Set Theory
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Das Wesen der Mathematik
liegt in ihrer Freiheit (Georg Cantor)

Aus dem Paradies, das Cantor uns geschaffen,
soll uns niemand vertreiben können (David Hilbert)

1 Introduction

What is Mathematics?

This a grand question and the main challenge for the philosophy of mathematics. A modest attempt would be to say that mathematics deals with *mathematical objects* and describes them in a *mathematical language*. Mathematical objects are usually considered or treated as objectively existing, like numbers or geometric objects, or as created by definitions or existence postulates. It seems that numbers like 0, 1, 2, ... are simply "there" and can be treated like physical objects, although this pragmatic viewpoint becomes problematic on closer inspection. One can define complicated objects like the collection of all twin primes, a collection which appears to be very concrete and very illusive at the same time. Properties of existing and/or created mathematical objects are expressed in a precise formal language which for the layman is a hallmark of mathematics.

By all accounts, existence in mathematics is different from ordinary existence. Even if the collection of twin primes "exists", we do not attempt to grasp it by our usual means of perceptions. We do not go near it to measure or see it, like we would with all other kinds of physical objects. If somebody claims that the collection is infinite, we would only accept a *mathematical proof* of this statement and not any other kind of evidence like physical experiments or observations, historical records, or public opinions. So even if we hold that the collection of twin primes exists, we treat it in a hypothetical and formal way: if T *were* the collection of twin primes then T *would* be infinite.

Depending on personal world views a similar criticism, though not very pragmatic, could im principle be held against ordinary existence of objects like tables and chairs. The natural sciences, confronted with a myriad of different phenomena have always tried to understand them through abstraction and unification. One can see tables and chairs made up from common building blocks like molecules and atoms. This analysis surely misses some aspects of a chair, but has been incredibly helpful for the advancement of knowledge. In physics the search is always on for encompassing world formulas. It is conceivable that such formulas do not speak about concrete physical objects but about some illusive notions like spaces, fields, symmetries, whose existence might be as problematic as that of mathematical objects.

Can we thus proceed to find mathematical "world formulas" which talk about some fundamental objects and which govern the behaviour or familiar higher objects like numbers and figures? A unified foundation for all of mathematics would explain and secure the applicability of one mathematical theory to another one. If number theory and geometry are parts of one bigger consistent theory then it becomes clear why one cannot derive results in number theory which contradict results in geometry.

What should the fundamental objects for mathematics be? Since we cannot expect them to exist in naive ways there will be many degrees of freedom, and the choice between various proposals is a matter of practicality, aesthetics, and agreement among the practitioneers. It is fair to say that *set theory* has been accepted as the basic theory for mathematics almost universally. We shall see that numbers, spaces, functions, figures, etc. can be considered as sets and that appropriate set theoretical assumptions imply that numbers, spaces, etc. have their familiar properties. This is analogous to the explanation of macroscopic friction, say, by assuming the the surfaces of bodies are made up from atoms with mutual attractions and repulsions. The microscopic theory is justified if it can provide macroscopic theories which correspond to our expectations. Set theory satisfies that criterion to a very high degree.

After the establishment of a successful theory, that theory poses questions of its own. In a theory of atoms one can ask what happens when particles collide with high velocities in a vacuum, although this question may seem irrelevant to the friction of tables and chairs. Similarly, set theory has developed its own questions. These are mostly concerned with *infinity*, transcending the concrete infinities given by collections of numbers or space points. One can view the *theory of infinity* as going to extreme frontiers of mathematics and mathematical knowledge.

At these frontiers there are many mathematical questions which cannot be decided by the standard assumptions and methods. In contrast to DAVID HILBERT's "es gibt kein *ignorabimus*" one is able to show rigorously that certain statement cannot be proved or disproved from the axioms. This is analogous to the undecidability of the parallel postulate from the other geometrical axioms. In set theory we shall encounter and prove such phenomena in connection with the axiom of choice. It is a tribute to mathematics that mathematics has often been able to *mathematically prove* its limitations, be it in geometry, algebra, logic, or set theory.

Elements of Set Theory

We want to exemplify some aspects of set theory which will be crucial for the further development.

Set-theoretic Reconstructions of Mathematical Objects

In current mathematics, *many* notions are explicitely defined using sets, e.g., a geometric figure is a set of points, usually given by some definition. The following example indicates that also notions which are not set theoretical *prima facie* can be construed set theoretically:

> f is a real funktion $\equiv$ f is a **set** of ordered pairs $(x, f(x))$ of real numbers;
> (x, y) is an ordered pair $\equiv$ (x, y) is a **set** ...$\{x, y\}$... ;
> x is a real number $\equiv$ x is a left half of a DEDEKIND cut in $\mathbb{Q}$ $\equiv$ x is a **subset** of $\mathbb{Q}$, such that ... ;
> r is a rational number $\equiv$ r is an **ordered pair** of integers, such that ... ;
> z is an integer $\equiv$ z is an **ordered pair** of natural numbers (= non-negative integers);
> $\mathbb{N} = \{0, 1, 2, \ldots\}$;
> 0 is the empty **set**;
> 1 is the **set** $\{0\}$;
> 2 is the **set** $\{0, 1\}$; etc. etc.

We shall see that *all* mathematical notions can be reduced to the notion of *set*.

Sets

GEORG CANTOR characterized sets as follows:

> Unter einer *Menge* verstehen wir jede Zusammenfassung M von bestimmten, wohlunterschiedenen Objekten m unsrer Anschauung oder unseres Denkens (welche die "Elemente" von M genannt werden) zu einem Ganzen.

If m is an *element* of M one writes $m \in M$. If all mathematical objects are reducible to sets, *both sides* of these relation have to be sets. This means that set theory studies the $\in$-relation $m \in M$ for arbitrary *sets* m and M. As it turns out, this is also sufficient for the purposes of set theory and mathematics. In set theory variables range over the class of all sets, the $\in$-relation is the only undefined structural component, every other notion will be defined from the $\in$-relation. Basically, set theoretical statement will thus be of the form

$$\ldots\forall x\ldots\exists y\ldots\ldots x \in y\ldots u \equiv v\ldots,$$

belonging to the first-order predicate language with the only given predicate $\in$. To deal with the complexities of set theory and mathematics one develops a comprehensive and intuitive aparatus of abbreviations and definitions which allows to write familiar statements like

$$e^{i\pi} = -1$$

and to view them as a statement of set theory. The language of set theory may be seen as a low-level, internal language. The language of mathematics possesses high-level "macro" expressions which abbreviate low-level statements in an efficient and intuitive way.

Infinite Cardinalities

The infinite possesses unusual and partially counterintuitive properties. The set $\mathbb{E} = \{0, 2, 4, 6, \ldots\}$ of even numbers has just as many elements as the set $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ of all natural numbers since the map

$$f: \mathbb{N} \leftrightarrow \mathbb{E}, n \mapsto 2 \cdot n$$

is a bijection between $\mathbb{N}$ and $\mathbb{E}$. Note that all these notions like natural number, odd, even, map, bijection will later be redefined as set-theoretical notions. One says that $\mathbb{N}$ and $\mathbb{E}$ have the same *cardinality*, and that they are both *countable*. This equality contradicts the traditional principle that "the part is smaller than the whole".

On December 12, 1873, GEORG CANTOR made a crucial discovery which may be taken as the beginning of set theory:

Theorem 1. *The set* $\mathbb{R}$ *of real numbers is* not *countable.*

Proof. Assume that $f: \mathbb{N} \leftrightarrow \mathbb{R}$ is bijective. Define a decimal number

$$r = 0, r_0 r_1 r_2\ldots$$

by:

$$r_i = 0\text{, if the } i\text{-th decimal position of } f(i) \text{ is a 1, and } r_i = 1 \text{ otherwise.}$$

Since f is surjective, take a natural number n such that $r = f(n)$. By definition $r_n \neq$ the n-th decimal position of $f(n)$ and thus

$$r_n \neq \text{the } n\text{-th decimal position of } r = r_n.$$

Contradiction. □

By this fundamental theorem the cardinality of the set $\mathbb{R}$ is strictly bigger than the cardinality of $\mathbb{N}$. CANTOR's *diagonal argument* can be applied to other sets, yielding higher and higher cardinalities. Thus the theory of infinite sets encompasses a rich theory of higher infinities. Infinitary combinatorics has some unusual properties. E.g., the real line $\mathbb{R}$ and the real plane $\mathbb{R} \times \mathbb{R}$ have the same cardinality, higher dimension does not necessarily lead to higher cardinalities.

The Axiom of Choice

One often constructs sequences $(a_i)_{i \in I}$ as follows:

for $i \in I$ choose a_i such that a_i satisfies ... .

This seems to be a straightforward generalization of an unproblematic procedure for finite sets I. The possibility of *infinitely many* simultaneous choices cannot be proved directly and is usually postulated as the *axiom of choice* (AC). The axiom of choice is important in many areas of mathematics. It is crucial for a smooth theory of infinite cardinalities. On the other hand it has problematic, counterintuitive consequences.

The ZERMELO-FRAENKEL *Axioms of Set Theory*

The critical discussion of the axiom of choice and related principles lead to the first axiomatization of set theory by ERNST ZERMELO. We shall deal with the axiom system ZF named after ZERMELO and ABRAHAM FRAENKEL. It is now recognized as the standard and universal axiomatization of set theory. After sufficiently developing ZERMELO-FRAENKEL set theory we shall study axiomatic aspects of set theory.

The Continuum Hypothesis

After proving that the set $\mathbb{R}$ is uncountable, CANTOR formulated the *continuum hypothesis* according to which $\mathbb{R}$ possesses the next cardinality above the cardinality of $\mathbb{N}$:

$$\mathrm{card}(\mathbb{N}) = \aleph_0 \text{ and } \mathrm{card}(\mathbb{R}) = \aleph_1 .$$

This may be reformulated as:

> for every infinite set $X \subseteq \mathbb{R}$ of real numbers, either there exists a bijection $X \leftrightarrow \mathbb{N}$ or there exists a bijection $X \leftrightarrow \mathbb{R}$.

The continuum and its generalizations imply strong combinatorial properties.

Independence

The ZF-axioms are subject to the *incompleteness theorems* of KURT GÖDEL: there are set theoretical statements which cannot be proved or disproved from the axioms. This independence does not only affect self-referential statements similar to the *liar paradoxon* ("this sentence cannot be proved from the axioms"), but also natural properties like the axiom of choice or the continuum hypothesis. We shall prove this by constructing differing *models of set theory* in which these statements are true or false.

2 The ZERMELO-FRAENKEL Axioms

CANTOR's naive description of the notion of set suggests that for any mathematical statement in $\varphi(x)$ in one free variable x there is a *set* y such that

$$x \in y \leftrightarrow \varphi(x),$$

i.e., y is the collection of all sets x which satisfy φ. Setting $\varphi(x)$ to be $x \notin x$ this becomes

$$x \in y \leftrightarrow x \notin x ,$$

and in particular for $x = y$:

$$y \in y \leftrightarrow y \notin y .$$

This *contradiction* is usually denoted RUSSELL's paradox. It shows that the formation of sets as collections of sets by arbitrary formulas is not consistent. ZERMELO's main idea was to restrict the formulas allowed in the formation of sets. The following axiom system extends the original ZERMELO axioms by contributions by FRAENKEL, MIRIMANOFF, and SKOLEM.

Definition 2. *The system* ZF *of the* ZERMELO-FRAENKEL axioms of set theory *consists of the following axioms:*

a) *The* set existence axiom (Ex)*:*

$$\exists x \forall y \neg y \in x$$

- *there is a set without elements, the* empty set*.*

b) *The* axiom of extensionality (Ext)*:*

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$$

- *a set is determined by its elements, sets having the same elements are identical.*

c) *The* pairing axiom (Pair)*:*

$$\forall x \forall y \exists z \forall w (u \in z \leftrightarrow u = x \vee u = y).$$

- *z is the unordered pair of x and y.*

d) *The* union axiom (Union)*:*

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (w \in x \wedge z \in w))$$

- *y is the union of all elements of x.*

e) *The* separation schema (Sep) *postulates for every $\in$-formula $\varphi(z, x_1, ..., x_n)$:*

$$\forall x_1 ... \forall x_n \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \varphi(z, x_1, ..., x_n))$$

- *this is an infinite scheme of axioms, the set z consists of all elements of x which satisfy φ.*

f) *The* powerset axiom (Pow)*:*

$$\forall x \exists y \forall z (z \in y \leftrightarrow \forall w (w \in z \rightarrow w \in x))$$

- *y consists of all subsets of x.*

g) *The* replacement schema (Rep) *postulates for every $\in$-formula $\varphi(x, y, x_1, ..., x_n)$:*

$$\forall x_1 ... \forall x_n (\forall x \forall y \forall y' ((\varphi(x, y, x_1, ..., x_n) \wedge \varphi(x, y', x_1, ..., x_n)) \rightarrow y = y') \rightarrow \\ \forall u \exists v \forall y (y \in v \leftrightarrow \exists x (x \in u \wedge \varphi(x, y, x_1, ..., x_n))))$$

- *v is the image of u under the map defined by φ.*

h) *The* axiom of infinity (Inf)*:*

$$\exists x (\exists y (y \in x \wedge \forall z \neg z \in y) \wedge \forall y (y \in x \rightarrow \exists z (z \in x \wedge \forall w (w \in z \leftrightarrow w \in y \vee w = y))))$$

- *by the closure properties of x, x has to be infinite.*

i) *The* foundation schema (Found) *postulates for every $\in$-formula $\varphi(x, x_1, ..., x_n)$:*

$$\forall x_1 ... \forall x_n (\exists x \varphi(x, x_1, ..., x_n) \rightarrow \exists x (\varphi(x, x_1, ..., x_n) \wedge \forall x' (x' \in x \rightarrow \neg \varphi(x', x_1, ..., x_n))))$$

- *if φ is satisfiable then there are $\in$-minimal elements satisfying φ.*

Note that the axiom system is an infinite *set* of axioms. It seems unavoidable that we have to go back to some previously given set notions to be able to define the collection of set theoretical axioms - another example of the frequent circularity in foundational theories.

