4. Convergence

The notion of convergence in arbitrary topological spaces is intricately intertwined with the notion of a filter. Convergence in the usual setting (i.e., convergence of sequences) does not suffice to describe convergence phenomena in arbitrary topological spaces; we shall give an example of a topological space where sequences don't describe topological closure.

4.1. DEFINITION. Let $\langle X, \tau\rangle$ be a topological space, $A \subseteq X$, $\mathcal{F}$ a filter on X and $x \in X$. We denote the set of all neighbourhoods of x by $\mathcal{N}(x)$. We will also call $\mathcal{N}(x)$ the **neighbourhood system** of x. We say x is a **cluster point** for $\mathcal{F}$ iff $U \cap F \neq \emptyset$ for all $U \in \mathcal{N}(x)$ and all $F \in \mathcal{F}$. We say $\mathcal{F}$ **converges to** x (or, x is a **limit point** of $\mathcal{F}$) iff $\{A \cap U\ ;\ U \in \mathcal{N}(x)\} \subseteq \mathcal{F}$, (in other words: if $\mathcal{F}$ is finer then $\mathcal{N}(x)$).

4.2. REMARK (Filters and sequences). We can see filters as generalized sequences. If we let $\langle a_n\ ;\ n \in \mathbb{N}\rangle$ be a sequence in A and let $E_n := \{a_k\ ;\ k \geq n\}$ be the endsegment of $\langle a_n\rangle$ from k on. If we let $\mathcal{B} := \{E_n\ ;\ n \in \mathbb{N}\}$, then $\emptyset \notin \mathcal{B}$ and $\mathcal{B}$ is closed under finite intersection, so $\mathcal{B}$ is a filter basis. We say the filter $\mathcal{F}(\mathcal{B})$ is the **filter generated by the sequence** $\langle a_n\rangle$.

EXERCISE 13. Check that x is an accumulation point of the sequence $\langle x_i\ ;\ i \in \omega\rangle$ iff x is a cluster point of the filter generated by $\langle x_i\ ;\ i \in \omega\rangle$.

EXERCISE 14. Let X be a topological space, $A \subseteq X$, $\mathcal{F}$ a filter on X and $x \in X$. Then the following are equivalent:

(a) x is a cluster point of $\mathcal{F}$, and
(b) there is a filter $\mathcal{G} \supseteq \mathcal{F}$, such that $\mathcal{G}$ converges to x.

4.3. COROLLARY. Let $\mathcal{F}$ be an ultrafilter. Then the following are equivalent:

(a) x is a cluster point of $\mathcal{F}$,
(b) $\mathcal{F}$ converges to x.

PROOF. $(a) \implies (b)$: Follows directly from the definitions.
$(b) \implies (a)$: Let $\mathcal{F}$ be an ultrafilter with cluster point x. We use Exercise 14 and get a filter $\mathcal{G} \supseteq \mathcal{F}$ which converges to x. But by Proposition 3.5, $\mathcal{F}$ is maximal, hence $\mathcal{G} = \mathcal{F}$. q.e.d.

4.4. DEFINITION. We say a topological space is a **Hausdorff space** iff for all $x \neq y \in X$ there exist open sets P and Q such that $x \in P$, $y \in Q$ and $P \cap Q = \emptyset$. This property is also called the separation axiom T_2.

EXERCISE 15. Let $\langle X, \tau\rangle$ be a Hausdorff space and let $A \subseteq A$. Show that A together with the subspace topology on A is a Hausdorff space as well.

4.5. PROPOSITION. Let $\langle X, \tau\rangle$ be a topological space, then the following are equivalent:

(a) $\langle X, \tau\rangle$ is a Hausdorff space,
(b) every filter on X has at most one cluster point.

PROOF. $(a) \implies (b)$: Let $\langle X, \tau\rangle$ be a Hausdorff space and let $\mathcal{F}$ be a filter on X with two distinct limit points x and y. Then by definition we have $\mathcal{N}(x) \subseteq \mathcal{F}$ and

$\mathcal{N}(y) \subseteq \mathcal{F}$. Now let P and Q be two open sets seperating x and y. Then $P \in \mathcal{N}(x)$ and $Q \in \mathcal{N}(y)$ so $P, Q \in \mathcal{F}$. But then $\emptyset = P \cap Q \in \mathcal{F}$, a contradiction.

$(b) \implies (a)$: Suppose X is not a Hausdorff space. Then there are two points which cannot be seperated by open sets of τ, i.e.,

$$(*) \qquad (\forall P \in \mathcal{N}(x))(\forall Q \in \mathcal{N}(y))(P \cap Q \neq \emptyset).$$

Now let $\mathcal{B}$ be the closure under finite intersection of $\mathcal{N}(x) \cup \mathcal{N}(y)$. From $(*)$ we know $\emptyset \notin \mathcal{B}$, hence $\mathcal{F}(\mathcal{B})$ is a filter. Clearly we have $\mathcal{N}(x), \mathcal{N}(y) \subseteq \mathcal{F}$, which means $\mathcal{F}$ is a filter on X with two distinct cluster points, namely x and y, a contradiction. q.e.d.

