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Against (Maddian) Naturalized Platonism†

Mark Balaguer*

1. Introduction

In 1973, Paul Benacerraf presented philosophers of mathematics with a dilemma, arguing that our best theory of the semantics of mathematics is incompatible with our best epistemology. The epistemology he was referring to—a certain causal theory of knowledge—has since been widely rejected; but the dilemma is still with us. If we take mathematics at face value, we are led to *platonism*, the view that mathematics is about an abstract mathematical realm, *i.e.*, a collection of aspatial, atemporal, acausal, mind-independent mathematical objects. But it's not clear how this can be squared with what seems to be our best theory of the nature of human beings, namely, that we are wholly naturalistic creatures, hopelessly ensnared in space-time, and that our only ultimate sources of knowledge are perception and introspection. (This naturalistic view of human beings does not *entail* the falsity of platonism, but it *does* seem to raise a *prima facie* worry about platonism. For if we have no *epistemic access* to the mathematical realm, then it's unclear how we could know anything about it; and if we can't know anything about the mathematical realm, then we ought not to follow platonists in claiming that our mathematical knowledge is knowledge of such a realm.)

Most platonists have tried to deal with this problem by either (a) rejecting the naturalistic view of human beings and claiming that we possess a faculty of mathematical intuition that provides us with some sort of *epistemic access* to (*i.e.*, *contact* with) the mathematical realm;[1] or (b) arguing that we can attain knowledge of the mathematical realm in spite of the fact that we have no epistemic access to it. But in recent years, Penelope Maddy

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[1] This is Gödel's view; see his [1964].

has developed a third strategy: she wants to *naturalize* platonism, *i.e.*, alter the traditional platonistic standpoint so that we can attain epistemic access to mathematical objects, even though we *don't* have any epistemic access to an aspatial, atemporal mathematical realm. Thus, she writes:

> I intend to reject the traditional platonist's characterization of mathematical objects... [and] bring them into the world we know and into contact with our familiar cognitive faculties.[2]

Maddy is thinking mainly of *sets* here. Thus, her two central claims are that sets are *spatio-temporally located*; (a set of eggs, for instance, is located right where the eggs are); and that sets are *perceptible*, *i.e.*, they can be seen, heard, smelled, felt and tasted in the usual way.

I will raise a few problems for the claim of spatio-temporality, but my central argument will be that Maddy cannot take sets to be perceptible, because this is incompatible with the *abstractness* of sets, to which Maddy is also committed. In short, then, my argument is that Maddy needs to both be and not be a platonist. Thus, while her view is certainly original[3] and in many ways ingenious, it is, at bottom, unintelligible. (Note that my argument is intended to work against not just *Maddy's* view, but any version of naturalized platonism, *i.e.*, any view that takes mathematical objects to be simultaneously abstract and perceptible.)

(It should be noted that Maddy does not think that the Benacerrafian problem of knowledge is solved by the bare claim that sets are perceptible; she tells a long story about how we are led from perceptions of sets to Zermelo-Fraenkel set theory (ZF).[4] I will ignore all of this, because I *do* think that the Benacerrafian problem is solved by the bare claim of perceptibility. If we can perceive mathematical objects, then we needn't worry that knowledge of such objects is impossible.)

2. Is Maddy's View Platonistic?

Platonists who set out to explore Maddy's philosophy in the hopes of finding an agreeable epistemology are apt to come away with the feeling that

[2] Maddy [1990], p. 48. The view that I will be concerned with is laid out in Maddy [1980] and [1990]. It is worth noting, however, that Maddy has since given up on this view; see her [1992].

[3] Obviously, Maddy isn't the *first* to bring abstract mathematical objects into space-time. Aside from Aristotle, David Armstrong—[1978], chapter 18, section V—attempts this, although his view is certainly quite different from Maddy's.

[4] There are two basic parts to Maddy's story. The first—which she gives in chapter two of her book—describes how we are led from perceptions of sets to *mathematical intuitions* about sets and how these intuitions lead us to accept certain axioms of ZF; the second part—which she gives in chapter four—describes how *pragmatic considerations* can lead us to accept certain axioms of ZF. (Knowledge of the theorems of ZF can, of course, be accounted for by appealing to proof.) I won't say much of anything about intuition, but—in the last section of the paper—I will say a few words about why Maddy's appeal to pragmatic considerations cannot solve the problems that I raise for her view.

she is the sort of friend who obviates the need for enemies. For since most platonists take the belief in abstract objects to be the very *core* of platonism and since they take abstract objects to be, by *definition*, aspatial and atemporal, they are apt to consider Maddy an anti-platonist. Thus, on this view, to naturalize platonism by bringing abstract objects into space-time is like naturalizing theism by taking God to be the Lincoln Tunnel. We're not naturalizing here, we're *abandoning*.

