2. Topological and measure-theoretical constructions

We know the objects and morphism of the category of topological spaces **Top**. Our next goal is to learn more about this category. We want to know whether there are constructions of subobjects, products and so on.

In this account, we restrict ourselves to those objects that we shall need later on.

2.1. DEFINITION (Category-theoretic version). Let in the following I be an arbitrary index set, X an arbitrary set and for each $i \in I$ let $\langle X_i, \tau_i \rangle$ be topological spaces, $\langle X_i, \mathfrak{A}_i \rangle$ be measure spaces and $\pi_i : X \to X_i$ and $\phi_i : X_i \to X$ be mappings. Then we call a topology τ on X:

(a) **initial for** $\langle \langle \pi_i, \tau_i \rangle \ ; \ i \in I \rangle$ iff for all $g : \langle Y, \tau_Y \rangle \to \langle X, \tau \rangle$ the following holds:

$$g \text{ is continuous } \iff (\forall i \in I)(\pi_i \circ g \text{ is continuous}),$$

(b) **final for** $\langle \langle \phi_i, \tau_i \rangle \ ; \ i \in I \rangle$ iff for all $g : \langle X, \tau \rangle \to \langle Y, \tau_Y \rangle$ the following holds:

$$g \text{ is continuous } \iff (\forall i \in I)(g \circ \phi_i \text{ is continuous}).$$

Similar we call a σ-algebra:

(c) **initial for** $\langle \langle \pi_i, \mathfrak{A}_i \rangle \ ; \ i \in I \rangle$ iff for all $g : \langle Y, \mathfrak{A}_Y \rangle \to \langle X, \mathfrak{A} \rangle$ the following holds:

$$g \text{ is measurable } \iff (\forall i \in I)(\pi_i \circ g \text{ is measurable}),$$

(d) **final for** $\langle \langle \phi_i, \mathfrak{A}_i \rangle \ ; \ i \in I \rangle$ iff for all $g : \langle X, \mathfrak{A} \rangle \to \langle Y, \mathfrak{A}_Y \rangle$ the following holds:

$$g \text{ is measurable } \iff (\forall i \in I)(g \circ \phi_i \text{ is measurable}).$$

Unless you are already well-versed in categorical thinking, this definition doesn't tell you too much. In order to see that these objects exist and are unique, and in order to understand their meaning, we have to go into tedious details.

A first property of initial topologies is easily seen, though:

2.2. PROPOSITION. Let τ^* be an initial topology for $\langle \langle \pi_i, \tau_i \rangle \ ; \ i \in I \rangle$. Then all functions $\pi_i : X \to X_i$ are τ^*-τ_i-continuous.

PROOF. For all $\langle Y, \tau_Y \rangle$ and all $g : \langle Y, \tau_Y \rangle \to \langle X, \tau^* \rangle$ we have g is continuous if and only if $\forall i \in I(\pi_i \circ g$ is continuous). If we set $Y := X$, $\tau_Y := T^*$ and $g := \mathrm{id}$, the above becomes

$$\begin{aligned} \mathrm{id} \text{ is continuous } &\iff (\forall i \in I)(\pi_i \circ \mathrm{id} \text{ is continuous}) \\ &\iff (\forall i \in I)(\pi_i \text{ is continuous}) \end{aligned}$$

and from Proposition 1.10 we know that id is continuous. q.e.d.

2.3. DEFINITION. Let X be an arbitrary set, $\langle X_i, \tau_i \rangle$ topological spaces, I an index set and $\pi_i : X \to X_i$ and $\phi_i : X_i \to X$ mappings for each $i \in I$. We then define:

- $\tau_{\text{IN}}(\langle \langle \pi_i, \tau_i \rangle \ ; \ i \in I \rangle) := \tau(\{\pi_i^{-1}[O] \ ; \ i \in I \wedge O \in \tau_i\})$,
- $\tau_{\text{FIN}}(\langle \langle \phi_i, \tau_i \rangle \ ; \ i \in I \rangle) := \{A \subseteq X \ ; \ (\forall i \in I)(\phi_i^{-1}[A] \in \tau_i\}$.