We shall discuss the axioms one by one and simultaneously introduce the logical language and useful conventions.

2.1 Set Existence

The *set existence axiom*

$$\exists x \forall y \neg y \in x,$$

like all axioms, is expressed in a language with quantifiers $\exists$ ("there exists") and $\forall$ ("for all"), which is familiar from the ϵ-δ-statements in analysis. The *language of set theory* uses variables $x, y, ...$ which may satisfy the binary relations $\in$ or $=$: $x \in y$ ("x is an *element of* y") or $x = y$. These elementary *formulas* may be connected by the *propositional connectives* $\wedge$ ("and"), $\vee$ ("or"), $\rightarrow$ ("implies"), $\leftrightarrow$ ("is equivalent"), and $\neg$ ("not"). The use of this language will be demonstrated by the subsequent axioms.

2.2 Extensionality

The *axiom of extensionality*

$$\forall x \forall x' (\forall y (y \in x \leftrightarrow y \in x') \rightarrow x = x')$$

expresses that a set is exactly determined by the collection of its elements. This allows to prove that there is exactly one empty set.

Lemma 3. $\forall x \forall x' (\forall y\, \neg y \in x \wedge \forall y\, \neg y \in x' \rightarrow x = x')$.

Proof. Consider x, x' such that $\forall y\, \neg y \in x \wedge \forall y\, \neg y \in x'$. Consider y. Then $\neg y \in x$ and $\neg y \in x'$. This implies $\forall y (y \in x \leftrightarrow y \in x')$. The axiom of extensionality implies $x = x'$. □

Note that this is a *formal proof* in the sense of mathematical logic. The sentences of the proof can be derived from earlier ones by purely formal deduction rules. The rules of natural deduction correspond model common sense figures of argumentation which treat hypothetical objects as if they would concretely exist.

2.3 Pairing

The *pairing axiom*

$$\forall x \forall y \exists z \forall u (u \in z \leftrightarrow u = x \vee u = y)$$

postulates that for all sets x, y there is set z which may be denoted as

$$z = \{x, y\}.$$

This notation abbreviates the formula

$$\forall u (u \in z \leftrightarrow u = x \vee u = y).$$

The language of mathematics which we are about to introduce will consist of many such abbreviations. The abbreviations are chosen for intuitive, pragmatic, or historical reasons. Using the notation for unordered pairs, the pairing axiom may be written as

$$\forall x \forall y \exists z\ z = \{x, y\}.$$

By the axiom of extensionality, the term-like notation has the expected behaviour. E.g.:

Lemma 4. $\forall x \forall y \forall z \forall z' (z = \{x, y\} \wedge z' = \{x, y\} \rightarrow z = z')$.

Proof. Exercise. □

Note that we use a number of notational conventions: variables have to be chosen in a reasonable way, for example the symbols z and z' in the lemma have to be taken different and different from x and y. We also assume some operator priorities to reduce the number of brackets: we let $\wedge$ bind stronger than $\vee$, and $\vee$ stronger than $\rightarrow$ and $\leftrightarrow$.

We used the "term" $\{x, y\}$ to occur within set theoretical formulas. This abbreviation is than to be expanded in a natural way, so that officially all mathematical formulas are formulas in the "pure" $\in$-language. We want to see the notation $\{x, y\}$ as an example of a *class term*. We define uniform notations and convention for such abbreviation terms.

Definition 5. *A* class term *is of the form* $\{x|\varphi\}$ *where* x *is a variable and* $\varphi \in L^{\in}$. *If* $\{x|\varphi\}$ *and* $\{y|\psi\}$ *are class terms then*

- $u \in \{x|\varphi\}$ *stands for* $\varphi\frac{u}{x}$, *where* $\varphi\frac{u}{x}$ *is obtained from* φ *by (resonably) substituting the variable* x *by the variable* u;
- $u = \{x|\varphi\}$ *stands for* $\forall v (v \in u \leftrightarrow \varphi\frac{v}{x})$;
- $\{x|\varphi\} = u$ *stands for* $\forall v (\varphi\frac{v}{x} \leftrightarrow v \in u)$;

- $\{x|\varphi\} = \{y|\psi\}$ *stands for* $\forall v(\varphi\frac{v}{x} \leftrightarrow \psi\frac{v}{y})$;
- $\{x|\varphi\} \in u$ *stands for* $\exists v(v \in u \wedge v = \{x|\varphi\}$;
- $\{x|\varphi\} \in \{y|\psi\}$ *stands for* $\exists v(\psi\frac{v}{y} \wedge v = \{x|\varphi\})$.

A term *is either a variable or a class term.*

We shall further extend this notation give suggestive or traditional names to important formulas and class terms.

Definition 6.

a) $\emptyset := \{x|x \neq x\}$ *is the* empty set*;*

b) $V := \{x|x = x\}$ *is the* universe *(of all sets);*

c) $\{x, y\} := \{u|u = x \vee u = y\}$ *is the* unordered pair *of* x *and* y.

Lemma 7.

a) $\emptyset \in V$.

b) $\forall x, y \, \{x, y\} \in V$.

Proof. a) $\emptyset \in V$ abbreviates the formula

$$\exists v(v = v \wedge v = \emptyset).$$

This is equivalent to $\exists v \, v = \emptyset$ which again is an abbreviation for

$$\exists v \, \forall w (w \in v \leftrightarrow w \neq w).$$

This is equivalent to $\exists v \forall w \, w \notin v$ which is equivalent to the axiom of set existence. So $\emptyset \in V$ is another way to write the axiom of set existence.
b) $\forall x, y \, \{x, y\} \in V$ abbreviates the formula

$$\forall x, y \exists z(z = z \wedge z = \{x, y\}).$$

This can be expanded equivalently to the pairing axiom

$$\forall x, y \exists z \forall u(u \in z \leftrightarrow u = x \vee u = y). \qquad \square$$

So a) and b) are concise equivalent formulations of the axiom Ex and Pair.
We also introduce *bounded quantifiers* to simplify notation.

Definition 8. *Let A be a term. Then $\forall x \in A \, \varphi$ stands for $\forall x(x \in A \rightarrow \varphi)$ and $\exists x \in A \, \varphi$ stands for $\exists x (x \in A \wedge \varphi)$.*

Definition 9. *Let $x, y, z, \ldots$ be variables and $X, Y, Z, \ldots$ be class terms. Define*

a) $X \subseteq Y := \forall x \in X \; x \in Y$, *$X$ is a* subclass *of Y;*

b) $X \cup Y := \{x|x \in X \vee x \in Y\}$ *is the* union *of X and Y;*

c) $X \cap Y := \{x|x \in X \wedge x \in Y\}$ *is the* intersection *of X and Y;*

d) $X \setminus Y := \{x|x \in X \wedge x \notin Y\}$ *is the* difference *of X and Y;*

e) $\bigcup X := \{x|\exists y \in X \; x \in y\}$ *is the* union *of X;*

f) $\bigcap X := \{x|\forall y \in X \; x \in y\}$ *is the* intersection *of X;*

g) $\mathcal{P}(X) := \{x|x \subseteq X\}$ *is the* power class *of X;*

h) $\{X\} := \{x|x = X\}$ *is the* singleton set *of X;*

i) $\{X, Y\} := \{x|x = X \vee x = Y\}$ *is the* (unordered) pair *of X and Y;*

j) $\{X_0, \ldots, X_{n-1}\} := \{x|x = X_0 \vee \ldots \vee x = X_{n-1}\}$.

One can prove the well-known boolean properties for these operations. We only give a few examples.

Proposition 10. *$X \subseteq Y \wedge Y \subseteq X \rightarrow X = Y$.*

Proposition 11. *$\bigcup \{x, y\} = x \cup y$.*

Proof. We show the equality by two inclusions:
$(\subseteq)$. Let $u \in \bigcup \{x, y\}$. $\exists v(v \in \{x, y\} \wedge u \in v)$. Let $v \in \{x, y\} \wedge u \in v$. $(v = x \vee v = y) \wedge u \in v$.
Case 1. $v = x$. Then $u \in x$. $u \in x \vee u \in y$. Hence $u \in x \cup y$.
Case 2. $v = y$. Then $u \in y$. $u \in x \vee u \in y$. Hence $u \in x \cup y$.
Conversely let $u \in x \cup y$. $u \in x \vee u \in y$.
Case 1. $u \in x$. Then $x \in \{x, y\} \wedge u \in x$. $\exists v(v \in \{x, y\} \wedge u \in v)$ and $u \in \bigcup \{x, y\}$.
Case 2. $u \in y$. Then $x \in \{x, y\} \wedge u \in x$. $\exists v(v \in \{x, y\} \wedge u \in v)$ and $u \in \bigcup \{x, y\}$. $\square$

Exercise 1. Show: a) $\bigcup V = V$. b) $\bigcap V = \emptyset$. c) $\bigcup \emptyset = \emptyset$. d) $\bigcap \emptyset = V$.

Combining objects into ordered pairs (x, y) is taken as an undefined fundamental operation of mathematics. We cannot use the unordered pair $\{x, y\}$ for this purpose, since it does not respect the order of entries:

$$\{x, y\} = \{y, x\}.$$

We have to introduce some asymmetry between x and y to make them distinguishable. Following KURATOWSKI and WIENER we define:

Definition 12. $(x, y) := \{\{x\}, \{x, y\}\}$ *is the* ordered pair *of x and y.*

The definition involves substituting class terms within class terms. We shall see in the following how these class terms are eliminated to yield pure $\in$-formulas.

Lemma 13. *$\forall x \forall y \exists z\, z = (x, y)$.*

Proof. Consider sets x and y. By the pairing axiom choose u and v such that $u = \{x\}$ and $v = \{x, y\}$. Again by pairing choose z such that $z = \{u, v\}$. We argue that $z = (x, y)$. Note that
$(x, y) = \{\{x\}, \{x, y\}\} = \{w | w = \{x\} \vee w = \{x, y\}\}$.
Then $z = (x, y)$ is equivalent to
$\forall w(w \in z \leftrightarrow w = \{x\} \vee w = \{x, y\})$,
$\forall w(w = u \vee w = v \leftrightarrow (w = \{x\} \vee w = \{x, y\})$,
and this is true by the choice of u and v. $\square$

The KURATOWSKI-pair satisfies the fundamental property of ordered pairs:

Lemma 14. *$(x, y) = (x', y') \rightarrow x = x' \wedge y = y'$.*

Proof. Assume $(x, y) = (x', y')$, i.e.,
(1) $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$.
Case 1. $x = y$. Then
$\{x\} = \{x, y\}$,
$\{\{x\}, \{x, y\}\} = \{\{x\}, \{x\}\} = \{\{x\}\}$,
$\{\{x\}\} = \{\{x'\}, \{x', y'\}\}$,
$\{x\} = \{x'\}$ and $x = x'$,
$\{x\} = \{x', y'\}$ and $y' = x$.
Hence $x = x'$ and $y = x = y'$ as required.
Case 2. $x \neq y$. (1) implies
$\{x'\} = \{x\}$ or $\{x'\} = \{x, y\}$.
The right-hand side would imply $x = x' = y$, contradicting the case assumption. Hence
$\{x'\} = \{x\}$ and $x' = x$.
Then (1) implies
$\{x, y\} = \{x', y'\} = \{x, y'\}$ and $y = y'$. $\square$

Exercise 2.

a) Show that $\langle x,\, y\rangle := \{\{x,\, \emptyset\},\, \{y,\, \{\emptyset\}\}\}$ also satisfies the fundamental property of ordered pairs (F. HAUSDORFF).

b) Can $\{x, \{y, \emptyset\}\}$ be used as an ordered pair?