At times, Maddy seems to imply that such criticisms can be simply ignored. She writes:

> On some terminological conventions, this means that sets no longer count as 'abstract'. So be it; I attach no importance to the term.[5]

Now, of course, this sort of response is often acceptable. Questions about whether a view gets counted as x-ism or y-ism are usually not important; all that matters is whether the view is correct. But in this particular case, the question of whether Maddy's view is platonistic is important. The reason is simple: platonism is supposed to be doing some *work* for Maddy; it's supposed to enable her to steer clear of the semantic horn of Benacerraf's dilemma.

Now, one might object that platonism is, in fact, *not* needed to avoid the semantic horn of the dilemma; for as Benacerraf described the situation, the semantic horn is avoided so long as we adopt a Tarskian semantics for the language of mathematics, and this is accomplished by *any* view that posits mathematical objects, regardless of whether these objects are taken to be abstract.[6]

The problem with this objection is that Benacerraf's characterization of the semantic horn is too weak. An adequate philosophy of mathematics must not only provide Tarskian truth *conditions* for mathematics; it must also account for why our mathematical theories are actually *true*, *i.e.*, for how it is that the truth conditions are actually *satisfied*. Thus, in order to avoid platonism, one would need to specify a collection of concrete objects which actually satisfy our mathematical theories. The two most obvious suggestions here are that mathematical theories are satisfied by mental objects and that they are satisfied by ordinary physical objects. Now, if Maddy were to endorse a thoroughgoing anti-platonism, her view would be of the latter sort; thus, she would need to claim that sets are ordinary physical objects and that they satisfy the axioms of set theory. But this view is basically just the Millian view that sets are aggregates of physical stuff; but Mill's view is wildly implausible, and Maddy knows it.

There are numerous problems with Mill's aggregate theory of sets (ATS).

[5] Maddy [1990], p. 59.

[6] A view that posits mathematical objects is a version of *mathematical realism*; if, in addition, those objects are taken to be abstract, then the view is a version of platonism.

Maddy uses a Fregean argument here: sets could not be mere physical aggregates, because while a set has a determinate number of members, an aggregate of physical stuff doesn't. For instance, if we have three eggs in a carton, then 3 is the only number that applies to the *set*; but the aggregate of egg-stuff consists of 'three eggs,... many more molecules, [and] even more atoms'.[7] Another problem with ATS (which Maddy doesn't mention) is that it is incapable of distinguishing the set containing the eggs from the set containing the set containing the eggs; thus, if all sets were aggregates, we wouldn't be able to get off the ground floor of the set-theoretic hierarchy.

In short, then, my response to Maddy's 'so be it' remark reduces to this: Maddy cannot afford to give up on platonism (*i.e.*, the abstractness of sets) altogether, because if she does, she will collapse into a view (*viz.*, Mill's) that she knows is false. Maddy might be able to give up on abstractness in the *traditional* sense (*i.e.*, she might be able to give up on the claim that sets exist outside of space and time) but she *needs* her sets to remain abstract in some *non-traditional* sense. Thus, she needs to find some middle ground between physical aggregates and full-blown aspatial and atemporal objects. And, indeed, this is exactly what she does: she claims that a singleton containing an egg is identical not with the aggregate of the egg-stuff, but with the *egg-as-individuated-thing*.[8] Now, she doesn't say exactly how this view is to be extended to pairs, triples, *etc.*, but presumably a pair of eggs would be identical with the *two-eggs-as-individuated-things-taken-together*, or some such thing. In brief, then, Maddy's view is that, while the set of eggs and the aggregate of egg-stuff are made of the same matter and share the same location, they are *not* identical, because they are *structured* differently.