2.4. REMARK. From the definitions of τ_{IN} and τ_{FIN} it follows that the π_i are τ_{IN}-τ-continuous and the ϕ_i are τ-τ_{FIN}-continuous and it is easily seen that τ_{IN} smallest topology with this property, and τ_{FIN} is the largest.

2.5. THEOREM. Let X be an arbitrary set, $\langle X_i, \tau_i \rangle$ topological spaces, I an index set and $\pi_i : X \to X_i$ and $\phi_i : X_i \to X$ mappings for each $i \in I$. Then the following are equivalent:

$$\text{(a)} \begin{cases} (i) & \tau^* \text{ is initial for } \langle \langle \pi_i, \tau_i \rangle \ ; \ i \in I \rangle \\ (ii) & \tau^* = \tau_{\text{IN}}(\langle \langle \pi_i, \tau_i \rangle \ ; \ i \in I \rangle), \end{cases}$$

$$\text{(b)} \begin{cases} (i) & \tau^* \text{ is final for } \langle \langle \phi_i, \tau_i \rangle \ ; \ i \in I \rangle \\ (ii) & \tau^* = \tau_{\text{FIN}}(\langle \langle \phi_i, \tau_i \rangle \ ; \ i \in I \rangle). \end{cases}$$

2.6. REMARK. The previous theorem gives us the existence and uniqueness of the initial and final topologies and allows us to say *the* initial topology for $\langle \langle \pi_i, \tau_i \rangle \ ; \ i \in I \rangle$ is the smallest topology such that all π_i are continuous.

PROOF OF THEOREM (2.5). We prove (a), the proof of (b) is similar.

First we show $(i) \Longrightarrow (ii)$: We assume τ^* is initial for $\langle \pi_i \ ; \ i \in I \rangle$ and we want to show $\tau^* = \tau_{\text{IN}}(\langle \langle \pi_i, \tau_i \rangle \ ; \ i \in I \rangle)$.

"$\tau_{\text{IN}}(\langle \langle \pi_i, \tau_i \rangle \ ; \ i \in I \rangle) \subseteq \tau^*$":

From Proposition (2.2) we know that all π_i are continuous. But $\tau_{\text{IN}}(\langle \pi_i \ ; \ i \in I \rangle)$ is the smallest such topology, so $\tau_{\text{IN}}(\langle \langle \pi_i, \tau_i \rangle \ ; \ i \in I \rangle) \subseteq \tau^*$.

"$\tau^* \subseteq \tau_{\text{IN}}(\langle \pi_i \ ; \ i \in I \rangle)$":

Assume not, so $\tau^* \setminus \tau_{\text{IN}}(\langle \pi_i \ ; \ i \in I \rangle) \neq \emptyset$. Let $A \in \tau^* \setminus \tau_{\text{IN}}(\langle \pi_i \ ; \ i \in I \rangle)$.

The mapping $\langle X, \tau_{\text{IN}} \rangle \to \langle X, \tau^* \rangle$ is not continuous, as $\text{id}^{-1}[A] = A \notin \tau_{\text{IN}}(\langle \pi_i \ ; \ i \in I \rangle)$. But for all $i \in I$ we have $\pi_i \circ \text{id} = \pi_i$ and π_i is continuous by Proposition (2.2). So τ^* is not continuous, a contradiction.

We now proceed to show $(ii) \Longrightarrow (i)$, so we want to show that $\tau_{\text{IN}}(\langle \langle \pi_i, \tau_i \rangle \ ; \ i \in I \rangle)$ is initial.

"g is continuous $\Longrightarrow \forall i \in I(\pi_i$ is continuous)":

By definition of τ_{IN}, all maps π_i are continuous, so the map $\pi_i \circ g$ is continuous as well.

"$\forall i \in I(\pi_i$ is continuous) $\Longrightarrow g$ is continuous":

Let P be an open set of X, *i.e.*, $P \in \tau_{\text{IN}}$. We want to show $g^{-1}[P] \in \tau_Y$. We look at a topology base for the topology τ_{IN}. Since $\mathcal{S} := \{\pi_i^{-1}[O] \ ; \ i \in I \land O \in \tau_i\}$ is in general not closed under finite intersections, we have to take its closure. Let $\mathcal{B}$ be the closure of $\mathcal{S}$. Consequently, each element of $\mathcal{B}$ is of the following form:

$$B = \pi_{i_0}^{-1}[O_0] \cap \dots \pi_{i_k}^{-1}[O_k],$$

for open sets $O_j \in \tau_j$.

