CHAPTER I

Topology & Measure Theory

1. Fundaments of topology and measure theory

1.1. DEFINITION. Let X be an arbitrary set.

(a) We say a system $\tau \subseteq \mathcal{P}(X)$ is a **topology** iff
  (i) $\emptyset, X \in \tau$,
  (ii) $\forall A, B \in \tau(A \cap B \in \tau)$,
  (iii) $\forall \langle A_i \mid i \in I \rangle (\forall i \in I(A_i \in \tau) \longrightarrow \bigcup_{i \in I} A_i \in \tau)$.

In this situation, we call $\langle X, \tau \rangle$ a **topological space**. The elements of τ will be called **open sets**, or (if we need to specify the topology to avoid ambiguities) τ**-open sets**. The complements of sets in τ shall be called **closed sets** or τ**-closed sets**.

Let $x \in X$. If $A \subseteq X$ has the properties that $x \in A$ and that there is an open set O with $x \in O$ and $O \subseteq A$, then we call A a **neighbourhood** of x.

Let $S \in X$. We say $\mathrm{cl}_{\langle X,\tau \rangle}(S) := \bigcap \{A \mid S \subseteq A \wedge A\ \tau\text{-closed}\}$ is the **closure of** A and $\mathrm{int}_{\langle X,\tau \rangle}(S) := \bigcup \{A \mid A \subseteq S \wedge A\ \tau\text{-open}\}$ is the **interior of** A.

(b) We say a system $\mathfrak{A} \subseteq P(X)$ is a σ**-algebra** iff
  (i) $\emptyset, X \in \mathfrak{A}$,
  (ii) $\forall A \in \mathfrak{A}(X \setminus A \in \mathfrak{A})$,
  (iii) $\forall \langle A_i \mid i \in \mathbb{N} \rangle (\forall i \in \mathbb{N}(A_i \in \mathfrak{A}) \longrightarrow \bigcup_{i \in \mathbb{N}} A_i \in \mathfrak{A})$.

In this situation, we call $\langle X, \mathfrak{A} \rangle$ a **measurable space** and the elements of $\mathfrak{A}$ **measurable** of $\mathfrak{A}$**-measurable**.

We call every set $A \in \mathfrak{A}$ **measurable**.

(c) We say a system $\mathcal{F} \subseteq \mathcal{P}(X)$ is a **filter** iff
  (i) $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$,
  (ii) $\forall A \in \mathcal{F}(\forall B \subseteq X(\subseteq B \longrightarrow B \in \mathcal{F})$,
  (iii) $\forall A, B \in \mathcal{F}(A \cap B \in \mathcal{F})$.

A filter is called an **ultrafilter** when $(\forall A \subseteq X)(A \in \mathcal{F} \vee X \setminus A \in \mathcal{F})$.

(d) We say a system $\mathcal{I} \subseteq \mathcal{P}(X)$ is a **ideal** iff
  (i) $\emptyset \in \mathcal{I}$ and $X \notin \mathcal{I}$,
  (ii) $(\forall A \in \mathcal{I})(\forall B \subseteq X)(B \subseteq A \longrightarrow B \in \mathcal{I})$,
  (iii) $(\forall A, B \in \mathcal{I})(A \cup B \in \mathcal{I})$.

EXERCISE 3. Filters and ideals are dual notions. Prove that:

(a) If $\mathcal{F}$ is a filter on X, then $\{A \subseteq X \ ; \ X \setminus A \in \mathcal{F}\}$ is an ideal.
(b) If $\mathcal{I}$ is an ideal on X, then $\{A \subseteq X \ ; \ X \setminus A \in \mathcal{F}\}$ is a filter.

1.2. EXAMPLE. (a) For every set X we have the following topologies and σ-algebras:

- The **discrete topology**: $\tau_{\text{discrete}} := \mathcal{P}(X)$,
- The **discrete σ-algebra** : $\mathfrak{A}_{\text{discrete}} := \mathcal{P}(X)$,
- The **indiscrete topology**: $\tau_{\text{indiscrete}} := \{\emptyset, X\}$,
- The **indiscrete σ-algebra** : $\mathfrak{A}_{\text{indiscrete}} := \{\emptyset, X\}$.