Maddy thinks that this structural difference between sets and aggregates provides her with the non-traditional sense of 'abstract' that she needs. Indeed, it seems to me that in claiming that sets are structurally different from aggregates, Maddy *commits* herself to the claim that sets are abstract, in some relevant sense. One reason for thinking this is that every physical aggregate is associated with *infinitely many* sets. Our aggregate of egg-stuff, for instance, shares its location with not only the set of three eggs, but the set containing this set, the set containing the set containing this set, the pair containing these two singletons, *etc.* Clearly, the differences between all of these objects—all of which are made of exactly the same matter—are, in some sense or other, abstract or non-physical. (The only other option would be to claim that sets are somehow *mental* objects, *i.e.*, that sets are individuated and taken together *by us in our minds*. But Maddy certainly doesn't want to make this move, because she would then collapse into psychologism, which (she is aware) is no more plausible than Millianism.[9] Maddy says just what she *should* say here, *viz.*, that sets exist *objectively*, *i.e.*, that the eggs in the above set are individuated and taken together *in themselves*, independently of us.)

So Maddy's view is that sets exist in space-time, but are nonetheless abstract, because they are *structured in a non-homogeneous way*. Our question, then, is this: by finding a middle ground between traditional platonistic views and traditional anti-platonistic views, is Maddy able to slip through the horns of the Benacerrafian dilemma? More precisely, are Maddian sets (e.g., eggs-as-individuated-things-taken-together) the sorts of objects that can both satisfy the axioms of set theory *and* be seen by human beings? I will argue that they aren't, that they can't do both of these things at the same time.

3. Problems for Maddy's Naturalized Platonism

There are actually *two* views contained in Maddy's writings, between which she remains neutral. The first is *physicalistic platonism*, *i.e.*, the view that *all* sets are spatio-temporally located in the way described above. The second I will call *hybrid platonism*; this is the view that *some* sets (*viz.*, *impure sets*, *i.e.*, sets of physical objects, sets of sets of physical objects, *etc.*[10]) exist in space-time while *others* (*viz.*, *pure sets*, *i.e.*, sets in the iterative hierarchy that are built up from the null set via set-creating operations like the power-set operation) exist outside of space-time.

3.1 Hybrid Platonism

I don't want to say very much here about hybrid platonism. I merely want to point out that, from a naturalistic point of view, it's no better than traditional platonism. That it's no better *ontologically* is entirely obvious: hybrid platonism still commits to full-blown abstract objects, *i.e.*, objects that are abstract in the traditional sense. But one might think that hybrid platonism offers an *epistemological* advance over traditional platonism. In particular, one might argue as follows. The mathematical theory ZF is about pure sets, and these are abstract objects in the traditional sense. But, by adopting hybrid platonism, we can account for how human beings acquire knowledge of such objects. First, they acquire perceptual knowledge of impure sets, which are spatio-temporally located; and then they proceed to knowledge of pure sets via some sort of theoretical inference.

[7] Maddy [1990], p. 60; see also Frege [1884], section 23.

[8] Maddy [1990], pp. 152–153.

[9] The most famous arguments against psychologism are again given by Frege; *cf.*, Frege [1893–1903], pp. 12–15.

[10] An 'impure set' is usually defined as a set that has at least one non-set in its transitive closure. A Maddian impure set is different in two ways: *everything* in its transitive closure is a *physical object*.

The problem with this is that traditional platonists can (if they want to) make virtually the same move: they can claim that knowledge of pure sets is obtained by theoretical inference from perceptual knowledge of aggregates of physical objects. The fact that, on the hybrid platonist view, we get perceptual knowledge of things that are *called* 'sets' is quite irrelevant; for it doesn't follow from this that these things (*i.e.*, Maddian impure sets) are any more similar to pure sets than aggregates are. Maddy might, at this point, claim that the hybrid platonist's inference (from facts about Maddian impure sets to facts about pure sets) is justified, while the traditional platonist's inference (from facts about aggregates to facts about pure sets) is not, because impure sets and pure sets are of the same *natural kind*, whereas aggregates and pure sets are not; her evidence here would be grounded on the claim that impure sets and pure sets obey the same laws (*i.e.*, that hierarchies of the two sorts of sets are isomorphic). But Maddy cannot help herself to this claim without argument. The whole point of the Benacerrafian challenge to platonism is that we cannot know what aspatial, atemporal objects are *like*; thus, Benacerraf would just ask the hybrid platonist how she knows that pure sets and impure sets are of the same kind (*i.e.*, how she knows that they obey the same laws, or that the two sorts of hierarchies are isomorphic). Since we only have epistemic access to impure sets (*i.e.*, since pure sets exist outside of space-time) we *cannot* know that they are of the same kind. Thus, the hybrid platonist's inference is every bit as unjustified as is the (misguided) traditional platonist's inference. To assume otherwise is to flagrantly beg the question.