Exercise 3. Give a set-theoretical formalization of an ordered-*triple* operation.

Ordered pairs allow to introduce *relations* and *functions* in the usual way. One has to distinguish between *sets* which are relations and functions, and *class terms* which are relations and functions.

Definition 15. *A term R is a* relation *if all elements of R are ordered pairs, i.e., $R \subseteq V \times V$. Also write Rxy or xRy instead of $(x, y) \in R$. If A is a term and $R \subseteq A \times A$ then R is a* relation *on A.*

Note that this definition is really an *infinite schema* of definitions, with instances for all terms R and A. The subsequent extensions of our language are also infinite definition schemas. We extend the term language by parametrized collections of terms.

Definition 16. *Let $t(\vec{x})$ be a term in the variables $\vec{x}$ and let φ be an $\in$-formula. Then $\{t(\vec{x})|\varphi\}$ stands for $\{z|\exists\vec{x}\,(\varphi \wedge z = t(\vec{x}))\}$.*

Definition 17. *Let R, S, A be terms.*

a) The domain *of R is* $\mathrm{dom}(R) := \{x \mid \exists y\, x\, R\, y\}$.

b) The range *of R is* $\mathrm{ran}(R) := \{y \mid \exists x\, x\, R\, y\}$.

c) The field *of R is* $\mathrm{field}(R) := \mathrm{dom}(R) \cup \mathrm{ran}(R)$.

d) The restriction *of R to A is* $R \upharpoonright A := \{(x, y)|xRy \wedge x \in A\}$.

e) The image *of A under R is* $R[A] := R''A := \{y|\exists x \in A\, xRy\}$.

f) The preimage *of A under R is* $R^{-1}[A] := \{x|\exists y \in A\, xRy\}$.

g) The composition *of S and R ("S after R") is* $S \circ R := \{(x, z)|\exists y\,(xRy \wedge ySz)\}$.

h) The inverse *of R is* $R^{-1} := \{(y, x)|xRy\}$.

Relations can play different roles in mathematics.

Definition 18. *Let R be a relation.*

a) R is reflexive *iff* $\forall x \in \mathrm{field}(R)\; xRx$.

b) R is irreflexive *iff* $\forall x \in \mathrm{field}(R)\; \neg xRx$.

c) R is symmetric *iff* $\forall x, y\,(xRy \rightarrow yRx)$.

d) R is antisymmetric *iff* $\forall x, y\,(xRy \wedge yRx \rightarrow x = y)$.

e) R is transitive *iff* $\forall x, y, z\,(xRy \wedge yRz \rightarrow xRz)$.

f) R is connex *iff* $\forall x, y \in \mathrm{field}(R)\,(xRy \vee yRx \vee x = y)$.

g) R is an equivalence relation *iff R is reflexive, symmetric and transitive.*

h) Let R be an equivalence relation. Then $[x]_R := \{y|yRx\}$ is the equivalence class *of x modulo R.*

It is possible that an equivalence class $[x]_R$ is not a set: $[x]_R \notin V$. Then the formation of the collection of all equivalence classes modulo R may lead to contradictions. Another important family of relations is given by *order relations*.

Definition 19. *Let R be a relation.*

a) R is a partial order *iff R is reflexive, transitive and antisymmetric.*

b) *R is a* linear order *iff R is a connex partial order.*

c) *Let A be a term. Then R is a* partial order on A *iff R is a partial order and* $\mathrm{field}(R) = A$.

d) *R is a* strict partial order *iff R is transitive and irreflexive.*

e) *R is a* strict linear order *iff R is a connex strict partial order.*

Partial orders are often denoted by symbols like $\leqslant$, and strict partial orders by $<$. A common notation in the context of (strict) partial orders R is to write

$$\exists pRq\varphi \text{ and } \forall pRq\varphi \text{ for } \exists p(pRq \wedge \varphi) \text{ and } \forall p(pRq \rightarrow \varphi) \text{ resp.}$$

One of the most important notions in mathematics is that of a *function*.

Definition 20. *Let F be a term. Then F is a* function *if it is a relation which satisfies*

$$\forall x, y, y' \, (xFy \wedge xFy' \rightarrow y = y').$$

If F is a function then

$$F(x) := \{u | \forall y (xFy \rightarrow u \in y)\}$$

is the value *of F at x.*

If F is a function and xFy then $y = F(x)$. If there is no y such that xFy then $F(x) = V$; the "value" V at x may be read as "undefined". A function can also be considered as the (indexed) sequence of its values, and we also write

$$(F(x))_{x \in A} \text{ or } (F_x)_{x \in A} \text{ instead of } F: A \rightarrow V.$$

We define further notions associated with functions.

Definition 21. *Let F, A, B be terms.*

a) *F is a* function from A to B, *or F: $A \rightarrow B$, iff F is a function,* $\mathrm{dom}(F) = A$, *and* $\mathrm{range}(F) \subseteq B$.

b) *F is a* partial function from A to B, *or $F: A \rightharpoonup B$, iff F is a function,* $\mathrm{dom}(F) \subseteq A$, *and* $\mathrm{range}(F) \subseteq B$.

c) *F is a* surjective *function from A to B iff $F: A \rightarrow B$ and* $\mathrm{range}(F) = B$.

d) *F is an* injective *function from A to B iff $F: A \rightarrow B$ and*

$$\forall x, x' \in A \, (x \neq x' \rightarrow F(x) \neq F(x'))$$

e) *F is a* bijective *function from A* to *B, or $F: A \leftrightarrow B$, iff $F: A \rightarrow B$ is surjective and injective.*

f) ${}^{A}B := \{f | f: A \rightarrow B\}$ *is the class of all functions from A to B.*

One can check that these functional notions are consistent and agree with common usage:

Exercise 4. Define a relation $\sim$ on V by

$$x \sim y \longleftrightarrow \exists f \, f: x \leftrightarrow y.$$

One say that x and y are *equinumerous* or *equipollent*. Show that $\sim$ is an equivalence relation on V. What is the equivalence class of $\emptyset$? What is the equivalence class of $\{\emptyset\}$?

Exercise 5. Consider functions $F: A \rightarrow B$ and $F': A \rightarrow B$. Show that

$$F = F' \text{ iff } \forall a \in A \; F(a) = F'(a).$$

2.4 Unions

The *union axiom* reads

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (w \in x \wedge z \in w)).$$

Lemma 22. *The union axiom is equivalent to* $\forall x \bigcup x \in V$.

Proof. Observe the following equivalences:
$\forall x \bigcup x \in V$
$\leftrightarrow \forall x \exists y (y = y \wedge y = \bigcup x)$
$\leftrightarrow \forall x \exists y \forall z (z \in y \leftrightarrow z \in \bigcup x)$
$\leftrightarrow \forall x \exists y \forall z (z \in y \leftrightarrow \exists w \in x\, z \in w)$
which is equivalent to the union axiom. □

Note that the union of x is usually viewed as the union of all *elements* of x:

$$\bigcup x = \bigcup_{w \in x} w,$$

where we define

$$\bigcup_{a \in A} t(a) = \{z | \exists a \in A\, z \in t(a)\}.$$

Combining the axioms of pairing and unions we obtain:

Lemma 23. $\forall x_0, \ldots, x_{n-1} \{x_0, \ldots, x_{n-1}\} \in V$.

Note that this is a *schema* of lemmas, one for each ordinary natural number n. We prove the schema by complete induction on n.

Proof. For $n = 0, 1, 2$ the lemma states that $\emptyset \in V$, $\forall x \{x\} \in V$, and $\forall x, y \{x, y\} \in V$ resp., and these are true by previous axioms and lemmas. For the induction step assume that the lemma holds for n, $n \geqslant 1$. Consider sets $x_0, \ldots, x_n$. Then

$$\{x_0, \ldots, x_n\} = \{x_0, \ldots, x_{n-1}\} \cup \{x_n\}.$$

The right-hand side exists in V by the inductive hypothesis and the union axiom. □

2.5 Separation

It is common to form a subset of a given set consisting of all elements which satisfy some condition. This is codified by the *separation schema*. For every $\in$-formula $\varphi(z, x_1, \ldots, x_n)$ postulate:

$$\forall x_1 \ldots \forall x_n \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \varphi(z, x_1, \ldots, x_n)).$$

Using class terms the schema can be reformulated as: for every term A postulate

$$\forall x\, A \cap x \in V.$$

The crucial point is the restriction to the given set x. The unrestricted, Fregean version $A \in V$ for every term A leads to the Russell antinomy. We turn the antinomy into a consequence of the separation schema:

Theorem 24. $V \notin V$.

Proof. Assume that $V \in V$. Then $\exists x\, x = V$. Take x such that $x = V$. Let R be the Russellian class:

$$R := \{x | x \notin x\}.$$

By separation, $y := R \cap x \in V$. Note that $R \cap x = R \cap V = R$. Then

$$y \in y \leftrightarrow y \in R \leftrightarrow y \notin y,$$

contradiction. □

This simple but crucial theorem leads to the distinction:

Definition 25. *Let A be a term. Then A is a* proper class *iff* $A \notin V$.

Set theory deals with sets and proper classes. Sets are the favoured objects of set theory, the axiom mainly state favorable properties of sets and set existence. Sometimes one says that a term A *exists* if $A \in V$. The intention of set theory is to construe important mathematical classes like the collection of natural and real numbers as sets so that they can be treated set-theoretically. ZERMELO observed that this is possible by requiring some set existences together with the *restricted* separation principle.

Exercise 6. Show that the class $\{\{x\}|x \in V\}$ of *singletons* is a proper class.

2.6 Power Sets

The *power set axiom* in class term notation is

$$\forall x \mathcal{P}(x) \in V.$$

The power set axiom yields the existence of function spaces.

Definition 26. *Let A, B be terms. Then*

$$A \times B := \{(a, b)|a \in A \wedge b \in B\}$$

is the cartesian product *of A and B.*

Exercise 7.

By the specific implementation of KURATOWSKI ordered pairs:

Lemma 27. $A \times B \subseteq \mathcal{P}(\mathcal{P}(A \cup B))$.

Proof. Let $(a, b) \in A \times B$. Then

$$\begin{aligned} a, b &\in A \cup B \\ \{a\}, \{a, b\} &\subseteq A \cup B \\ \{a\}, \{a, b\} &\in \mathcal{P}(A \cup B) \\ (a, b) = \{\{a\}, \{a, b\}\} &\subseteq \mathcal{P}(A \cup B) \\ (a, b) = \{\{a\}, \{a, b\}\} &\in \mathcal{P}(\mathcal{P}(A \cup B)) \end{aligned}$$

□

Theorem 28.

a) $\forall x, y\ x \times y \in V$.

b) $\forall x, y\ {}^x y \in V$.

Proof. Let x, y be sets. a) Using the axioms of pairing, union, and power sets, $\mathcal{P}(\mathcal{P}(x \cup y)) \in V$. By the previous lemma and the axiom schema of separation,

$$x \times y = (x \times y) \cap \mathcal{P}(\mathcal{P}(x \cup y)) \in V.$$

b) ${}^x y \subseteq \mathcal{P}(x \times y)$ since a function $f: x \to y$ is a subset of $x \times y$. By the separation schema,

$$^x y = ^x y \cap \mathcal{P}(x \times y) \in V.$$

□

Note that to "find" the sets in this theorem one has to apply the power set operation repeatedly. We shall see that the universe of all sets can be obtained by iterating the power set operation.

The power set axiom leads to higher *cardinalities*. The theory of cardinalities will be developed later, but we can already prove CANTOR's theorem:

Theorem 29. *Let $x \in V$.*

a) There is an injective map $f: x \to \mathcal{P}(x)$.

b) *There does not exist an injective map* $g: \mathcal{P}(x) \to x$.

Proof. a) Define the map $f: x \to \mathcal{P}(x)$ by $u \mapsto \{u\}$. This is a set since

$$f = \{(u, \{u\}) | u \in x\} \subseteq x \times \mathcal{P}(x) \in V.$$

f is injective: let $u, u' \in x$, $u \neq u'$. By extensionality,

$$f(u) = \{u\} \neq \{u'\} = f(u').$$

b) Assume there were an injective map $g: \mathcal{P}(x) \to x$. Define the CANTORean set

$$c = \{u | u \in x \wedge u \notin g^{-1}(u)\} \in P(x).$$

Let $u_0 = g(c)$. Then $g^{-1}(u_0) = c$ and

$$u_0 \in c \leftrightarrow u_0 \notin g^{-1}(u_0) = c.$$

Contradiction. □

2.7 Replacement

If every element of a set is definably *replaced* by another set, the result is a set again. The *schema of replacement* postulates for every term F:

$$F \text{ is a function } \to \forall x\, F[x] \in V.$$

Lemma 30. *The replacement schema implies the separation schema.*

Proof. Let A be a term and $x \in V$.
Case 1. $A \cap x = \emptyset$. Then $A \cap x \in V$ by the axiom of set existence.
Case 2. $A \cap x \neq \emptyset$. Take $u_0 \in A \cap x$. Define a map $F: x \to x$ by

$$F(u) = \begin{cases} u\,, \text{ if } u \in A \cap x \\ u_0\,, \text{ else} \end{cases}$$

Then by replacement

$$A \cap x = F[x] \in V$$

as required. □

2.8 Infinity

All the axioms so far can be realized in a domain of finite sets, see exercise ???. The true power of set theory is set free by postulating the existence of *one* infinite set and continuing to assume the axioms. The *axiom of infinity* expresses that the set of “natural numbers” exists. To this end, some “number-theoretic” notions are defined.