(b) If $\langle X, d\rangle$ is a metric space, then the family of open sets as defined in any course of analysis forms a topology.

EXERCISE 4. (a) A topology is discrete if and only if every singleton is open.

(b) A σ-algebra on a countable set is discrete if and only if every singleton is open.

These examples can be called "trivial examples" with some right. Most students have known them for a long time and if these were the only interesting examples, the notion of a topology wouldn't be very interesting. As the reader probably knows, there are lots of other examples, and they fill the textbooks on general and also algebraic topology.

Since this course is not a topology course but emphasizes the set theoretic character of certain parts of topology, we will focus on those examples for topological spaces that are normally (unjustly) neglected by topologists but highly regarded by set theorists: ordinal spaces, ultrafilter spaces, and the general notion of a Polish space.

1.3. DEFINITION. Let X be an arbitrary set, $\mathcal{S} \subseteq \mathcal{P}(X)$. We say

$$\tau(\mathcal{S}) := \bigcap\{\tau' \ ; \ \mathcal{S} \subseteq \tau' \ \wedge \ \tau' \text{ topology}\}$$

is the **topology generated by** $\mathcal{S}$ and we say

$$\mathfrak{A}(\mathcal{S}) := \bigcap\{\mathfrak{A}' \ ; \ \mathcal{S} \subseteq \mathfrak{A}' \ \wedge \ \mathfrak{A}' \ \sigma\text{-algebra}\}$$

is the **σ-algebra generated by** $\mathcal{S}$.

1.4. DEFINITION. Let τ be a topology on X. We call $\mathfrak{B}(\tau) := \mathfrak{A}(\tau)$ the **Borel-σ-algebra generated by** τ .

1.5. PROPOSITION. $\tau(\mathcal{I})$ is a topology and $\mathfrak{A}(\mathcal{I})$ is a σ-algebra.

PROOF. The second part is Exercise 5, so here only the first part will be shown. Define $T_{\mathcal{I}} := \{\tau' \mid \tau' \text{ is a topology } \wedge \mathcal{I} \subseteq \tau'\}$. We will check the defining properties:

(i) $(\forall \tau' \in T_{\mathcal{I}})(\emptyset, X \in \tau')$, so $\emptyset, X \in \tau$,

(ii) Let $A, B \in \bigcap T_{\mathcal{I}}$, i.e., $(\forall \tau' \in T_{\mathcal{I}})(A, B \in \tau')$, then $(\forall \tau' \in T_{\mathcal{I}})(A \cap B \in T_{\mathcal{I}})$. So $A \cap B \in \bigcap T_{\mathcal{I}}$,

(iii) Let $\langle A_i \mid i \in I\rangle$ be an arbitrary family in $\bigcap T_{\mathcal{I}}$, i.e., $(\forall \tau' \in T_{\mathcal{I}})(\{A_i \mid i \in I\} \subseteq \tau')$, then $(\forall \tau' \in T_{\mathcal{I}})(\bigcup_{i\in I} A_i \in \tau')$. So $\bigcup_{i\in I} \in \bigcap T_{\mathcal{I}}$.

So $\bigcap T_{\mathcal{I}} = \tau(\mathcal{I})$ is a topology. q.e.d.

EXERCISE 5. (a) Prove the second claim of Proposition 1.5.

(b) Let $\mathcal{S}$ be a family of sets that is closed under finite intersections. We call such an $\mathcal{S}$ a **topology base**. Then the following holds:

$$\tau(\mathcal{S}) = \{A \subseteq X \ ; \ \exists S \subseteq \mathcal{S} \, (A = \bigcup_{B \in S} B)\ \} \cup \{X\}.$$

(c) Let τ be the discrete topology on X. Then $\mathfrak{B}(\tau)$ is the discrete σ-algebra.
(d) Show that the following are equivalent for any topological space $\langle X, \tau\rangle$:
  (i) $\tau = \mathfrak{B}(\tau)$ (in words: "every Borel set is open"), and
  (ii) every τ-open set is τ-closed.