I, therefore, conclude that hybrid platonism offers no naturalistic advance over traditional platonism. If Maddy's naturalism is going to help us avoid the epistemic horn of Benacerraf's dilemma, she is going to have to claim that the objects we perceive *are* the objects of set theory. Otherwise, she will face the same epistemic gap that traditional platonists face: she will need an explanation of how we can know what the objects of set theory are like, given that we're not causally related to them.

3.2 Physicalistic Platonism

I now proceed to the meat of my argument, *i.e.*, the attack on physicalistic platonism. (This, recall, is the view that there are no pure sets, that *all* sets are Maddian impure sets, *i.e.*, that all sets are both perceptible and abstract, where 'abstract' is taken in the non-traditional sense described above.)

The first thing I want to say about physicalistic platonism is that it does not avoid all of the traditional arguments against Mill's view. It *does* avoid the perils of identifying sets with aggregates, but that is not the only problem with Mill's view. I will just briefly state some of the problems that Maddy will encounter here.

One problem is that physicalistic platonists might not be able to account for the truth of the axiom of infinity. Now, Maddy may be able to avoid this criticism, either by taking space-time points as physical objects (and, hence, procuring the result that there are uncountably many physical objects) or by arguing that, even if there are only finitely many urelements, there are infinitely many physical objects, because there are infinitely many sets in the iterative hierarchy. But the corresponding objection about the axiom of the null set is a bit more troubling. Since there is no physical object that could be the null set, Maddy will have to reject standard set theories like ZF and replace them with set theories in which there is no null set.[11] But even if her 'radically impure set theory' can work (and there are reasons to doubt this[12]) it seems undesirable to have to say that ZF is false. After all, we might wonder whether it's legitimate to salvage a philosophical interpretation of a given branch of science by singling out various claims of that science and rejecting them.[13]

Another problem is that Maddy's physicalistic platonist has to claim that mathematics is empirical, *i.e.*, that sentences like '2 + 2 = 4' are empirical hypotheses that could turn out false. Maddy tries to make this more palatable in the same way that Mill did, *i.e.*, by claiming that mathematical laws like '2 + 2 = 4' only *seem* necessary and *a priori* because we see them confirmed so *frequently*.[14] I think there are numerous problems with this response (such as that some physical claims are just as frequently confirmed and that some mathematical axioms are *never* confirmed and would, if taken empirically, be highly controversial) but I cannot embark, at this point, upon an argument against empiricism in the philosophy of mathematics. I merely wish to point out that physicalistic platonists are

11 See Maddy [1990a], appendix and Maddy [1990], p. 157.

12 There is nothing technically wrong with Maddy's impure hierarchy, but there may be philosophical problems with it. Most of them result from Maddy's claim that a physical object and the set containing it are one and the same thing, *i.e.*, that $x = \{x\} = \{\{x\}\} = \{\{\{x\}\}\} = \ldots$. Maddy cannot take this line with non-singleton sets; if she did, then she would have to admit that $x, y = \{x, y\} = \{\{x, y\}\} = \ldots$, and she would be unable to get off the ground floor of the iterative hierarchy, *i.e.*, she would be unable to countenance sets of higher rank). But the question is whether the double standard that Maddy employs here can be tolerated. Is it acceptable to claim that some singletons collapse into their transitive closures, while pairs and other singletons don't, and while certain other sets—*e.g.*, {{{ Madonna}}, {{Madonna, Quine }}}—suffer *partial* collapses? What non-*ad hoc* reason could Maddy give for thinking that seemingly similar sets behave so differently?

13 One might think that this problem is a red herring, since the null set axiom is entailed by the axiom of separation (together with the assertion that there is at least one set). But this doesn't *solve* the problem for the null set axiom; rather, it *extends* the problem to the axiom of separation. Since physicalistic platonists don't recognize a null set, they cannot endorse the axiom of separation (although, they *can* endorse an altered version of it).

14 See Mill [1843], Book II, chapter V, section 5; and Maddy [1990a], p. 276.

committed to empiricism and that there are well known reasons to be leery of this.