Definition 31.

a) $0 := \emptyset$ *is the number* zero.

b) *For any term* t, $t + 1 := t \cup \{t\}$ *is the* successor *of* t.

These notions are reasonable in the later formalization of the natural numbers. The axiom of infinity postulates the existence of a set which contains 0 and is closed under successors

$$\exists x\, (0 \in x \wedge \forall n \in x\; n + 1 \in x).$$

Intuitively this says that there is a set which contains all natural numbers. Let us define set-theoretic analogues of the standard natural numbers:

Definition 32. *Define*

a) $1 := 0 + 1$*;*

b) $2:=1+1$;

c) $3:=2+1$; ...

From the context it will be clear, whether "3", say, is meant to be the standard number "three" or the set theoretical object

$$\begin{aligned} 3 &= 2\cup\{2\} \\ &= (1+1)\cup\{1+1\} \\ &= (\{\emptyset\}\cup\{\{\emptyset\}\})\cup\{\{\emptyset\}\cup\{\{\emptyset\}\}\} \\ &= \{\emptyset,\{\emptyset\},\{\emptyset\}\cup\{\{\emptyset\}\}\}. \end{aligned}$$

The set-theoretic axioms will ensure that this interpretation of "three" has all important number-theoretic properties of "three".

2.9 Foundation

The *axiom schema of foundation* provides structural information about the set theoretic universe V. It can be reformulated by postulating, for any term A:

$$A\neq\emptyset\rightarrow\exists x\in A\,A\cap x=\emptyset.$$

Viewing $\in$ as some kind of order relation this means that every non-empty class has an $\in$-minimal element $x\in A$ such that the $\in$-predecessors of x are not in A. Foundation excludes circles in the $\in$-relation:

Lemma 33. *Let n be a natural number $\geqslant 1$. Then there are* no $x_0,\ldots,x_{n-1}$ *such that*

$$x_0\in x_1\in\ldots\in x_{n-1}\in x_0\,.$$

Proof. Assume not and let $x_0\in x_1\in\ldots\in x_{n-1}\in x_0$. Let

$$A=\{x_0,\ldots,x_{n-1}\}.$$

$A\neq\emptyset$ since $n\geqslant 1$. By foundation take $x\in A$ such that $A\cap x=\emptyset$.
Case 1. $x=x_0$. Then $x_{n-1}\in A\cap x=\emptyset$, contradiction.
Case 2. $x=x_i$, $i>0$. Then $x_{i-1}\in A\cap x=\emptyset$, contradiction. □

Exercise 8. Show that $x\neq x+1$.

Exercise 9. Show that the successor function $x\mapsto x+1$ is injective.

Exercise 10. Show that the term $\{x,\{x,y\}\}$ may be taken as an ordered pair of x and y.

Theorem 34. *The foundation scheme is equivalent to the following,* PEANO*-type, induction scheme: for every term B postulate*

$$\forall x\,(x\subseteq B\rightarrow x\in B)\rightarrow B=V.$$

This says that if a "property" B is inherited by x if all elements of x have the property B, then every set has the property B.

Proof. ($\rightarrow$) Assume B were a term which did not satisfy the induction principle:

$$\forall x\,(x\subseteq B\rightarrow x\in B)\text{ and } B\neq V.$$

Set $A=V\setminus B\neq\emptyset$. By foundation take $x\in A$ such that $A\cap x=\emptyset$. Then

$$u\in x\rightarrow u\notin A\rightarrow u\in B,$$

i.e., $x\subseteq B$. By assumption, B is inherited by x: $x\in B$. But then $x\notin A$, contradiction.
($\leftarrow$) Assume A were a term which did not satisfy the foundation scheme:

$$A\neq\emptyset\text{ and }\forall x\in A\,A\cap x\neq\emptyset.$$

Set $B = V \setminus A$. Consider $x \subseteq B$. Then $A \cap x = \emptyset$. By assumption, $x \notin A$ and $x \in B$. Thus $\forall x (x \subseteq B \rightarrow x \in B)$. The induction principle implies that $B = V$. Then $A = \emptyset$, contradiction. □

This proof shows, that the induction principle is basically an equivalent formulation of the foundation principle. The $\in$-relation is taken as some binary relation without reference to specific properties of this relation. This leads to:

Exercise 11. A relation R on a domain D is called *wellfounded*, iff for all terms A

$$\emptyset \neq A \wedge A \subseteq D \rightarrow \exists x \in A\, A \cap \{y \mid yRx\} = \emptyset.$$

Formulate and prove a principle for R-induction on D which coressponds to the assumption that R is wellfounded on D.

2.10 Axiom Systems

Using class terms, the ZF can be formulated concisely:

Theorem 35. *The* ZF *axioms are equivalent to the following system; we take all free variables of the axioms to be universally quantified:*

a) *Ex:* $\emptyset \in V$.

b) *Ext:* $x \subseteq y \wedge y \subseteq x \rightarrow x = y$.

c) *Pair:* $\{x, y\} \in V$.

d) *Union:* $\bigcup x \in V$.

e) *Sep:* $A \cap x \in V$.

f) *Pow:* $\mathcal{P}(x) \in V$.

g) *Rep:* F *is a function* $\rightarrow F[x] \in V$.

h) *Inf:* $\exists x (0 \in x \wedge \forall n \in x\; n + 1 \in x)$.

i) *Found:* $A \neq \emptyset \rightarrow \exists x \in A\; A \cap x = \emptyset$.

This axiom system can be used as a foundation for nearly all of mathematics. Axiomatic set theory considers various axiom systems of set theory.

Definition 36. *The axiom system* ZF^- *consists of the* ZF*-axioms except the power set axiom. The system* EML *("elementary set theory") consists of the axioms* Ex*,* Ext*,* Pair*, and* Union*.*

3 Ordinal Numbers

We had defined the "natural numbers" in set theory. Recall that

$$\begin{array}{rcl} 0 & = & \emptyset \\ 1 & = & 0+1 = 0 \cup \{0\} = \{0\} \\ 2 & = & 1+1 = 1 \cup \{1\} = \{0,1\} \\ 3 & = & 2+1 = 2 \cup \{2\} = \{0,1,2\} \\ & \vdots & \end{array}$$

We would then like to have $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. To obtain a set theoretic formalization of numbers we note some properties of the informal presentation:

1. "Numbers" are ordered by the $\in$-relation:

   $$m < n \text{ iff } m \in n.$$

   E.g., $1 \in 3$ but not $3 \in 1$.

2. On each "number", the $\in$-relation is a *strict linear order*: $3 = \{0, 1, 2\}$ is strictly linearly ordered by $\in$.

3. "Numbers" are "complete" with respect to smaller "numbers"

$$i < j < m \rightarrow i \in m.$$

   This can be written with the $\in$-relation as

$$i \in j \in m \rightarrow i \in m.$$

Definition 37.

a) A *is* transitive, $\mathrm{Trans}(A)$, *iff* $\forall y \in A \forall x \in y\, x \in A$.

b) x *is an* ordinal (number), $\mathrm{Ord}(x)$, *if* $\mathrm{Trans}(x) \wedge \forall y \in x\, \mathrm{Trans}(y)$.

c) *Let* $\mathrm{Ord} := \{x | \mathrm{Ord}(x)\}$ *be the class of all ordinal numbers.*

We shall use small greek letter $\alpha, \beta, ...$ as variables for ordinals. So $\exists \alpha \varphi$ stands for $\exists \alpha \in \mathrm{Ord}\ \varphi$, and $\{\alpha | \varphi\}$ for $\{\alpha | \mathrm{Ord}(\alpha) \wedge \varphi\}$.

Exercise 12. Show that arbitrary unions and intersections of transitive sets are again transitive.

We shall see that the ordinals extend the standard natural numbers. Ordinals are particularly adequate for enumerating infinite sets.

Theorem 38.

a) $0 \in \mathrm{Ord}$.

b) $\forall \alpha\ \alpha + 1 \in \mathrm{Ord}$.

Proof. a) $\mathrm{Trans}(\emptyset)$ since formulas of the form $\forall y \in \emptyset ...$ are tautologously true. Similarly $\forall y \in \emptyset\, \mathrm{Trans}(y)$.
b) Assume $\alpha \in \mathrm{Ord}$.
(1) $\mathrm{Trans}(\alpha + 1)$.
Proof. Let $u \in v \in \alpha + 1 = \alpha \cup \{\alpha\}$.
Case 1. $v \in \alpha$. Then $u \in \alpha \subseteq \alpha + 1$, since α is transitive.
Case 2. $v = \alpha$. Then $u \in \alpha \subseteq \alpha + 1$. *qed*(1)
(2) $\forall y \in \alpha + 1\, \mathrm{Trans}(y)$.
Proof. Let $y \in \alpha + 1 = \alpha \cup \{\alpha\}$.
Case 1. $y \in \alpha$. Then $\mathrm{Trans}(y)$ since α is an ordinal.
Case 2. $y = \alpha$. Then $\mathrm{Trans}(y)$ since α is an ordinal. □

Exercise 13.

a) Let $A \subseteq \mathrm{Ord}$ be a term, $A \neq \emptyset$. Then $\bigcap A \in \mathrm{Ord}$.

b) Let $x \subseteq \mathrm{Ord}$ be a set. Then $\bigcup A \in \mathrm{Ord}$.

Theorem 39. Trans(Ord).

Proof. This follows immediately from the transitivity definition of Ord. □

Exercise 14. Show that Ord is a proper class. (Hint: if $\mathrm{Ord} \in V$ then $\mathrm{Ord} \in \mathrm{Ord}$.)

Theorem 40. *The class* Ord *is strictly linearly ordered by* $\in$, *i.e.,*

a) $\forall \alpha, \beta, \gamma\, (\alpha \in \beta \wedge \beta \in \gamma \rightarrow \alpha \in \gamma)$.

b) $\forall \alpha\, \alpha \notin \alpha$.

c) $\forall \alpha, \beta\, (\alpha \in \beta \vee \alpha = \beta \vee \beta \in \alpha)$.

Proof. a) Let $\alpha, \beta, \gamma \in \mathrm{Ord}$ and $\alpha \in \beta \wedge \beta \in \gamma$. Then γ is transitive, and so $\alpha \in \gamma$.
b) follows immediately from the non-circularity of the $\in$-relation.
c) Assume that there are "incomparable" ordinals. By the foundation schema choose $\alpha_0 \in \mathrm{Ord}$ $\in$-minimal such that $\exists \beta \neg(\alpha_0 \in \beta \vee \alpha_0 = \beta \vee \beta \in \alpha_0)$. Again, choose $\beta_0 \in \mathrm{Ord}$ $\in$-minimal such that $\neg(\alpha_0 \in \beta_0 \vee \alpha_0 = \beta_0 \vee \beta_0 \in \alpha_0)$. We obtain a contradiction by showing that $\alpha_0 = \beta_0$:

Let $\alpha \in \alpha_0$. By the $\in$-minimality of α_0, α is comparable with β_0: $\alpha \in \beta_0 \vee \alpha = \beta_0 \vee \beta_0 \in \alpha$. If $\alpha = \beta_0$ then $\beta_0 \in \alpha_0$ and α_0, β_0 would be comparable, contradiction. If $\beta_0 \in \alpha$ then $\beta_0 \in \alpha_0$ by the transitivity of α_0 and again α_0, β_0 would be comparable, contradiction. Hence $\alpha \in \beta_0$.

For the converse let $\beta \in \beta_0$. By the $\in$-minimality of β_0, β is comparable with α_0: $\beta \in \alpha_0 \vee \beta = \alpha_0 \vee \alpha_0 \in \beta$. If $\beta = \alpha_0$ then $\alpha_0 \in \beta_0$ and α_0, β_0 would be comparable, contradiction. If $\alpha_0 \in \beta$ then $\alpha_0 \in \beta_0$ by the transitivity of β_0 and again α_0, β_0 would be comparable, contradiction. Hence $\beta \in \alpha_0$.