1.6 (The order topology). Let $\langle X, <\rangle$ be a strict linear order. Define for $x \in X$: $L_x := \{y \in X \; ; \; y < x\}$, the points "left" of x and $R_x := \{y \in X \; ; \; x < y\}$, the points "right" of x. We call the topology

$$\tau_<^X := \tau(\{L_x \; ; \; x \in X\} \cup \{R_x \; ; \; x \in X\})$$

the **order-topology**. The topological space $\langle X, \tau_<^X\rangle$ will be denoted as $\mathrm{Top}_<(X)$.

1.7. REMARKS.

(i) If $\langle \mathbb{R}, <\rangle$ is the usual order on the real, then $\tau_<^{\mathbb{R}}$ is the usual topology on $\mathbb{R}$.
(ii) Let $\langle \mathbb{N}, <\rangle$ be the usual order on the natural numbers. What is $\tau_<^{\mathbb{N}}$? We have $L_1 = \{0\}, L_{n+2} = \{0, 1, \ldots, n+1\}, R_n = \{n+1, n+2, \ldots\}$. So $\{n+1\} = L_{n+2} \cap R_n \in \tau_<$. Thus every singleton is in $\tau_<^{\mathbb{N}}$, i.e., $\tau_<^{\mathbb{N}}$ is the discrete topology.
(iii) Clearly, the argument shows that the order topology on each single natural number is the discrete topology as well. Yet, we will later see that the above does not generalize to $\langle \omega + 1, <\rangle$.

1.8 (Morphisms). The morphisms of the category of the topological spaces are the **continuous functions** .

Let $\langle X, \tau\rangle$ and $\langle Y, \tau^*\rangle$ be topological spaces. We then call a function $f : X \to Y$ **continuous** or **τ-τ^*-continuous** (if we want to or have to mention the topologies) if

$$\forall A \in \tau^*(f^{-1}[A] \in \tau).$$

The morphisms of the category of measurable spaces are the **measurable functions**.

Let $\langle X, \mathfrak{A}\rangle$ and $\langle Y, \mathfrak{A}^*\rangle$ be measurable spaces. We then call a function $f : X \to Y$ **measurable** or **$\mathfrak{A}$-$\mathfrak{A}^*$-measurable** (if we want to or have to mention the σ-algebras) if

$$\forall A \in \mathfrak{A}^*(f^{-1}[A] \in \mathfrak{A}).$$

EXERCISE 6. Let X be a set, τ the discrete topology on X and τ^* the indiscrete topology on X. Let $\langle Y, \hat{\tau}\rangle$ be an arbitrary topological space. Then:

(a) Every function $\pi : X \to Y$ is τ-$\hat{\tau}$-continuous.
(b) Every function $\pi : Y \to X$ is $\hat{\tau}$-τ^*-continuous.

1.9. REMARK. Let $f : X \to Y$ be a mapping with $\langle X, \tau\rangle$, $\langle Y, \tau'\rangle$ topological spaces. For f to be continuous is a property that tells us something about the topologies τ and τ': by making τ' larger we make it harder for the function f to be continuous. On the other hand, if we make τ larger, we make it easier for f to be continuous.

Exercise 6 contains the extreme cases of this: if τ' contains the minimal sets allowed, then we don't have to meet any requirements. Moreover, if τ contains all sets, then all requirements are automatically met.

1.10. PROPOSITION. Let X be a set and τ, τ' topologies on X. Then the following are equivalent:

(a) $\tau \subseteq \tau'$,
(b) $\mathrm{id} : X \to X$ is τ'-τ-continuous.