Finally, physicalistic platonism seems to inherit the *perversity* of Mill's view; in particular, it implies that mathematics is about eggs and pebbles and biscuits. Now, Maddy might be able to make this a bit more plausible by pointing out that set theory gives us general laws that are obeyed by *all* impure sets. But where does this leave non-set-theoretic branches of mathematics? Granted, such branches reduce (in the technical sense) to set theory, but that doesn't mean they're *about* sets. As Maddy realizes, her view leaves us wholly incapable of accounting for the fact that people had mathematical knowledge before set theory was born.

But let us assume, for the sake of argument, that Maddy can solve all of these problems that arise from bringing mathematical objects into space-time. I want to argue that physicalistic platonism is, nonetheless, unintelligible: we cannot claim, without contradiction, that Maddian impure sets are simultaneously abstract and perceptible. To regain intelligibility, Maddy has to either abandon abstractness (and, hence, impale herself on the semantic horn of Benacerraf's dilemma) or else abandon perceptibility (and, hence, impale herself on the epistemic horn of the dilemma).

Now, I have already argued that Maddy is correct in taking her impure sets to be abstract. Thus, what I need to argue is that we cannot perceive these sets. I begin by asking whether we can perceive the structural difference between an aggregate and a set. That is, when we look into the egg carton, can we see the aggregate *and* the set? Now, obviously, we can't see *all* of the infinitely many sets in the carton, but Maddy claims that we *can* see the set containing the three eggs. But how is this possible? Since the set and the aggregate are made of the same matter, both lead to the same retinal stimulation; Maddy herself admits this.[15] But if we receive only one retinal stimulation, then the perceptual data about the set is identical to the perceptual data about the aggregate. Thus, we cannot perceive the difference between the aggregate and the set.[16] But since it is pretty obvious that we *can* perceive the aggregate, and since there *is* a difference between the aggregate and the set, it follows that we cannot perceive the set.

Before looking at how Maddy might respond to my argument, it is worth emphasizing that what we're encountering here is just Benacerraf's problem all over again: there's no way that we could know what the objects of set theory are like, because we have no access to them. We have perceptual knowledge of what aggregates are like, but any epistemic jump from aggregates to sets is unwarranted, because we receive no data about the difference between these two sorts of objects. Maddy might be right that

[15] Maddy [1990], p. 65.

[16] Chihara has given a different argument for the same conclusion. See his [1990], pp. 202-204.

the eggs in her carton generate an infinite hierarchy of sets; but she has said nothing to block the Benacerrafian worry that we could have no good reason for claiming that this hierarchy has one nature rather than another.

Let us now consider how Maddy might respond to my argument. I think there are two different responses suggested in her writings. The first is that the set/aggregate case is analogous to the psychologist's case in which we see one and the same picture first as a young woman and then, suddenly, as an old woman.[17] Maddy tries to back this up with a bit of neurophysiology, giving a scientific explanation for *why* we can (with the same retinal stimulation) sometimes see a set and sometimes see an aggregate. The explanation relies upon the notion of a *cell-assembly*. A cell-assembly is, basically, a 'neural recognizer': every time I recognize an object of type X, it is because my X-cell-assembly is activated. (Thus, cell-assemblies correspond to concepts: I have one for horses, one for cars, one for triangles, *etc*. Moreover, the formation of a cell-assembly corresponds to the acquisition of a concept; after repeated perceptual experience with objects of a given kind, a cell-assembly is formed in my brain and I acquire the corresponding concept.) Maddy's claim, then, is that whether we see a set or an aggregate on a given occasion depends upon whether a set cell-assembly or an aggregate cell-assembly is activated.

I have two responses to all of this. First of all, the set/aggregate case is simply *not* analogous to the case of the psychologist's picture. With the picture, it is a confusion to say that we see two different things: the picture of the old woman and the picture of the young woman are the same object. All we're doing here is seeing one object in two different *ways*. But, according to Maddy, the set and the aggregate are not identical; we have two different objects here, and the question we want to answer is whether we can see the difference between them. Thus, since the case of the psychologist's picture isn't even a case of seeing two different objects, it is entirely irrelevant for us. Second of all, with respect to the cell-assemblies, Maddy cannot assume that human beings even *have* such things for sets, for to do so is to beg the question. According to the above story, a cell-assembly for objects of kind X is only formed after repeated perceptions of such objects. Thus, a cell-assembly for sets could only be formed if we could perceive sets. But since this is precisely what is at issue, Maddy cannot assume that human beings have such cell-assemblies. (What really needs to be determined is whether human beings can *form* cell-assemblies for sets; thus, obviously, Maddy cannot rely in her argument upon the claim that we do have such cell-assemblies. Of course, I've already argued that human beings *can't* form cell-assemblies for sets—or, at least, for *Maddian* sets—because I've already argued that they can't *perceive* Maddian sets.)