Using Exercise 5 (2) we can find a family of finite sequences $\langle \nu(i, 0), \dots, \nu(i, m(i)) \rangle$ such that

$$P = \bigcup_{i \in I} \pi_{\nu(i,0)}^{-1}[O_{\nu(i,0)}] \cap \dots \cap \pi_{\nu(i,m(i))}^{-1}[O_{\nu(i,m(i))}].$$

Now we can compute

$$\begin{aligned}
g^{-1}[P] &= g^{-1}\left[\bigcup_{i\in I}\pi^{-1}_{\nu(i,0)}[O_{\nu(i,0)}]\cap\ldots\cap\pi^{-1}_{\nu(i,m(i))}[O_{\nu(i,m(i))}]\right]\\
&= \bigcup_{i\in I} g^{-1}\left[\pi^{-1}_{\nu(i,0)}[O_{\nu(i,0)}]\cap\ldots\cap\pi^{-1}_{\nu(i,m(i))}[O_{\nu(i,m(i))}]\right]\\
&= \bigcup_{i\in I} g^{-1}[\pi^{-1}_{\nu(i,0)}[O_{\nu(i,0)}]]\cap\ldots\cap g^{-1}[\pi^{-1}_{\nu(i,m(i))}[O_{\nu(i,m(i))}]]\\
&= \bigcup_{i\in I}(\pi_{\nu(i,0)}\circ g)^{-1}[O_{\nu(i,0)}]\cap\ldots\cap(\pi_{\nu(i,m(i))}\circ g)^{-1}[O_{\nu(i,m(i))}].
\end{aligned}$$

By assumption this last union is open. q.e.d.

2.7. EXAMPLE. Let I be an index set and for each $i \in I$ let $\langle X_i, \tau_i\rangle$ be a topological space. Let $X := \prod_{i\in I} X_i$ be the cartesian product and let $x := \langle x_i \ ; \ i \in I\rangle$ be an element of X.

If we let $\pi_i(x) = X_i$ be the i-th projection (*i.e.*, $\pi_i(\langle x_j \ ; \ j \in I\rangle) := x_i$), we call $\tau_{\text{IN}}(\langle\langle\pi_i, \tau_i\rangle \ ; \ i \in I\rangle)$ the **product topology**.

In the special case where $X_i = X_0$ for all $i \in I$, we say $X := \prod_{i\in I} X_i$ the **I-power of X_0** which we also denote as X_0^I.

2.8. PROPOSITION. Let $\langle A_i \ ; \ i \in I\rangle$ be a sequence of open subsets of X. We say $\langle A_i \ ; \ i \in I\rangle$ is **good** iff $A_i = X$ for all but finitely many $i \in I$. If we let $\mathcal{S} := \{\prod_{i\in I} A_i \ ; \ \langle A_i \ ; \ i \in I\rangle \text{ is good}\}$, then is $\tau(\mathcal{S})$ the product topology on X^I.

PROOF. The product topology is the smallest topology for which the projections π_i are continuous, and after Theorem (2.5)) we know that the initial topology is unique. So it is enough to show $\tau(\mathcal{S})$ is the smallest topology for which the π_i are continuous.

$\tau(\mathcal{S})$ makes all π_i continuous: Let $P \subseteq X$ be open, then $\pi_i^{-1}[P] = \prod_{j\in I} B_j$ where $b_j = X$ for $j \neq i$ and $B_j = P$ for $j = i$. Clearly $\langle B_j \ ; \ j \in I\rangle$ is good, so $\pi_i^{-1}[P] \in \tau(\mathcal{J})$.