But then $\alpha_0 = \beta_0$ contrary to the choice of β_0. □

Definition 41. *Let* $<\,:= \in \cap (\mathrm{Ord} \times \mathrm{Ord}) = \{(\alpha, \beta) | \alpha \in \beta\}$ *be the natural strict linear ordering of* Ord *by the* $\in$*-relation.*

Theorem 42. *Let* $\alpha \in \mathrm{Ord}$. *Then* $\alpha + 1$ *is the immediate successor of* α *in the* $\in$*-relation:*

a) $\alpha < \alpha + 1$;

b) if $\beta < \alpha + 1$, *then* $\beta = \alpha$ *or* $\beta < \alpha$.

Definition 43. *Let* α *be an ordinal.* α *is a* successor ordinal, $\mathrm{Succ}(\alpha)$, *iff* $\exists \beta\, \alpha = \beta + 1$. α *is a limit ordinal,* $\mathrm{Lim}(\alpha)$, *iff* $\alpha \neq 0$ *and* α *is not a successor ordinal. Also let*

$$\mathrm{Succ} := \{\alpha | \mathrm{Succ}(\alpha)\} \text{ and } \mathrm{Lim} := \{\alpha | \mathrm{Lim}(\alpha)\}.$$

The existence of limit ordinals will be discussed together with the formalization of the natural numbers.

3.1 Induction

Ordinals satisfy an *induction theorem* which generalizes *complete induction* on the integers:

Theorem 44. *Let* $\varphi(x, v_0, \ldots, v_{n-1})$ *be an* $\in$*-formula and* $x_0, \ldots, x_{n-1} \in V$. *Assume that the property* $\varphi(x, x_0, \ldots, x_{n-1})$ *is* inductive, *i.e.,*

$$\forall \alpha (\forall \beta \in \alpha\ \varphi(\beta, x_0, \ldots, x_{n-1}) \rightarrow \varphi(\alpha, x_0, \ldots, x_{n-1})).$$

Then φ *holds for all ordinals:*

$$\forall \alpha\, \varphi(\alpha, x_0, \ldots, x_{n-1}).$$

Proof. Apply Theorem 34 to the term

$$B = \{x | x \in \mathrm{Ord} \rightarrow \varphi(x, x_0, \ldots, x_{n-1})\}.$$

Assume not. This means that there are x satisfying the property:

$$x \in \mathrm{Ord} \wedge \neg \varphi(x, x_0, \ldots, x_{n-1}).$$

According to the schema of foundation one can take an $\in$-minimal x with that property:

$$x \in \mathrm{Ord} \wedge \neg \varphi(x, x_0, \ldots, x_{n-1}) \wedge \forall y (y \in x \rightarrow \neg y \in \mathrm{Ord} \wedge \neg \varphi(y, x_0, \ldots, x_{n-1})).$$

The clause $y \in \mathrm{Ord}$ is redundant since $x \subseteq \mathrm{Ord}$:

$$x \in \mathrm{Ord} \wedge \neg \varphi(x, x_0, \ldots, x_{n-1}) \wedge \forall y (y \in x \rightarrow \varphi(y, x_0, \ldots, x_{n-1})).$$

By the inductivity of φ the right-hand clause implies $\varphi(x, x_0, \ldots, x_{n-1})$ and so

$$x \in \mathrm{Ord} \wedge \neg \varphi(x, x_0, \ldots, x_{n-1}) \wedge \varphi(x, x_0, \ldots, x_{n-1}).$$

Contradiction. □

Induction can be formulated in various forms:

Exercise 15. Prove the following transfinite induction principle: Let $\varphi(x) = \varphi(x, v_0, \ldots, v_{n-1})$ be an $\in$-formula and $x_0, \ldots, x_{n-1} \in V$. Assume

a) $\varphi(0)$ (the initial case),

b) $\forall\alpha(\varphi(\alpha)\rightarrow\varphi(\alpha+1))$ (the successor step),

c) $\forall\lambda\in\mathrm{Lim}\ (\forall\alpha<\lambda\,\varphi(\alpha)\rightarrow\varphi(\lambda))$ (the limit step).

Then $\forall\alpha\,\varphi(\alpha)$.

3.2 Natural Numbers

We have 0, 1, ... $\in$ Ord. We shall now define and study the set of *natural numbers/integers*. Recall the axiom of infinity:

$$\exists x\,(0\in x\wedge\forall u\in x\; u+1\in x).$$

The set of natural numbers should be the $\subseteq$-smallest such x.

Definition 45. *Let* $\omega=\bigcap\ \{x|0\in x\wedge\forall u\in x\; u+1\in x\}$ *be the* set of natural numbers. *Sometimes we write* $\mathbb{N}$ *instead of* ω.

Theorem 46.

a) $\omega\in V$.

b) $\omega\subseteq\mathrm{Ord}$.

c) $(\omega,0,+1)$ *satisfy the* second order PEANO axiom*, i.e.,*

$$\forall x\subseteq\omega\,(0\in x\wedge\forall n\in x\; n+1\in x\rightarrow x=\omega).$$

d) $\omega\in\mathrm{Ord}$.

e) ω *is a limit ordinal.*

Proof. a) By the axiom of infinity take a set x_0 such that

$$0\in x_0\wedge\forall u\in x_0\; u+1\in x_0\,.$$

Then

$$\omega=\bigcap\ \{x|0\in x\wedge\forall u\in x\; u+1\in x\}=x_0\cap\bigcap\ \{x|0\in x\wedge\forall u\in x\; u+1\in x\}\in V$$

by the separation schema.
b) By a), $\omega\cap\mathrm{Ord}\in V$. Obviously $0\in\omega\cap\mathrm{Ord}\wedge\forall u\in\omega\cap\mathrm{Ord}\; u+1\in\omega\cap\mathrm{Ord}$. So $\omega\cap\mathrm{Ord}$ is one factor of the intersection in the definition of ω and so $\omega\subseteq\omega\cap\mathrm{Ord}$. Hence $\omega\subseteq\mathrm{Ord}$.
c) Let $x\subseteq\omega$ and $0\in x\wedge\forall u\in x\; u+1\in x$. Then x is one factor of the intersection in the definition of ω and so $\omega\subseteq x$. This implies $x=\omega$.
d) By b), every element of ω is transitive and it suffices to show that ω is transitive. Let

$$x=\{n|n\in\omega\wedge\forall m\in n\; m\in\omega\}\subseteq\omega.$$

We show that the hypothesis of c) holds for x. $0\in x$ is trivial. Let $u\in x$. Then $u+1\in\omega$. Let $m\in u+1$. If $m\in u$ then $m\in\omega$ by the assumption that $u\in x$. If $m=u$ then $m\in x\subseteq\omega$. Hence $u+1\in x$ and $\forall u\in x\; u+1\in x$. By b), $x=\omega$. So $\forall n\in\omega\, n\in x$, i.e.,

$$\forall n\in\omega\forall m\in n\; m\in\omega.$$

e) Of course $\omega\neq 0$. Assume for a contradiction that ω is a successor ordinal, say $\omega=\alpha+1$. Then $\alpha\in\omega$. Since ω is closed under the $+1$-operation, $\omega=\alpha+1\in\omega$. Contradiction. □

Thus the axiom of infinity implies the existence of the set of natural numbers, which is also the smallest limit ordinal. The axiom of infinity can now be reformulated equivalently as:

h) Inf: $\omega\in V$.

3.3 Recursion

Recursion, often called induction, over the natural numbers is a ubiquitous method for defining mathematical object. We prove the following *recursion theorem* for ordinals.

Theorem 47. *Let $G: V \to V$. Then there is a canonical class term F, given by the subsequent proof, such that*

$$F\colon \mathrm{Ord} \to V \text{ and } \forall\alpha\, F(\alpha) = G(F \upharpoonright \alpha).$$

We then say that F is defined recursively *(over the ordinals) by the* recursion rule *G. F is unique in the sense that if another term F' satisfies*

$$F'\colon \mathrm{Ord} \to V \text{ and } \forall\alpha\, F'(\alpha) = G(F' \upharpoonright \alpha)$$

then $F = F'$.

Proof. Let

$$\tilde{F} := \{f \mid \exists\delta \in \mathrm{Ord}\, (f\colon \delta \to V \text{ and } \forall\alpha < \delta\, f(\alpha) = G(f \upharpoonright \alpha))\}$$

be the class of all *approximations* to the desired function F. We show properties of $\tilde{F}$ using the induction theorem.
(1) Let $f, g \in \tilde{F}$. Then f, g are *compatible*, i.e., $\forall\alpha \in \mathrm{dom}(f) \cap \mathrm{dom}(g)\ f(\alpha) = g(\alpha)$.
Proof. We want to show that

$$\forall\alpha \in \mathrm{Ord}\ (\alpha \in \mathrm{dom}(f) \cap \mathrm{dom}(g) \to f(\alpha) = g(\alpha)).$$

By the induction theorem it suffices to show that $\alpha \in \mathrm{dom}(f) \cap \mathrm{dom}(g) \to f(\alpha) = g(\alpha)$ is inductive, i.e.,

$$\forall\alpha \in \mathrm{Ord}\, (\forall y \in \alpha\ (y \in \mathrm{dom}(f) \cap \mathrm{dom}(g) \to f(y) = g(y)) \to (\alpha \in \mathrm{dom}(f) \cap \mathrm{dom}(g) \to f(\alpha) = g(\alpha))).$$

So let $\alpha \in \mathrm{Ord}$ and $\forall y \in \alpha\ (y \in \mathrm{dom}(f) \cap \mathrm{dom}(g) \to f(y) = g(y))$. Let $\alpha \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$. Since $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$ are ordinals, $\alpha \subseteq \mathrm{dom}(f)$ and $\alpha \subseteq \mathrm{dom}(g)$. By assumption

$$\forall y \in \alpha\ f(y) = g(y).$$

Hence $f \upharpoonright \alpha = g \upharpoonright \beta$. Then

$$f(\alpha) = G(f \upharpoonright \alpha) = G(g \upharpoonright \alpha) = g(\alpha).$$

qed(1)

By the compatibility of the approximation functions the union

$$F = \bigcup \tilde{F}$$

is a function defined on a subclass of the ordinals. We show that F satisfies the recursion rule G where F is defined:
(2) $\forall\alpha \in \mathrm{dom}(F)\, (\alpha \subseteq \mathrm{dom}(F) \wedge F(\alpha) = G(F \upharpoonright \alpha))$.
Proof. Let $\alpha \in \mathrm{dom}(F)$. Take some approximation $f \in \tilde{F}$ such that $\alpha \in \mathrm{dom}(f)$. Since $\mathrm{dom}(f)$ is an ordinal and transitive, we have

$$\alpha \subseteq \mathrm{dom}(f) \subseteq \mathrm{dom}(F).$$

Moreover

$$F(\alpha) = f(\alpha) = G(f \upharpoonright \alpha) = G(F \upharpoonright \alpha).$$

qed(2)
(3) $\forall\alpha\ \alpha \in \mathrm{dom}(F)$.
Proof. By induction on the ordinals. We have to show that $\alpha \in \mathrm{dom}(F)$ is inductive in the variable α. So let $\alpha \in \mathrm{Ord}$ and $\forall y \in \alpha\ y \in \mathrm{dom}(F)$. Hence $\alpha \subseteq \mathrm{dom}(F)$. Let

$$f = F \upharpoonright \alpha \cup \{(\alpha, G(F \upharpoonright \alpha))\}.$$

f is a function with $\mathrm{dom}(f) = \alpha + 1 \in \mathrm{Ord}$. Let $\alpha' < \alpha + 1$. If $\alpha' < \alpha$ then

$$f(\alpha') = F(\alpha') = G(F \upharpoonright \alpha') = G(f \upharpoonright \alpha').$$

if $\alpha' = \alpha$ then also

$$f(\alpha') = f(\alpha) = G(F \upharpoonright \alpha) = G(f \upharpoonright \alpha) = G(f \upharpoonright \alpha').$$

Hence $f \in \tilde{F}$ and $\alpha \in \mathrm{dom}(f) \subseteq \mathrm{dom}(F)$. *qed*(3)

Now if F': Ord $\rightarrow V$ also satisfies the *recursion equation* $\forall\alpha\, F'(\alpha) = G(F' \upharpoonright \alpha)$ then $F = F'$ can be proved just like (1). □

Theorem 48. *Let* $a_0 \in V$, G_{succ}: Ord $\times V \rightarrow V$, *and* $G_{\lim}$: Ord $\times V \rightarrow V$. *Then there is a canonically defined class term* F: Ord $\rightarrow V$ *such that*

a) $F(0) = a_0$;

b) $\forall\alpha\, F(\alpha + 1) = G_{\text{succ}}(\alpha, F(\alpha))$;

c) $\forall\lambda \in \text{Lim}\ F(\lambda) = G_{\lim}(\lambda, F \upharpoonright \lambda)$.

Again F *is unique in the sense that if some* F' *also satisfies a)-c) then* $F = F'$.

