3. Filter Spaces

3.1. DEFINITION. Let X be an arbitrary set. We say a system $\mathcal{B} \subseteq \mathcal{P}(X)$ is a **filter basis** iff

(a) $\emptyset \notin \mathcal{B}$,
(b) $(\forall A, B \in \mathcal{B})(A \cap B \in \mathcal{B})$.

A filter basis $\mathcal{B}$ induces a filter $\mathcal{F}(\mathcal{B}) := \{A \,;\, \exists B \in \mathcal{B}(B \subseteq A)\}$.

3.2. DEFINITION. We say a filter $\mathcal{F}$ is a **principal filter** iff there exists a $A \subseteq X$ such that $\mathcal{F} = \mathcal{F}(\{A\})$.

3.3. PROPOSITION. Let $x \in X$, then the following are equivalent:

(a) $\mathcal{F} = \mathcal{F}(\{\{x\}\})$,
(b) $\{x\} \in \mathcal{F}$.

PROOF. As $\mathcal{F}(\{\{x\}\}) = \{A \,;\, \{x\} \subseteq A\}$, $(a) \implies (b)$ is evident. For $(b) \implies (a)$, let $\mathcal{F}$ be a filter such that $\{x\} \in \mathcal{F}$ and let $A \in \mathcal{F}$, then $A \cap \{x\} \in \mathcal{F}$. So $A \cap \{x\} \neq \emptyset$, which means that $x \in A$, i.e., $\{x\} \subseteq A$. Therefore we have $\mathcal{F} \subseteq \mathcal{F}(\{\{x\}\})$. The other inclusion is obvious, as $\{x\} \in \mathcal{F}$ and a filter is closed under supersets. q.e.d.

3.4. PROPOSITION. Let $\mathcal{F}$ be a principal filter, i.e., $\mathcal{F} = \mathcal{F}(\{A\})$, then the following are equivalent:

(a) $\mathcal{F}$ is an ultrafilter,
(b) There is a $x \in X$ such that $A = \{x\}$.

PROOF. $(a) \implies (b)$: Let A be a set with at least two distinct elements, i.e., there are $a_0 \neq a_1$ both in A. As every set in $\mathcal{F}$ must be a superset of $\{a_0, a_1\}$, we have $\{a_0\} \notin \mathcal{F}$ and $X \backslash \{a_0\} \notin \mathcal{F}$, contradiction the assumption that $\mathcal{F}$ is an ultrafilter.

$(b) \implies (a)$: Let $A \subseteq X$ be arbitrary, then we have either $x \in A$ or $x \notin A$. If the first holds, then is $\{x\} \subseteq A$, so by Proposition 3.3 we have $A \in \mathcal{F}$. If the second holds, then $x \in X \backslash A$, so we have similar as before $X \backslash A \in \mathcal{F}$. q.e.d.

3.5. PROPOSITION. Let $\mathcal{F}$ be a filter, then the following are equivalent:

(a) $\mathcal{F}$ is an ultrafilter,
(b) $\mathcal{F}$ is maximal, i.e., if $\mathcal{G} \supseteq \mathcal{F}$ is a filter, then $\mathcal{G} = \mathcal{F}$.

PROOF. $(a) \implies (b)$: Let $\mathcal{F}$ be an ultrafilter, $\mathcal{G} \supseteq \mathcal{F}$ a filter and let $A \in \mathcal{G} \backslash \mathcal{F}$. As $\mathcal{F}$ is an ultrafilter and $A \notin \mathcal{F}$, we have $X \backslash A \in \mathcal{F} \subseteq \mathcal{G}$. So we find $\emptyset = A \cap X \backslash A \in \mathcal{G}$, a contradiction.

$(b) \implies (a)$: Suppose $\mathcal{F}$ is not an ultrafilter, then there exists a set $A \subseteq X$ such that $A \notin \mathcal{F}$ and $X \backslash A \notin \mathcal{F}$. Let $\mathcal{B} := \mathcal{F} \cup \{B \,;\, (\exists F \in \mathcal{F})(B = A \cap F)\}$. Then $\mathcal{B}$ is obviously closed under finite intersections and $\emptyset \notin \mathcal{B}$: If $A \cap F = \emptyset$ for a certain $F \in \mathcal{F}$, then $F \subseteq X \backslash A$ and $X \backslash A \in \mathcal{F}$, a contradiction.
Therefore is $\mathcal{B}$ a filter basis and $\mathcal{F}(\mathcal{B})$ a filter. But $\mathcal{F} \subsetneqq \mathcal{F}(\mathcal{B})$, contradicting the maximality of $\mathcal{F}$. q.e.d.

3.6. DEFINITION. Let X be a set and $A \subseteq X$. We define

- $\text{IO}(X) := \{\mathcal{F} \,;\, \mathcal{F}$ is an ultrafilter on $X\}$,
- $O_A := \{\mathcal{F} \in \text{IO}(X) \,;\, A \in \mathcal{F}\}$,
- $\mathcal{S}_{\text{Stone}} := \{O_A \,;\, \emptyset \neq A \subseteq X\}$, and
- $\tau_{\text{Stone}} := \tau(\mathcal{S}_{\text{Stone}})$.

We call the topological space $\langle \text{IO}(X), \tau_{\text{Stone}} \rangle$ the **Stone space of** X. This space is also called the **Stone-Čech-compactification** or **Čech-compactification** of X and is often written as βX.

3.7. REMARKS.

(1) Every set in the basis $\mathcal{S}_{\text{Stone}}$ is clopen. We call spaces with this property **zero-dimensional**.

In particular these spaces are not connected.

(2) There is a canonical embedding $j : X \to \text{IO}(X); x \mapsto \mathcal{F}(\{\{x\}\})$ (compare this with Proposition 3.4). In case X is equipped with the discrete topology, this embedding is continuous.

EXERCISE 11. Let X be equipped with the discrete topology and let $\mathcal{S}_{\text{Stone}}$ be the basis of the Stone space $\text{IO}(X)$.

(a) Prove Remark 3.7 (1).

(b) Show that one-element sets $\{\mathcal{F}\}$ are closed.

(c) Show further that the one-element sets $\{\mathcal{F}(\{\{x\}\})\}$ which comtain an ultrafilter are both open and closed.

(d) Show that for all $U \subseteq \text{IO}(X)$ the following holds:

$$\text{cl}(U) = \{\mathcal{F} \,;\, \forall A \in \mathcal{F}(\exists \mathcal{F}^* \in U(A \in \mathcal{F}^*))\}.$$

(e) We let $j : X \to \text{IO}(X); x \mapsto \mathcal{F}(\{\{x\}\})$ be the canonical embedding as defined in Remark 3.7(2) and look at its image $j[X]$ with the subspace topology. Show that $j[X]$ and X are homeomorphic (in other words, X is densely embedded in $\text{IO}(X)$).

3.8. REMARK (Ultrafilters and measures). There are the following connections between measures and ultrafilters:

(1) Let μ be a measure on $\mathfrak{A}_{\text{discrete}}$ of X which takes only the values 0 and 1, i.e., $(\forall A \subseteq X)(\mu(A) \in \{0,1\})$. Then $A \in U_\mu :\iff \mu(A) = 1$ defines an ultrafilter.

(2) If U is an ultrafilter on X, then

$$\mu_U(A) := \begin{cases} 1 & \text{if } A \in U \\ 0 & \text{if } A \notin U \end{cases}$$

defines a two-valued measure on $\mathfrak{A}_{\text{discrete}}$.

EXERCISE 12. Prove Remark 3.8.