[17] See Maddy [1990], pp. 64-65.

(Perhaps, Maddy will claim that children can be *taught* to form two different kinds of cell-assemblies by simply being *told* that corresponding to every aggregate there is a set. But this is no better than a traditional platonist claiming that people are taught what the mathematical realm is like. Both responses just move the problem back a generation; the question now arises how the *teacher* has knowledge of sets. My point here is, of course, *not* that one can't be taught the difference between sets and aggregates; it's that the teacher's knowledge of this distinction can no more be grounded in the perception of Maddian sets than the pupil's can.)

Maddy's second response to my argument is this: the fact that we can see a set, even when we only receive data from the corresponding aggregate, is analogous to the fact that we can see a physical object, even when we only receive data from the front side of a time-slice of the object.[18] But this case is no more analogous to the set/aggregate case than is the psychologist's picture. The front side of, say, an egg is a *part* of the egg; thus, to see the front side of the egg at a particular time just *is* to see the egg at that time. But since an aggregate is distinct from its corresponding sets and not a *part of* them (in anything like the sense in which the front side of the egg is a part of the egg) we are not inclined to say that to see an aggregate *is* to see its corresponding sets.[19]

I conclude, then, that Maddy cannot block my argument and that we cannot see her impure sets. And it is worth emphasizing here that the invisibility of these sets is a direct result of their abstractness. The reason we can't see them is that there is something abstract about them, over and above the aggregates they correspond to. To fix this, to make sets wholly concrete, would be to collapse right back into Mill's implausible version of anti-platonism; *i.e.*, it would be to run headlong into the semantic horn of our dilemma. In short, my point is that Maddy cannot steer a course between the horns of Benacerraf's dilemma by wedding abstractness and perceptibility, because these two properties are simply incompatible.

4. Extrinsic Modes of Justification

In this section, I would like to consider whether Maddy's situation is improved by her claim that mathematical axioms can be justified not only *intrinsically* (*i.e.*, via perception-based intuition) but also *extrinsically* (*i.e.*, via pragmatic considerations). Now, this is *not* the Quinean claim that mathematical theories can be justified by their usefulness in empirical science.[20] For Maddy, pragmatic reasons for accepting an axiom can arise

[18] See Maddy [1990], p. 49.

[19] Hale makes essentially this point on p. 81 of his [1987].

[20] See Quine [1951], section 6. Quine's epistemology is inadequate, because it doesn't account for our knowledge of *unapplied* mathematical theories.

within mathematics. For instance, we might be inclined to accept a certain axiom if it provided a solution to an open mathematical question.[21]

I do not wish to deny that axioms can be justified pragmatically. I merely want to point out that platonists cannot use this fact to respond to Benacerraf. I have two arguments here. The first is identical to the argument against using *proof* to respond to Benacerraf. Benacerraf's point is that, if platonism were true, we couldn't *get started* mathematically. Neither the method of proof nor (intra-mathematical) pragmatic considerations are relevant to this point, because both of these modes of justification rely upon previous mathematical knowledge. (This is why Maddy also wants a faculty of intuition; but, as we've seen, her appeal to such a faculty fails.) The second argument is that Maddy has said nothing to explain *why* pragmatic modes of justification are legitimate in mathematics (*i.e.*, about why pragmatic considerations lead us to true rather than false mathematical belief). That such an explanation is needed is obvious: Benacerraf's argument can no more be answered by merely *asserting* that we can attain mathematical knowledge via pragmatic considerations than by *asserting* that we can attain mathematical knowledge via intuition. In both cases, the platonist has to say *how* we are led to mathematical truth.