We say that F *is* recursively defined *by the properties a)-c).*

Proof. We incorporate a_0, G_{succ}, and $G_{\lim}$ into a single recursion rule $G: V \rightarrow V$,

$$G(f) = \begin{cases} a_0 \text{ , if } f = \emptyset, \\ G_{\text{succ}}(\alpha, f(\alpha)) \text{ , if } f: \alpha + 1 \rightarrow V, \\ G_{\lim}(\lambda, f) \text{ , if } f: \lambda \rightarrow V \text{ and } \text{Lim}(\lambda), \\ \emptyset \text{ , else.} \end{cases}$$

Then the term F: Ord $\rightarrow V$ defined recursively by the recursion rule G satisfies the theorem. □

In many cases, the *limit rule* will just require to form the union of the previous values so that

$$F(\lambda) = \bigcup_{\alpha < \lambda} F(\alpha).$$

Such recursions are called *continuous* (at limits).

3.4 Ordinal Arithmetic

We extend the recursion rules of standard integer arithmetic continously to obtain transfinite version of the arithmetic operations. The initial operation of ordinal arithmetic is the $+1$-operation defined before. Ordinal arithmetic satisfies some but not all laws of integer arithmetic.

Definition 49. *Define* ordinal addition $+$: Ord $\times$ Ord $\rightarrow$ Ord *recursively by*

$$\begin{aligned} \delta + 0 &= \delta \\ \delta + (\alpha + 1) &= (\delta + \alpha) + 1 \\ \delta + \lambda &= \bigcup_{\alpha < \lambda} (\delta + \alpha) \text{ , \textit{for limit ordinals} } \lambda \end{aligned}$$

Definition 50. *Define* ordinal multiplication $\cdot$: Ord $\times$ Ord $\rightarrow$ Ord *recursively by*

$$\begin{aligned} \delta \cdot 0 &= 0 \\ \delta \cdot (\alpha + 1) &= (\delta \cdot \alpha) + \delta \\ \delta \cdot \lambda &= \bigcup_{\alpha < \lambda} (\delta \cdot \alpha) \text{ , \textit{for limit ordinals} } \lambda \end{aligned}$$

Definition 51. *Define* ordinal exponentiation $\_^{-}$: Ord $\times$ Ord $\rightarrow$ Ord *recursively by*

$$\begin{aligned} \delta^0 &= 1 \\ \delta^{\alpha+1} &= \delta^\alpha \cdot \delta \\ \delta^\lambda &= \bigcup_{\alpha < \lambda} \delta^\alpha \text{ , \textit{for limit ordinals} } \lambda \end{aligned}$$

Exercise 16. Explore which of the standard *ring axioms* hold for the ordinals with addition and multiplication. Give proofs and counterexamples.

Exercise 17. Show that for any ordinal α , $\alpha + \omega$ is a limit ordinal. Use this to show that the class Lim of all limit ordinals is a proper class.

4 Number Systems

We are now able to give set-theoretic formalizations of the standard number systems with their arithmetic operations.

4.1 Natural Numbers

Definition 52. *The structure*

$$\mathbb{N} := (\omega, + \restriction (\omega \times \omega), \cdot \restriction (\omega \times \omega), < \restriction (\omega \times \omega), 0, 1)$$

is called the structure of natural numbers*, or* arithmetic*. We sometimes denote this structure by*

$$\mathbb{N} := (\omega, +, \cdot, <, 0, 1).$$

$\mathbb{N}$ is an adequate formalization of arithmetic within set theory since $\mathbb{N}$ satisfies all standard arithmetical axioms.

Exercise 18. Prove:

a) $+[\omega \times \omega] := \{m + n | m \in \omega \wedge n \in \omega\} \subseteq \omega$.

b) $\cdot [\omega \times \omega] := \{m \cdot n | m \in \omega \wedge n \in \omega\} \subseteq \omega$.

c) Addition and multiplication are commutative on ω .

d) Addition and multiplication satisfy the usual monotonicity laws with respect to $<$.

Definition 53. *We define the structure*

$$\mathbb{Z} := (\mathbb{Z}, +^{\mathbb{Z}}, \cdot^{\mathbb{Z}}, <^{\mathbb{Z}}, 0^{\mathbb{Z}}, 1^{\mathbb{Z}})$$

of integers *as follows:*

a) Define an equivalence relation $\approx$ *on* $\mathbb{N} \times \mathbb{N}$ *by*

$$(a, b) \approx (a', b') \text{ iff } a + b' = a' + b.$$

b) Let $a - b := [(a, b)]_{\approx}$ *be the equivalence class of* (a, b) *in* $\approx$ *. Note that every* $a - b$ *is a set.*

c) Let $\mathbb{Z} := \{a - b | a \in \mathbb{N} \wedge b \in \mathbb{N}\}$ *be the set of* integers*.*

d) Define the integer addition $+^{\mathbb{Z}} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ *by*

$$(a - b) +^{\mathbb{Z}} (a' - b') := (a + a') - (b + b').$$

e) Define the integer multiplication $\cdot^{\mathbb{Z}} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ *by*

$$(a - b) \cdot^{\mathbb{Z}} (a' - b') := (a \cdot a' + b \cdot b') - (a \cdot b' + a' \cdot b).$$

f) Define the strict linear order $<^{\mathbb{Z}}$ *on* $\mathbb{Z}$ *by*

$$(a - b) <^{\mathbb{Z}} (a' - b') \text{ iff } a + b' < a' + b.$$

g) Let $0^{\mathbb{Z}} := 0 - 0$ *and* $1^{\mathbb{Z}} := 1 - 0$.

Exercise 19. Check that the above definitions are *sound*, i.e., that they do not depend on the choice of representatives of equivalence classes.

Exercise 20. Check that $\mathbb{Z}$ satisfies (a sufficient number) of the standard axioms for rings.

The structure $\mathbb{Z}$ extends the structure $\mathbb{N}$ in a natural and familiar way: define an injective map $e: \mathbb{N} \rightarrow \mathbb{Z}$ by

$$n \mapsto n - 0.$$

The embedding e is a *homomorphism*:

a) $e(0) = 0 - 0 = 0^{\mathbb{Z}}$ and $e(1) = 1 - 0 = 1^{\mathbb{Z}}$;

b) $e(m+n) = (m+n) - 0 = (m+n) - (0+0) = (m-0) +^{\mathbb{Z}} (n-0) = e(m) +^{\mathbb{Z}} e(n)$;

c) $e(m \cdot n) = (m \cdot n) - 0 = (m \cdot n + 0 \cdot 0) - (m \cdot 0 + n \cdot 0) = (m-0) \cdot^{\mathbb{Z}} (n-0) = e(m) \cdot^{\mathbb{Z}} e(n)$;

d) $m < n \leftrightarrow m + 0 < n + 0 \leftrightarrow (m-0) <^{\mathbb{Z}} (n-0) \leftrightarrow e(m) <^{\mathbb{Z}} e(n)$.

By this injective homomorphism, one may consider $\mathbb{N}$ as a *substructure* of $\mathbb{Z}$: $\mathbb{N} \subseteq \mathbb{Z}$.

4.2 Rational Numbers

Definition 54. *We define the structure*

$$\mathbb{Q}_0^+ := (\mathbb{Z}, +^{\mathbb{Q}}, \cdot^{\mathbb{Q}}, <^{\mathbb{Q}}, 0^{\mathbb{Q}}, 1^{\mathbb{Q}})$$

of non-negative rational numbers *as follows:*

a) Define an equivalence relation $\backsimeq$ *on* $\mathbb{N} \times (\mathbb{N} \setminus \{0\})$ *by*

$$(a, b) \backsimeq (a', b') \text{ iff } a \cdot b' = a' \cdot b.$$

b) Let $\frac{a}{b} := [(a, b)]_{\backsimeq}$ *be the equivalence class of* (a, b) *in* $\backsimeq$. *Note that* $\frac{a}{b}$ *is a set.*

c) Let $\mathbb{Q}_0^+ := \{\frac{a}{b} | a \in \mathbb{N} \wedge b \in (\mathbb{N} \setminus \{0\})\}$ *be the set of* non-negative rationals.

d) Define the rational addition $+^{\mathbb{Q}}: \mathbb{Q}_0^+ \times \mathbb{Q}_0^+ \rightarrow \mathbb{Q}_0^+$ *by*

$$\frac{a}{b} +^{\mathbb{Q}} \frac{a'}{b'} := \frac{a \cdot b' + a' \cdot b}{b \cdot b'}.$$

e) Define the rational multiplication $\cdot^{\mathbb{Q}}: \mathbb{Q}_0^+ \times \mathbb{Q}_0^+ \rightarrow \mathbb{Q}_0^+$ *by*

$$\frac{a}{b} \cdot^{\mathbb{Q}} \frac{a'}{b'} := \frac{a \cdot a'}{b \cdot b'}.$$

f) Define the strict linear order $<^{\mathbb{Q}}$ *on* $\mathbb{Q}_0^+$ *by*

$$\frac{a}{b} <^{\mathbb{Q}} \frac{a'}{b'} \text{ iff } a \cdot b' < a' \cdot b.$$

g) Let $0^{\mathbb{Q}} := \frac{0}{1}$ *and* $1^{\mathbb{Q}} := \frac{1}{1}$.

Again one can check the soundness of the definitions and the well-known laws of standard non-negative rational numbers. Also one may assume $\mathbb{N}$ to be embedded into $\mathbb{Q}_0^+$ as a substructure. The transfer from non-negative to *all* rationals, including negative rationals can be performed in analogy to the transfer from $\mathbb{N}$ to $\mathbb{Z}$.

Definition 55. *We define the structure*

$$\mathbb{Q} := (\mathbb{Q}, +^{\mathbb{Q}}, \cdot^{\mathbb{Q}}, <^{\mathbb{Q}}, 0^{\mathbb{Q}}, 1^{\mathbb{Q}})$$

of rational numbers *as follows:*

a) Define an equivalence relation $\approx$ *on* $\mathbb{Q}_0^+ \times \mathbb{Q}_0^+$ *by*

$$(p, q) \approx (p', q') \text{ iff } p + q' = p' + q.$$

b) Let $p - q := [(p, q)]_{\approx}$ *be the equivalence class of* (p, q) *in* $\approx$.

c) Let $\mathbb{Q} := \{p - q | p \in \mathbb{Q}_0^+ \wedge p \in \mathbb{Q}_0^+\}$ *be the set of* rationals.

Exercise 21. Continue the definition of the structure $\mathbb{Q}$ and prove the relevant properties.

4.3 Real Numbers

Definition 56. $r \subseteq \mathbb{Q}_0^+$ *is a* positive real number *if*

a) $\forall p \in r \forall q \in \mathbb{Q}_0^+ (q <^{\mathbb{Q}} p \rightarrow q \in r)$, *i.e., r is an initial segment of* $(\mathbb{Q}_0^+, <^{\mathbb{Q}})$;

b) $\forall p \in r \exists q \in r\, p <^{\mathbb{Q}} q$, *i.e., r is right-open in* $(\mathbb{Q}_0^+, <^{\mathbb{Q}})$;

c) $0 \in r \neq \mathbb{Q}_0^+$, *i.e., r is nonempty and bounded in* $(\mathbb{Q}_0^+, <^{\mathbb{Q}})$.

Definition 57. *We define the structure*

$$\mathbb{R}^+ := (\mathbb{R}^+, +^{\mathbb{R}}, \cdot^{\mathbb{R}}, <^{\mathbb{R}}, 1^{\mathbb{R}})$$

of positive real numbers *as follows:*

a) *Let* $\mathbb{R}^+$ *be the set of* positive reals.

b) *Define the* real addition $+^{\mathbb{R}} : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ *by*

$$r +^{\mathbb{R}} r' = \{p +^{\mathbb{Q}} p' | p \in r \wedge p' \in r'\}.$$

c) *Define the* real multiplication $\cdot^{\mathbb{R}} : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ *by*

$$r \cdot^{\mathbb{R}} r' = \{p \cdot^{\mathbb{Q}} p' | p \in r \wedge p' \in r'\}.$$

d) *Define the strict linear order* $<^{\mathbb{R}}$ *on* $\mathbb{R}^+$ *by*

$$r <^{\mathbb{R}} r' \text{ iff } r \subseteq r' \wedge r \neq r'.$$

e) *Let* $1^{\mathbb{R}} := \{p \in \mathbb{Q}_0^+ | q <^{\mathbb{Q}} 1\}$.

We justify some details of the definition.

Lemma 58.

a) $\mathbb{R}^+ \in V$.

b) *If* $r, r' \in \mathbb{R}^+$ *then* $r +^{\mathbb{R}} r'$, $r \cdot^{\mathbb{R}} r' \in \mathbb{R}^+$.

c) $<^{\mathbb{R}}$ *is a strict linear order on* $\mathbb{R}^+$.

