DEGREES OF BELIEF AND KNOWLEDGE IN MATHEMATICS

EXTENDED ABSTRACT FOR *DEGREES OF BELIEF*, KONSTANZ (22-24 JULY 2004)

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> "But a proof is sometimes a fuzzy concept,
> subject to whim and personality."
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> Kenneth Chang, **New York Times**
> *(April 6, 2004)*

1. Introduction

Belief is very often analysed in terms of degrees, and since most human beings experience belief as a matter of degree, this kind of analysis has strong support. Much more controversial is the claim that knowledge, too, may be a matter of degree. The two questions are linked—after all, knowledge is, according to the traditional analyses, a special kind of belief. Many philosophers have analysed knowledge as belief that does not allow doubt and consequently, no degrees either.

This view has been challenged by some authors, among them David Lewis, who gives a contextualist account of knowledge in [**Lew96**]:

"S *knows* that P iff S's evidence eliminates every possibility in which not-P—Psst!—except for those possibilities that we are properly ignoring." The option of "properly ignoring possibilities" yields a spectrum of knowledge contexts from the loose standards of every-day usage ("I know that [my cat] Possum is not in my study without checking the closed drawers") to the demanding standards of epistemology (Cartesian Doubt). The weaker my criteria for knowledge are (or, equivalently, the more possibilities I am allowed to be properly ignoring in the context), the lower my *degree of knowledge* is.

In this spectrum of knowledge contexts, where does mathematics feature? Mathematics is a deductive science in which the notion of proof plays a crucial rôle. In philosophical contexts, mathematics is often used as an epistemological rôle model: mathematical knowledge is assumed to be absolute. However, we shall argue in this paper that even mathematical knowledge comes in degrees. Given the special status of mathematics, this result may lend support to the idea that knowledge in other areas is also a matter of degree.[1]

It is important to be clear about our methodology. We shall subscribe to a moderately naturalistic position: we check the philosophical analysis of knowledge for coherence with actual usage of the term "knowledge" in scientific and everyday contexts. Concretely, we shall be discussing standard attempts to describe mathematical knowledge (without degrees) and show that they are incompatible with natural language usage of knowledge attributions.

2. Mathematical knowledge

2.1. The naïve view. Compared to the other sciences, mathematics is an "epistemic exception":[2] Whereas empirical claims have relative support through other empirical claims, mathematical claims admit of proof, and as we learn early on in our education, a mathematical proof is either correct or incorrect. A proven mathematical statement is beyond doubt in a way that even a well-supported empirical claim is not: The notion of proof sets an absolute standard of support for a mathematical claim; proofs do not admit degrees of correctness. To use Keith Devlin's polemic words: "Surely, any math teacher can tell in ten minutes whether a solution to a math problem is right or wrong! [...] Come on folks, it's a simple enough question. Is his math right or wrong?[3] [**Dev03**]" This view could be seen as the naïve received view of mathematical knowledge:

> If a person A believes that φ in the right way, *viz.*, if A has available a proof of φ, then necessarily A's belief in φ is of the firmest kind possible, and A's belief that φ has the status of knowledge.

This view is implicitly or explicitly shared by a large number of philosophers ancient and modern, ranging from the famous παῖς example in Plato's *Meno* to Kant's analysis of mathematical truths as

[1] While we argue that both knowledge and belief have degrees, these degrees needn't be related directly: I can believe very strongly in an utterly false φ, which should not count as a high degree of knowledge of φ under any account.

[2] This has been an important topic in the sociology of science, discussed by Mannheim, Bloor [**Blo76**] and Livingston [**Liv86**]. Cf. [**Hei00**, Kapitel 1], and [**Pre01**, pp. 24f.].

[3] Just for the record: Of course, Devlin is playing the *advocatus* here.

synthetic *a priori* and Frege's Logicism. Arguably even more important is the fact that the naïve received view is deeply entrenched in the image of mathematics in the educated public. In light of this view, mathematical knowledge would be analysed in the following way:

$(\star)$ $\qquad A$ knows that φ iff A has available a proof of φ.

2.2. Proof *vs* derivation. Obviously, a crucial term in this analysis of mathematical knowledge is the term 'proof'. The (meta)mathematical community, following the lead of Frege, distinguishes between an informal notion of proof (the kind of proof that one finds, *e.g.*, in mathematical journals and in textbooks) and a formal, mathematically well-defined notion of proof or—as we shall say to mark the distinction—*derivation*. While "derivation" has a mathematical definition (whose exact details depend on the proof system used but rarely matter for the philosophical analysis), there cannot be a formal definition of the notion of informal proof.