$\tau(\mathcal{S})$ is the smallest such topology: Take an arbitrary element $\prod_{i\in I} A_i$ from $\mathcal{S}$. Since $\langle A_i \ ; \ i \in I\rangle$ is good, there is a finite number of indices $i(0), \ldots, i(m)$ such that A_i is not the whole space if and only if $i = i(k)$ for some k.

Thus, we have

$$\prod A_i = \pi^{-1}_{i(0)}[A_{i(0)}]\cap\cdots\cap\pi^{-1}_{i(m)}[A_{i(m)}],$$

so every $\prod_{i\in I} A_i$ is an element of $\tau_{\text{IN}}(\langle\langle\pi_i, \tau_i\rangle \ ; \ i \in I\rangle)$.

But since the topology $\tau(\mathcal{S})$ is the least topology containing all elements of $\mathcal{S}$, we get that $\tau(\mathcal{S}) \subseteq \tau_{\text{IN}}(\langle\langle\pi_i, \tau_i\rangle \ ; \ i \in I\rangle)$. q.e.d.

2.9. PROPOSITION. Let X have the discrete topology and let $s \in X^{<\omega}$, where $X^{<\omega} := \bigcup_{n\in\mathbb{N}}\{\vec{x} \; ; \; \exists n \in \omega(\vec{x}$ is a sequence of X of length $n\}$, the set of finite sequences of X. If we set $[s] := \{x \in X^{\omega} \; ; \; s \subseteq x\}$, then $\mathcal{B} := \{[s] \; ; \; s \in X^{<\omega}\}$ is a basis for the topology on X.

PROOF. From Proposition (2.8) we know that $\mathcal{S} := \{\prod_{i\in I} A_i \, ; \, \langle A_i \, ; \, i \in I\rangle$ is good$\}$ is a basis for the product topology, so we need to show $\tau(\mathcal{B}) = \tau(\mathcal{J})$. In other words, we have to show that every element in $\mathcal{J}$ can be constructed from elements of $\mathcal{B}$ using finite intersections and arbitrary unions.

Let $\langle A_i \; ; \; i \in I\rangle$ be an arbitrary good sequence, then there are sets $B_{i_0}, \ldots, B_{i_k}$, such that $A_i = X$ for $i \notin \{i_0, \ldots i_k\}$ and $A_i = B_{i_\ell}$ for $i = i_\ell$. If we set $C_\ell := X \times \cdots \times X \times B_{i_\ell} \times X \times \ldots$, we have $\prod_{i\in I} A_i := C_0 \cap \cdots \cap C_k$. But we can write

$$C_\ell = \bigcup_{s(i_l)\in B_{i_l}} [s_{i_l}],$$

so $\prod_{i\in I} A_i \in \tau(\mathcal{B})$ and we are done. q.e.d.

2.10. DEFINITION. Let $f : X \to Y$ be a map and τ a topology on Y. We then define the **pullback topology for** f as $\tau_{\mathrm{IN}}(\langle f\rangle)$ and we write $f^*\tau$.

2.11. EXAMPLE. In the special case where $X \subseteq Y$ and $f = \mathrm{id}\restriction X$, we call $f^*\tau$ the **subspace topology**

EXERCISE 9. Let $\langle X, \tau\rangle$ be a topological space and $A \subseteq X$. Show that

$$\{A \cap O \; ; \; O \in \tau\}$$

is the subspace topology of A.

Very similar to the pullback topologies are the notions of pullback for filters and ideals. These will be leading us into the next chapter:

EXERCISE 10. Let X and Y be sets, $\pi : X \to Y$ a map, $\mathcal{F}$ a filter on Y and $\mathcal{I}$ an ideal on Y. Define:

$$\pi^{-1}[A] \in \pi^*\mathcal{F} \;:\leftrightarrow\; A \in \mathcal{F}, \text{ and}$$
$$\pi^{-1}[A] \in \pi^*\mathcal{I} \;:\leftrightarrow\; A \in \mathcal{I}.$$

(a) Show that $\pi^*\mathcal{F}$ is a filter.
(b) Show that $\pi^*\mathcal{I}$ is an ideal.
(c) Suppose $\mathcal{F}$ is an ultrafilter. Show that then $\pi^*\mathcal{F}$ is an ultrafilter.