4.6. DEFINITION. Let $\mathcal{F}$ be a filter on X and let $A \subseteq X$. We call the set $\{A \cap F \; ; \; F \in \mathcal{F}\}$ the **trace filter** of $\mathcal{F}$ on A. Despite the name, a trace filter is *not* always a filter (because it may contain the empty set).

4.7. THEOREM. Let $\langle X, \tau \rangle$ be a topological space, $A \subseteq X$ and $x \in X$. Then the following are equivalent:

(a) $x \in \mathrm{cl}_X(A)$,
(b) there is a filter on A converging to x.

PROOF. $(a) \implies (b)$: Let $x \in \mathrm{cl}_X(A)$. Then we have that $U \cap A \neq \emptyset$ for all $U \in \mathcal{N}(x)$. But in this case the trace filter $\mathcal{T}$ of $\mathcal{N}(x)$ on A is a filter on A and x is a limit point for $\mathcal{T}$. From Exercise 14 we know that in this case there is a filter on A which converges to x.

$(b) \implies (a)$: This follows directly from the definitions. q.e.d.

4.8. THEOREM. There is a topological space $\langle X, \tau \rangle$ and an $A \subseteq X$ such that $\mathrm{cl}_X(A) \supsetneq \{x \in X \; ; \;$ there is a sequence in A converging to $x\}$ ($= \mathrm{LP}_\omega(A)$, see Exercise 16).

So in general topological spaces, the closure cannot be described by convergent sequences as it was in metric spaces.

PROOF. We prove the theorem by giving a counterexample to equality. Let $X := \omega_1 + 1$ and let $\tau = \tau_<$ be the order topology on X. We first note that ω_1 is a **regular cardinal** , i.e., for every countable sequence $\langle \alpha_i \; ; \; i \in \mathbb{N} \rangle$ of countable ordinals, we have $\bigcup_{i \in \mathbb{N}} \alpha_i < \omega_1$. Now let $\langle \alpha_i < \omega_1 \; ; \; i \in \mathbb{N} \rangle$ be an arbitrary sequence in $\{\alpha \; ; \; \alpha < \omega_1\} = \omega_1$. Then $\bigcup_{i \in \mathbb{N}} \alpha_i$ is the (unique) limit point of $\langle \alpha_i \; ; \; i \in \omega \rangle$. By the regularity of ω_1 we know that $\bigcup_{i \in \mathbb{N}} \alpha_i < \omega_1$.

On the other hand let us look at the neigbourhood filter of ω_1. For every $\beta < \omega_1$ is $\{\alpha \; ; \; \beta < \alpha\}$ a neighbourhood of $\mathcal{N}(\omega_1)$. Furthermore, these sets form a basis for $\mathcal{N}(\omega_1)$, in particular for all S in $\mathcal{N}(\omega_1)$, we have that $S \cap \{\alpha \; ; \; \alpha < \omega_1\} \neq \emptyset$, so $\mathcal{N}(\omega_1)$ converges to ω_1. Using the previous theorem we see this is equivalent to $\omega_1 \in \mathrm{cl}_{\omega_1+1}(\{\alpha \; ; \; \alpha < \omega_1\}) = \mathrm{cl}_{\omega_1+1}(\omega_1)$. q.e.d.

EXERCISE 16. We define another notion of convergence: Let $\langle X, \tau \rangle$ be a topological space and let $\vec{x} := \langle x_\alpha \ ; \ \alpha < \omega \rangle$ be a sequence of length ω_1. We say that $\vec{x}$ **converges to** x iff for all $U \in \mathcal{N}(x)$ there exists a $\beta < \omega_1$ such that $\{x_\alpha \ ; \ \alpha > \beta\}$.

Now let $A \subseteq X$. We look at the family of ω_1**-limits of** A:

$$\mathrm{LP}_{\omega_1}(A) := \{x \in X \ ; \ \exists \vec{x} \subseteq A(\vec{x} \text{ converges to } x)\}.$$

Show that:

(a) $\mathrm{LP}_{\omega_1}(A) \subseteq \mathrm{cl}_X(A)$.
(b) There exists a topological space $\langle X, \tau \rangle$ and an $A \subseteq X$ such that $\mathrm{LP}_{\omega_!}(A) \neq \mathrm{cl}_X(A)$. (Hint: Look at the proof of Theorem 4.8 and generalize it for $X = \omega_2 + 1$.)
(c) Generalize (b) to sequences of arbitrary length.