In this case τ' is said to be **finer** than τ.

1.11. DEFINITION. Let $\langle X, \tau\rangle$, $\langle Y, \tau'\rangle$ be topological spaces, $\langle X, \mathcal{A}\rangle$, $\langle Y, \mathcal{B}\rangle$ measure spaces and let $f : X \to Y$ be a mapping. We call f:

(a) a **homeomorphism** iff f is a bijection and both f and f^{-1} are continuous,
(b) a **measure-isomorphism** iff f is a bijection and both f and f^{-1} are measurable.

If there exists a homeomorphism between $\langle X, \tau\rangle$ and $\langle Y, \tau'\rangle$ we say $\langle X, \tau\rangle$ and $\langle Y, \tau'\rangle$ are **homeomorphic** and similar we call $\langle X, \mathcal{A}\rangle$ and $\langle Y, \mathcal{B}\rangle$ **measure-isomorphic** if there exists a measure-isomorphism between $\langle X, \mathcal{A}\rangle$ and $\langle Y, \mathcal{B}\rangle$. If no confusion can occur, we generally only refer to the domains of the structures and simple say X is homeomorphic (measure-isomorphic) with Y.

1.12. PROPOSITION. Let X and Y be equipollent sets, i.e., there exists a bijection between X and Y. Then is $\langle X, \tau_{\text{discrete}}\rangle$ isomorphic with $\langle Y, \tau_{\text{discrete}}\rangle$.

PROOF. Let f be a bijection between X and Y. Then we know from Exercise 6 that f and f^{-1} are both continuous. q.e.d.

1.13. PROPOSITION. The topological spaces $\langle \omega, \tau_<\rangle$ and $\langle \omega+1, \tau_<\rangle$ are not homeomorphic.

PROOF. Suppose $f : \omega + 1 \to \omega$ is such a homeomorphism. We know $\{\omega\}$ is not open in $\langle \omega + 1, \tau_<\rangle$ (Why?). But $\tau_<$ is the discrete topology on ω, so $\{f(\omega)\}$ is open in $\langle \omega, \tau_<\rangle$. But $f^{-1}[\{f(\omega)\}] = \{f^{-1}(x) \mid x \in \{f(\omega)\}\} = \{f^{-1}(f(\omega))\} = \{\omega\}$, so f is not continuous, a contradiction. q.e.d.

At this point we know that all ordinals up to $\omega+1$ have different topologies. This is not true after that point: $\langle \omega + 2, \tau_<\rangle$ and $\langle \omega + 1, \tau_<\rangle$ are homeomorphic: Both of them have countably many isolated points and exactly one limit point. Any bijection mapping isolated points to isolated points and fixing the limit point is a homeomorphism.

What about measure isomorphisms? If you look at any two countable ordinals, their Borel σ-algebras coincide as the following exercise shows:

EXERCISE 7. (a) Sei α eine beliebige abzählbare Ordinalzahl mit der Ordnungstopologie. Zeigen Sie, daß $\mathfrak{B}(\tau_<)$ die diskrete σ-Algebra ist.

(b) Listen Sie alle Topologien und σ-Algebren auf der Menge $3 = \{0, 1, 2\}$ auf. Welche der Topologien ist die Ordnungstopologie? Welche der σ-Algebren ist die von der Ordnungstopologie erzeugte σ-Algebra?

But this doesn't generalize to homeomorphisms. The type of limit points and their number allow us to distinguish between different topologies on ordinals:

EXERCISE 8. (a) Let n and m be natural numbers with $n \neq m$. Show: $\langle \omega \cdot n, \tau_< \rangle$ and $\langle \omega \cdot m, \tau_< \rangle$ are not homeomorphic.

(b) Show that none of the spaces $\langle \omega \cdot n, \tau_< \rangle$ is homeomorphic to $\langle \omega^2, \tau_< \rangle$.

Later on, using notions of set theory, we'll be able to completely classify all ordinal spaces up to homeomorphism.