Proof. a) If $r \in \mathbb{R}^+$ then $r \subseteq \mathbb{Q}_0^+$ and $r \in \mathcal{P}(\mathbb{Q}_0^+)$. Thus $\mathbb{R}^+ \subseteq \mathcal{P}(\mathbb{Q}_0^+)$, and $\mathbb{R}^+$ is a set by the power set axiom and separation.
b) Let $r, r' \in \mathbb{R}^+$. We show that

$$r \cdot^{\mathbb{R}} r' = \{p \cdot^{\mathbb{Q}} p' | p \in r \wedge p' \in r'\} \in \mathbb{R}^+.$$

Obviously $r \cdot^{\mathbb{R}} r' \subseteq \mathbb{Q}_0^+$ is a non-empty bounded initial segment of $(\mathbb{Q}_0^+, <^{\mathbb{Q}})$.

Consider $p \in r \cdot^{\mathbb{R}} r'$, $q \in \mathbb{Q}_0^+$, $q <^{\mathbb{Q}} p$. Let $p = \frac{a}{b} \cdot^{\mathbb{Q}} \frac{a'}{b'}$ where $\frac{a}{b} \in r$ and $\frac{a'}{b'} \in r'$. Let $q = \frac{c}{d}$. Then $\frac{c}{d} = \frac{c \cdot b'}{d \cdot a'} \cdot^{\mathbb{Q}} \frac{a'}{b'}$, where

$$\frac{c \cdot b'}{d \cdot a'} = q \cdot^{\mathbb{Q}} \frac{b'}{a'} <^{\mathbb{Q}} p \cdot^{\mathbb{Q}} \frac{b'}{a'} = \frac{a}{b} \cdot^{\mathbb{Q}} \frac{a'}{b'} \cdot^{\mathbb{Q}} \frac{b'}{a'} = \frac{a}{b} \in r.$$

Hence $\frac{c \cdot b'}{d \cdot a'} \in r$ and

$$\frac{c}{d} = \frac{c \cdot b'}{d \cdot a'} \cdot^{\mathbb{Q}} \frac{a'}{b'} \in r \cdot^{\mathbb{R}} r'.$$

Similarly one can show that $r \cdot^{\mathbb{R}} r'$ is open on the right-hand side.
c) The transitivity of $<^{\mathbb{R}}$ follows from the transitivity of the relation $\subsetneqq$. To show that $<^{\mathbb{R}}$ is connex, consider $r, r' \in \mathbb{R}^+$, $r \neq r'$. Then r and r' are different subsets of $\mathbb{Q}_0^+$. Without loss of generality we may assume that there is some $p \in r' \setminus r$. We show that then $r <^{\mathbb{R}} r'$, i.e., $r \subsetneqq r'$. Consider $q \in r$. Since $p \notin r$ we have $p \not<^{\mathbb{Q}} q$ and $q \leqslant^{\mathbb{Q}} p$. Since r' is an initial segment of $\mathbb{Q}_0^+$, $q \in r'$. $\square$

Exercise 22. Show that $(\mathbb{R}^+, \cdot^{\mathbb{R}}, 1^{\mathbb{R}})$ is a multiplicative group.

We can now construct the complete real line $\mathbb{R}$ from $\mathbb{R}^+$ just like we constructed $\mathbb{Z}$ from $\mathbb{N}$. Details are left to the reader. We can further assume that after some manipulations, the number systems form an ascending chain

$$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

Exercise 23. Formalize the structure $\mathbb{C}$ of complex numbers such that $\mathbb{R} \subseteq \mathbb{C}$.

Remark 59. In set theory the set $\mathbb{R}$ of reals is often identified with the sets ${}^{\omega}\omega$ or ${}^{\omega}2$, basically because all these sets have the same cardinality. We shall come back to this in the context of cardinality theory.

5 Sequences

The notion of a *sequence* is crucial in many contexts.

Definition 60.

a) *A set w is an α-*sequence *iff $w: \alpha \to V$; then α is called the* length *of the α-sequence w and is denoted by $|\alpha|$. w is a* sequence *iff it is an α-sequence for some α. A sequence w is called* finite *iff $|w| < \omega$.*

b) *A finite sequence $w: n \to V$ may be denoted by its enumeration $w_0, \ldots, w_{n-1}$ where we write w_i instead of $w(i)$. One also writes $w_0 \ldots w_{n-1}$ instead of $w_0, \ldots, w_{n-1}$, in particular if w is considered to be a* word *formed out of the* symbols *$w_0, \ldots, w_{n-1}$.*

c) *An ω-sequence $w: \omega \to V$ may be denoted by $w_0, w_1, \ldots$ where $w_0, w_1, \ldots$ suggests a definition of w.*

d) *Let $w: \alpha \to V$ and $w': \alpha' \to V$ be sequences. Then the* concatenation *$w^\frown w': \alpha + \alpha' \to V$ is defined by*

$$(w^\frown w') \upharpoonright \alpha = w \upharpoonright \alpha \text{ and } \forall i < \alpha' \, w^\frown w'(\alpha + i) = w'(i).$$

e) *Let $w: \alpha \to V$ and $x \in V$. Then the* adjunction *wx of w by x is defined as*

$$wx = w^\frown \{(0, x)\}.$$

Sequences and the concatenation operation satisfy the algebraic laws of a *monoid* with cancellation rules.

Proposition 61. *Let w, w', w'' be sequences. Then*

a) $(w^\frown w')^\frown w'' = w^\frown (w'^\frown w'')$.

b) $\emptyset^\frown w = w^\frown \emptyset = w$.

c) $w^\frown w' = w^\frown w'' \to w' = w''$.

There are many other operations on sequences. One can *permute* sequences, substitute elements of a sequence, etc.

5.1 (ω-)Sequences of Reals

ω-sequences are particularly prominent in analysis. One may now define properties like

$$\lim_{i \to \infty} w_i = z \text{ iff } \forall \varepsilon \in \mathbb{R}^+ \exists m < \omega \forall i < \omega \, (i \geqslant m \to (z - \varepsilon < w_i \wedge w_i < z + \varepsilon))$$

or

$$\forall x: \omega \to \mathbb{R} (\lim_{i \to \infty} x_i = a \to \lim_{i \to \infty} f(x_i) = f(a)).$$

If $x_0, x_1, \ldots$ is given then the partial sums

$$\sum_{i=0}^{n} x_i$$

are defined recursively as

$$\sum_{i=0}^{0} x_i = 0 \quad \text{and} \quad \sum_{i=0}^{n+1} x_i = (\sum_{i=0}^{n} x_i) + x_n\,.$$

The map $\varphi: {}^{\omega}2 \to \mathbb{R}$ defined by

$$\varphi((x_i)_{i<\omega}) = \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}} = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{x_i}{2^{i+1}}\,.$$

maps the function space ${}^{\omega}2$ surjectively onto the real interval

$$[0,1] = \{r \in \mathbb{R} \mid 0 \leqslant r \leqslant 1\}.$$

Such maps are the reason that one often identifies ${}^{\omega}2$ with $\mathbb{R}$ in set theory.

5.2 Symbols and Words

Languages are mathematical objects of growing importance. Mathematical logic takes terms and formulas as mathematical material. Terms and formulas are finite sequences of symbols from some alphabet. We represent the standard symbols $=$, $\in$, etc. by some set-theoretical terms $\dot{=}$, $\dot{\in}$, etc. Note that details of such a formalization are highly arbitrary.

Definition 62. *Formalize the* basic set-theoretical symbols *by*

a) $\dot{=} = 0,\ \dot{\in} = 1,\ \dot{\wedge} = 2,\ \dot{\vee} = 3,\ \dot{\rightarrow} = 4,\ \dot{\leftrightarrow} = 5,\ \dot{\neg} = 6,\ \dot{(} = 7,\ \dot{)} = 8,\ \dot{\exists} = 9,\ \dot{\forall} = 10.$

b) *Variables* $\dot{v}_n = (1,n)$ *for* $n<\omega$.

c) *Let* $L_{\in} = \{\dot{=}, \dot{\in}, \dot{\wedge}, \dot{\vee}, \dot{\rightarrow}, \dot{\leftrightarrow}, \dot{\neg}, \dot{(}, \dot{)}, \dot{\exists}, \dot{\forall}\} \cup \{(1,n) | n<\omega\}$ *be the* alphabet of set theory.

d) *A* word *over* $L_{\in}$ *is a finite sequence with values in* $L_{\in}$.

e) *Let* $L_{\in}^{*} = \{w \mid \exists n<\omega\ w: n \to L_{\in}\}$ *be the set of all words over* $L_{\in}$.

f) *If* φ *is a standard set-theoretical formula, we let* $\dot{\varphi} \in L_{\in}^{*}$ *denote the formalization of* φ. *E.g.,* $\dot{\mathrm{Ex}} = \dot{\exists}\dot{v}_0\dot{\forall}\dot{v}_1\dot{\neg}\dot{v}_1\dot{\in}\dot{v}_0$ *is the formalization of the set existence axiom. If the intention is clear, one often omits the formalization dots and simply writes* $\dot{\mathrm{Ex}} = \exists v_0 \forall v_1 \neg v_1 \in v_0$.

6 The von Neumann Hierarchy

We use ordinal recursion to obtain more information on the universe of all sets.

Definition 63. *Define the* von Neumann Hierarchy $(V_\alpha)_{\alpha\in\mathrm{Ord}}$ *by recursion:*

a) $V_0 = \emptyset$;

b) $V_{\alpha+1} = \mathcal{P}(V_\alpha)$;

c) $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ *for limit ordinals* λ.

We show that the von Neumann hierarchy is indeed a (fast-growing) hierarchy

Lemma 64. *Let* $\beta < \alpha \in \mathrm{Ord}$. *Then*

a) $V_\beta \in V_\alpha$

b) $V_\beta \subseteq V_\alpha$

c) V_α *is transitive*

Proof. We conduct the proof by a simultaneous induction on α.
$\alpha = 0$: $\emptyset$ is transitive, thus a)-c) hold at 0.
For the *successor case* assume that a)-c) hold at α. Let $\beta < \alpha + 1$. By the inductive assumption, $V_\beta \subseteq V_\alpha$ and $V_\beta \in \mathcal{P}(V_\alpha) = V_{\alpha+1}$. Thus a) holds at $\alpha + 1$. Consider $x \in V_\alpha$. By the inductive assumption, $x \subseteq V_\alpha$ and $x \in V_{\alpha+1}$. Thus $V_\alpha \subseteq V_{\alpha+1}$. Then b) at $\alpha + 1$ follows by the inductive assumption. Now consider $x \in V_{\alpha+1} = \mathcal{P}(V_\alpha)$. Then $x \subseteq V_\alpha \subseteq V_{\alpha+1}$ and $V_{\alpha+1}$ is transitive.
For the *limit case* assume that α is a limit ordinal and that a)-c) hold at all $\gamma < \alpha$. Let $\beta < \alpha$. Then $V_\beta \in V_{\beta+1} \subseteq \bigcup_{\gamma<\alpha} V_\gamma = V_\alpha$ hence a) holds at α. b) is trivial for limit α. V_α is transitive as a union of transitive sets. □

The V_α are nicely related to the ordinal α.

Lemma 65. *For every* α, $V_\alpha \cap \mathrm{Ord} = \alpha$.

Proof. Induction on α. $V_0 \cap \mathrm{Ord} = \emptyset \cap \mathrm{Ord} = \emptyset = 0$.
For the *successor case* assume that $V_\alpha \cap \mathrm{Ord} = \alpha$. $V_{\alpha+1} \cap \mathrm{Ord}$ is transitive, and every element of $V_{\alpha+1} \cap \mathrm{Ord}$ is transitive. Hence $V_{\alpha+1} \cap \mathrm{Ord}$ is an ordinal, say $\delta = V_{\alpha+1} \cap \mathrm{Ord}$. $\alpha = V_\alpha \cap \mathrm{Ord}$ implies that $\alpha \in V_{\alpha+1} \cap \mathrm{Ord} = \delta$ and $\alpha + 1 \leqslant \delta$. Assume for a contradiction that $\alpha + 1 < \delta$. Then $\alpha + 1 \in V_{\alpha+1}$ and $\alpha + 1 \subseteq V_\alpha \cap \mathrm{Ord} = \alpha$, contradiction. Thus $\alpha + 1 = \delta = V_{\alpha+1} \cap \mathrm{Ord}$.
For the *limit case* assume that α is a limit ordinal and that $V_\beta \cap \mathrm{Ord} = \beta$ holds for all $\beta < \alpha$. Then

$$V_\alpha \cap \mathrm{Ord} = (\bigcup_{\beta<\alpha} V_\beta) \cap \mathrm{Ord} = \bigcup_{\beta<\alpha} (V_\beta \cap \mathrm{Ord}) = \bigcup_{\beta<\alpha} \beta = \alpha.$$

□

The foundation schema implies that the V_α-hierarchy exhausts the universe V.