Now mathematical practice strongly supports the view that the important notion of proof in mathematics is not derivation, but informal proof. With few, very specialised exceptions,[4] you will not find derivations in mathematical publications. "The point of publishing a proof [...] is to communicate that proof to other mathematicians. [...] [T]he most efficient way [...] is not by laying out the entire sequence of propositions in excruciating detail [**Fal03**, p. 55]". Instead, mathematicians publish informal proofs, and it is not the case that they had a derivation in mind and transformed it into an informal proof for publication in order to reach a wider public; the entire procedure of doing research mathematics rests on doing informal proofs, and the proofs in mathematical research papers are so far removed from derivations that only a few experts could produce a derivation from them even if they wanted to (which they do not).

Informal proofs come in many flavours, from semi-formal textbook proofs for beginning students to highly informal research notes. Taking this into consideration, proof admits of degrees. Proofs may be more or less detailed, they may contain more or less gaps. If we wish to analyse mathematical knowledge by $(\star)$, we have two options. Either we accept that the degrees of proof induce degrees of mathematical knowledge, or we fix a level of detail and declare the corresponding type of proof as constitutive for mathematical knowledge via $(\star)$. In the next two

[4] *E.g.*, the *Journal of Formalized Mathematics*, which focuses on derivations in the specific proof system `MIZAR`; *cf.* `http://www.mizar.org/JFM/`.

sections we highlight the failure of standard attempts at making the latter option work.

2.3. The modal point of view. In the 20th century, philosophical accounts of mathematical knowledge centred on the notion of derivation. Reading "proof" in ($\star$) as "derivation" yields

($\star\star$) $\quad$ A knows that φ iff A has available a derivation of φ.

Interpreting ($\star\star$) literally, almost no mathematical knowledge exists in the world. Probably no living mathematician has seen a derivation of the Feit-Thompson Theorem, yet there are (many) mathematicians who know that every group of odd order is solvable.[5]

Several authors try to circumvent this conclusion by subscribing to what we call the "modal point of view".

(‡) $\quad$ A knows that φ iff
A could in principle generate a derivation of φ.

A classical example of this is Brouwer's *idealised mathematician.* More recent modal approaches can be found in [**Ste75**], [**Chi91**] and [**Fal03**]. Note that in (‡), the modality "could in principle generate" must be closely connected to the current skill level of the individual A: If we interpret the modality as logical possibility, then every sentient being would have an unlimited amount of mathematical knowledge; if we interpret it as practical possibility given unlimited time, then lots of bright school children would know a great number of statements φ that belong to the most complicated research mathematics; after all, it is possible that they go on to get a PhD in mathematics, become research mathematicians, and at age 35 are in the position to write down a derivation of φ (if they wanted to). Both consequences are clearly not intended.

2.4. Mathematical Knowledge without proof. We shall challenge the modal point of view by giving examples in which we commonly attribute mathematical knowledge to people whose current skill level does not suffice to produce a derivation of mathematical facts that they know. Only very good beginning mathematics students would be able to produce derivations, but nevertheless, in oral exams we attest all passing students some mathematical knowledge. Furthermore, in

[5]The original paper, [**FeiTho63**], has over 250 pages. Only specialists in finite group theory will know even an informal proof, let alone a derivation. On the other hand, the theorem is rather well known.

industrialised countries the majority of the public has mathematical knowledge of some kind, *e.g.* elementary algorithms of arithmetic, the Rule of Three etc., but of course only a tiny fraction of the public would satisfy any reasonable reading of (‡).

3. Conclusion

The examples of Section 2.4 show that the modal point of view according to (‡) cannot deal with some knowledge attributions that we consider reasonable in natural language. Arguing a little bit more the examples give more than just counterarguments against (‡):

Consider an analysis of mathematical knowledge

(‡‡) A knows that φ iff
A could in principle generate a proof* of φ,

where "proof*" is some diligently chosen notion of proof weaker than "derivation". The examples of mathematical skills in the public show that unless you want to break with our methodological maxim of accepting natural language as a guideline, "proof*" has to be so weak that it is unfit to differentiate between "Andrew Wiles knows that FLT is true" and "An average first year math graduate student knows that FLT is true".

This failure supports our main claim: In mathematics, both belief and knowledge are matters of degree.

Our brief analysis of linguistic usage of knowledge assertions in mathematical practice is connected with deep questions in the philosophy of mathematics: Is it possible to have (a high degree of) knowledge of φ by pure intuition without any formal proof in mind (the Ramanujan phenomenon)?[6] Unfolding (or potentially unfolding) a gappy proof into a more formal proof is connected to mathematical skills—how is this connected to non-propositional knowledge? It may well be necessary to develop a skill-model for mathematical knowledge in analogy to the Dreyfus-Dreyfus skill-acquisition model [**DreDre86**].

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[6]Cf. [**Thu94**].

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