Take $(a, A) \in G \cap D_f$. Let $m = \mathrm{dom}(a)$. Consider $n \in [m, \omega)$. Let $(a', A') \in G$ such that $n \in \mathrm{dom}(a')$. Since all elements of G are compatible we may assume that $(a', A') \leqslant (a, A)$. Then

$$h(n) = a'(n) > f(n).$$

Hence $h \geqslant^* f$. □

So under MA or CH all entries in the Cichon diagram are equal to $2^{\aleph_0}$.

In the Cohen model for $2^{\aleph_0} = \kappa > \aleph_1$ we have from our previous analysis:



We now determine that the values of $\mathfrak{b}$ and $\mathfrak{d}$ are consistent with the diagram:

Theorem 43. *Let M be a ground model of* ZFC + CH, *and let $M \vDash \kappa$ is a regular cardinal $> \aleph_1$. Let $M[G]$ be a generic extension of M by the partial order for adjoining κ* Cohen *reals using finite conditions. Then, in $M[G]$, $\mathfrak{b} = \aleph_1$ and $\mathfrak{d} = 2^{\aleph_0}$.*

Proof. We show that the first $\aleph_1$ Cohen reals are unbounded. On the other hand no family $< 2^{\aleph_0}$ can be cofinal in ${}^{\omega}\omega$ since there will always be a Cohen real which is not dominated. We have to decide whether we shall use 0/1-valued reals or ω-valued reals. □