Theorem 66.

a) $\forall x \subseteq \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha \, \exists \beta \, x \subseteq V_\beta$.

b) $V = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$.

Proof. a) Let $x \subseteq \bigcup_{\alpha \in \mathrm{Ord}}$. Define a function $f: x \to \mathrm{Ord}$ by

$$f(u) = \min \{\gamma \,|\, u \in V_\gamma\}.$$

By the axioms of replacement and union, $\beta = \bigcup \{f(u) + 1 | u \in x\} \in V$ and $\beta \in \mathrm{Ord}$. Let $u \in x$. Then $f(u) < f(u) + 1 \leqslant \beta$ and $u \in V_{f(u)} \subseteq V_\beta$. Thus $x \subseteq V_\beta$.
b) Let $B = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$. By the schema of $\in$-induction it suffices to show that

$$\forall x (x \subseteq B \rightarrow x \in B).$$

So let $x \subseteq B = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$. By a) take β such that $x \subseteq V_\beta$. Then $x \in V_{\beta+1} \subseteq \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha = B$. □

The V_α-hierarchy ranks the elements of V into levels.

Definition 67. *Define the* rank (function) $\mathrm{rk}: V \to \mathrm{Ord}$ *by*

$$x \in V_{\mathrm{rk}(x)+1} \setminus V_{\mathrm{rk}(x)}.$$

The rank function satisfies a recursive law.

Lemma 68. $\forall x \ \mathrm{rk}(x) = \bigcup_{y \in x} \mathrm{rk}(y) + 1$.

Proof. Let us prove the statement

$$\forall x \in V_\alpha \ \mathrm{rk}(x) = \bigcup_{y \in x} \mathrm{rk}(y) + 1$$

by induction on α. The case $\alpha = 0$ is trivial. The limit case is obvious since $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ for limit λ.

For the successor case assume that the statement holds for α . Consider $x \in V_{\alpha+1}$. If $x \in V_\alpha$ the statement holds by the inductive assumption. So assume that $x \in V_{\alpha+1} \setminus V_\alpha$. Then $\mathrm{rk}(x) = \alpha$. Let $y \in x \subseteq V_\alpha$. Then $y \in V_{\beta+1} \setminus V_\beta$ for some $\beta = \mathrm{rk}(y) < \alpha$. $\mathrm{rk}(y) + 1 \subseteq \alpha$. Thus $\bigcup_{y \in x} \mathrm{rk}(y) + 1 \subseteq \alpha$. Assume that $\gamma = \bigcup_{y \in x} \mathrm{rk}(y) + 1 < \alpha$. Let $y \in x$. Then $\mathrm{rk}(y) + 1 \leqslant \gamma$ and $y \in V_{\mathrm{rk}(y)+1} \subseteq V_\gamma$. Thus $x \subseteq V_\gamma$, $x \in V_{\gamma+1} \subseteq V_\alpha$, contradicting the assumption that $x \in V_{\alpha+1} \setminus V_\alpha$. □

Lemma 69. *Let A be a term. Then $A \in V$ iff $\exists \alpha\, A \subseteq V_\alpha$.*

Our analysis of the V_α-hierarchy suggest the following picture of the universe V.

7 Induction and Recursion on Wellfounded Relations

The axiom schema of foundation yields an induction theorem for the $\in$-relation, and in the previous section we have seen a recursive law for the rank-function. We shall generalize these observations to *wellfounded* relations.

Definition 70. *Let R be a relation on a domain D.*

a) R is wellfounded, *iff for all terms A*

$$\emptyset \neq A \wedge A \subseteq D \rightarrow \exists x \in A\; A \cap \{y \mid y R x\} = \emptyset.$$

b) R is strongly wellfounded *iff it is wellfounded and*

$$\forall x \in D\; \{y \in D \mid y R x\} \in V.$$

c) R is a wellorder *iff R is a wellfounded strict linear order.*

d) R is a strong wellorder *iff R is a strongly wellfounded wellorder.*

By the scheme of foundation, the $\in$-relation is strongly wellfounded. The ordinals are strongly wellordered by $<$. There are wellfounded relations which are *not* strongly wellfounded: e.g., let $R \subseteq \mathrm{Ord} \times \mathrm{Ord}$,

$$x R y \text{ iff } (x \neq 0 \wedge y \neq 0 \wedge x < y) \vee (y = 0 \wedge x \neq 0),$$

be a rearrangement of $(\mathrm{Ord}, <)$ with 0 put on top of all the other ordinals.

For strongly wellfounded relations, every element is contained in a *set-sized* initial segment of the relation.

Lemma 71. *Let R be a strongly wellfounded relation on D. Then*

$$\forall x \subseteq D \exists z (z \subseteq D \wedge x \subseteq z \wedge \forall u \in z \forall v R u\; v \in z).$$

*Moreover for all $x \subseteq D$, the R-*transitive closure

$$\mathrm{TC}_R(x) = \bigcap \{z \mid z \subseteq D \wedge x \subseteq z \wedge \forall u \in z \forall v R u\; v \in z\}$$

of x is a set. In case R is the $\in$-relation, we write $\mathrm{TC}(x)$ instead of $\mathrm{TC}_\in(x)$.

Proof. We prove by R-induction that

$$\forall x \in D\; \mathrm{TC}_R(\{x\}) \in V.$$

So let $x \in D$ and $\forall y R x\; \mathrm{TC}_R(\{y\}) \in V$. Then

$$z = \{x\} \cup \bigcup_{y R x} \mathrm{TC}_R(\{y\}) \in V$$

by replacement. z is a subset of D and includes $\{x\}$. z is R-closed, i.e., closed with respect to R-predecessors: each $\mathrm{TC}_R(\{y\})$ is R-closed, and if $y R x$ then $y \in \{y\} \subseteq \mathrm{TC}_R(\{y\}) \subseteq z$. So $\mathrm{TC}_R(\{x\})$ is the intersection of a non-empty class, hence a set.

Finally observe that we may set

$$\mathrm{TC}_R(x) = \bigcup_{y \in x} \mathrm{TC}_R(\{y\}).$$

$\square$

Exercise 24. Show that for an ordinal α, $\mathrm{TC}(\alpha) = \alpha$ and $\mathrm{TC}(\{\alpha\}) = \alpha + 1$.

For *strongly* wellfounded relations, the following recursion theorem holds:

Theorem 72. *Let R be a strongly wellfounded relation on D . Let $G: V \to V$. Then there is a canonical class term F, given by the subsequent proof, such that*

$$F: D \to V \text{ and } \forall x \in D\, F(x) = G(F \upharpoonright \{y \mid y R x\}).$$

We then say that F is defined by R-recursion with the recursion rule *G. F is unique in the sense that if another term F' satisfies*

$$F': D \to V \text{ and } \forall \alpha \in D\, F'(x) = G(F' \upharpoonright \{y \mid y R x\})$$

then $F = F'$.

Proof. We proceed as in the ordinal recursion theorem. Let

$$\tilde{F} := \{f \mid \exists z \subseteq D\, (\forall x \in z\ \{y \mid y R x\} \subseteq z\,,\ f: z \to V \text{ and } \forall x \in z\ f(x) = G(f \upharpoonright \{y \mid y R x\}))\}$$

be the class of all *approximations* to the desired function F.

(1) Let $f, g \in \tilde{F}$. Then f, g are *compatible*, i.e., $\forall x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)\ f(x) = g(x)$.

Proof. By induction on R. Let $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ and assume that $\forall y R x\ f(y) = g(y)$. Then $f \upharpoonright \{y \mid y R x\} = g \upharpoonright \{y \mid y R x\}$

$$f(x) = G(f \upharpoonright \{y \mid y R x\}) = G(g \upharpoonright \{y \mid y R x\}) = g(x).$$

qed(1)

By the compatibility of the approximation functions the union

$$F = \bigcup \tilde{F}$$

is a function defined on $\mathrm{dom}(F) \subseteq D$. $\mathrm{dom}(F)$ is R-closed since the domain of every approximation is R-closed.

(2) $\forall x \in \mathrm{dom}(F)\, (\{y \mid y R x\} \subseteq \mathrm{dom}(F) \wedge F(x) = G(F \upharpoonright \{y \mid y R x\}))$.

Proof. Let $x \in \mathrm{dom}(F)$. Take some approximation $f \in \tilde{F}$ such that $x \in \mathrm{dom}(f)$. Then $\{y \mid y R x\} \subseteq \mathrm{dom}(f) \subseteq \mathrm{dom}(F)$ and

$$F(x) = f(x) = G(f \upharpoonright \{y \mid y R x\}) = G(F \upharpoonright \{y \mid y R x\}).$$

qed(2)

(3) $D = \mathrm{dom}(F)$.

Proof. We show by R-induction that $\forall x \in D\ x \in \mathrm{dom}(F)$. Let $x \in D$ and assume that $\forall y R x\ y \in \mathrm{dom}(F)$. $\mathrm{TC}_R(\{y \mid y R x\}) \subseteq \mathrm{dom}(F)$ since $\mathrm{dom}(F)$ is R-closed. Then

$$f = (F \upharpoonright \mathrm{TC}_R(\{y \mid y R x\})) \cup \{(x, G(F \upharpoonright \{y \mid y R x\}))\}$$

is an approximation with $x \in \mathrm{dom}(f)$, and so $x \in \mathrm{dom}(F)$. $\square$

Exercise 25. Define set theoretic operations

$$x + y = x \cup \{x + z | z \in y\}$$

and

$$x \cdot y = \bigcup_{z \in y} (x \cdot z + x)$$

and study their arithmetic/algebraic properties. Show that they extend ordinal arithmetic.

Theorem 73. *Let R be a strongly wellfounded relation on D and suppose that R is* extensional, *i.e., $\forall x, y \in D\, (\forall u\, (u R x \leftrightarrow u R y) \to x = y)$. Then there is a transitive class $\bar{D}$ and an isomorphism π: $(D, R) \leftrightarrow (\bar{D}, \in)$. $\bar{D}$ and π are uniquely determined by R and D, they are called the* MOSTOWSKI-*collapse of R and D.*

Proof. Define $\pi: D \rightarrow V$ by R-recursion with

$$\pi(x) = \{\pi(y) | y R x\}.$$

Let $\bar{D} = \operatorname{rng}(\pi)$.
(1) $\bar{D}$ is transitive.
Proof. Let $\pi(x) \in \bar{D}$ and $u \in \pi(x) = \{\pi(y) | y R x\}$. Let $u = \pi(y)$, yRx . Then $u \in \operatorname{rng}(\pi) = \bar{D}$. *qed*(1)
(2) π is injective.
Proof. We prove by $\in$-induction that every $z \in \bar{D}$ has exactly one preimage under π . So let $z \in \bar{D}$ and let this property be true for all elements of z . Assume that $x, y \in D$ and $\pi(x) = \pi(y) = z$. Let $u R x$. Then $\pi(u) \in \pi(x) = \pi(y) = \{\pi(v) | v R y\}$. Take $v R y$ such that $\pi(u) = \pi(v)$. By the inductive assumption, $u = v$, and $u R y$. Thus $\forall u (u R x \rightarrow u R y)$. By symmetry, $\forall u (u R x \leftrightarrow u R y)$. Since R is extensional, $x = y$. So z has exactly one preimage under π . *qed*(2)
(3) π is an *isomorphism*, i.e., π is bijective and $\forall x, y \in D (x R y \leftrightarrow \pi(x) \in \pi(y))$.
Proof. Let x, $y \in D$. If $x R y$ then $\pi(x) \in \{\pi(u) | u R y\} = \pi(y)$. Conversely, if $\pi(x) \in \{\pi(u) | u R y\} = \pi(y)$ then let $\pi(x) = \pi(u)$ for some $u R y$. Since π is injective, $x = u$ and $x R y$. *qed*(3)

Uniqueness of the collapse $\bar{D}$ and π is given by the next theorem. □

Theorem 74. *Let X and Y be transitive and let* $\sigma: X \leftrightarrow Y$ *be an* $\in$ - $\in$ -isomorphism *between X and Y, i.e.,* $\forall x, y \in X (x \in y \leftrightarrow \sigma(x) \in \sigma(y))$. *Then* $\sigma = \operatorname{id} \upharpoonright X$ *and* $X = Y$.

Proof. We show that $\sigma(x) = x$ by $\in$ -induction over X. Let $x \in X$ and assume that $\forall y \in x\ \sigma(y) = y$.

Let $y \in x$. By induction assumption, $y = \sigma(y) \in \sigma(x)$. Thus $x \subseteq \sigma(x)$.

Conversely, let $v \in \sigma(x)$. Since $Y = \operatorname{rng}(\sigma)$ is transitive take $u \in X$ such that $v = \sigma(u)$. Since σ is an isomorphism, $u \in x$. By induction assumption, $v = \sigma(u) = u \in x$. Thus $\sigma(x) \subseteq x$. □

8 The Axiom of Choice

9 Cardinalities