This last point suggests a final argument against Maddy's philosophy of mathematics: pragmatic modes of justification are legitimate in mathematics; Maddy cannot account for this fact; therefore, her view is false. Now, in fairness to Maddy, I should note that she is quite aware that she needs an explanation here. But she claims that this problem is equally pressing for all philosophies of mathematics. Whether or not this is true, it is clearly a problem for all versions of *platonism* (and, more generally, for mathematical realism). For since platonists think that we should only accept a mathematical sentence if it corresponds to the *objective* mathematical facts, they have to explain how it could be legitimate to accept a sentence merely because it entailed a solution to some open problem. That is, they have to explain why pragmatic value is evidence for truth. (One might object that, since pragmatic considerations are relevant in empirical as well as mathematical theory construction, this problem is no worse for mathematical platonists than it is for scientific realists. But the two cases are radically different: if an empirical hypothesis is pragmatically useful, we seek independent (non-pragmatic) confirmation for it and, until such confirmation is obtained, the hypothesis is considered suspicious and *ad hoc*; but this is not true in mathematics.)[22]

[21] See Maddy [1990], chapter 4.

[22] I would like to point out that I have elsewhere shown how platonists can solve this problem (*i.e.*, the problem of explaining the legitimacy of pragmatic modes of justification) *while also solving Benacerraf's epistemological objection*. See my [forthcoming]. To give just a hint of the strategy here, the answer lies, as Maddy suspected, in altering

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ABSTRACT. It is argued here that mathematical objects cannot be simultaneously abstract and perceptible. Thus, naturalized versions of mathematical platonism, such as the one advocated by Penelope Maddy, are unintelligible. Thus, platonists cannot respond to Benacerrafian epistemological arguments against their view via Maddy-style naturalization. Finally, it is also argued that naturalized platonists cannot respond to this situation by abandoning abstractness (that is, platonism); they must abandon perceptibility (that is, naturalism).

the traditional platonist position. But the alteration needed is not a naturalization, or a *de-platonization*; rather, what we need is a *super-platonization*. Intuitively, the view I have in mind can be sloppily expressed by the slogan 'All possible mathematical objects exist'. More precisely (*i.e.*, without the *de re* possibility) the view is that the mathematical objects which *do* exist exhaust all of the possibilities. Of course, a good deal of argument is needed to show that this position solves our two problems, but the main idea (with respect to the Benacerraf problem, anyway) is that since all possible mathematical objects exist, we can attain knowledge of facts about the mathematical realm by merely constructing a consistent mathematical theory. Thus, since knowledge of consistency doesn't require access to the objects of the theory in question, the Benacerraf problem has been solved.

On the Most Open Question in the History of Mathematics: A Discussion of Maddy

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Imagine for a moment that the reknowned mathematician William Butler McLeod has been trying to prove a well known conjecture in the field of Absolute Geometry (AG). AG consists of theorems deduced from Euclid's first four postulates, and the famous conjecture, known as the Parallel Hypothesis (PH), is what we know as Euclid's fifth postulate. McLeod's method of attack consists in deducing consequences in the theory AG + ¬PH in the hope of deducing a contradiction. Professor McLeod has dedicated many years of his life to this endeavor, but to no avail. His perennial failure is explained when Professor Edna St. Vincent Smythe shows that PH is in fact independent of AG, and thus that McLeod's preliminary results are actually theorems in the consistent theory AG + ¬PH. Professor McLeod responds by saying that this state of affairs is manifestly impossible. He insists that 'there must be a fact of the matter' concerning PH, and that mathematicians must thus take up the task of finding proper axioms to add to AG in order to settle the question correctly and in accord with the 'truth' about PH. He declares that St. Vincent Smythe's results make the question of PH 'more open than any mathematical question in history', and that it is therefore most urgent that mathematicians find the solution.

Next imagine a similar situation in which the mathematician Dr. Mary Ann Evans has been trying to settle the old conjecture known as the Commutativity Hypothesis (CH) in the area of Group Theory (GT). The axioms of group theory are just our familiar four, and CH is the statement that multiplication in a group obeys the commutative law: $a \times b = b \times a$. As McLeod did, Evans labors at length without success, and as St. Vincent Smythe did, Professor Henry Handel Richardson dashes her hopes by proving that CH is independent of GT. Richardson's result is even more discouraging owing to his displaying infinitely many models of each of GT + ¬CH and GT + CH. Evans responds as did McLeod, insisting that there 'must be a fact of the matter'; CH is definitely either true or it is

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