TWIP Welch

Let $\langle \kappa, <, F_n \rangle_{n<\omega}$ (F_n - finitary, often we expand κ to H_κ or some H_θ, and F_n includes Skolem functions for the structure)

<u>DEF</u> κ has the <u>free subset-property</u> $\Longleftrightarrow$

$$\forall \mathcal{M} \ \exists X \text{ cof. in } \kappa \text{ which is free, namely}$$

$$\forall x \in X \quad x \notin SH^{\mathcal{M}}(X - \{x\})$$

κ regular $\overset{FSP}{\Longrightarrow}$ κ Jonsson

But we're interested in singular κ.

<u>THM</u> (Koepke) $\aleph_\omega$ has FSP $\overset{\text{Equic. ZFC}}{\Longleftrightarrow}$ $\exists \kappa$ (κ a measurable)

<u>DEF</u> $N \prec \mathcal{M}$, N is <u>Internally Approachable</u> of type μ (μ reg, $\mu < \kappa$), if $N = \bigcup_{\alpha<\mu} N_\alpha$ and $\forall \alpha < \mu \ \langle N_\beta \mid \beta \le \alpha \rangle \in N_{\alpha+1}$;

$\chi_N(\tau) = \sup(N \cap \tau)$ for $\tau \in N \cap \kappa$.

<u>DEF</u> (κ sing) (Pereira) κ has the <u>Approachable FSP</u>

$\Longleftrightarrow$ $\forall \mathcal{M}$ on κ $\exists N \prec \mathcal{M}$ N IA of type $<\kappa$ $\exists X \subseteq \operatorname{ran} \chi_N$

s.t. (i) X cofinal in κ

(ii) $otp(X) = cf(\kappa)$

(iii) $\forall x \in X \ x \notin SH^{\mathcal{M}}(N \cup X - \{x\})$

κ –

τ' –

$\chi_N(\tau') \uparrow$

τ

$\chi_N(\tau) \uparrow$

Why consider this?

<u>Recall</u> pcf - conj. If A is a set of reg. card., $|A| < \min A$

$$|pcf(A)| = |A|.$$

Shelah shows

<u>Thm</u> If A is as above, but $|pcf(A)| > |A|$, then $(*)$ $\forall N$ IA of suff. large type, that contains $pcf(A)$, there exists

$$\mu_n < \mu_{n+1} \cdots \quad (n < \omega) \quad \text{s.t.}$$

$$\forall \ell \ \forall \vec{n} = n_1 < n_2 < \ldots < n_m$$

$$\chi_N(\mu_\ell) \notin SH^{\mathcal{M}}(N \cup \{\chi_N(\mu_{n_1}), \ldots, \chi_N(\mu_{n_m})\})$$

<u>Cor</u> For example: $\aleph_\omega$ a strong limit, $\aleph_\omega^{\aleph_0} > \aleph_{\omega_1}$

$F : \aleph_\omega^{<\omega} \longrightarrow \aleph_\omega$ is an algebra, $F \in N$. Then we have $\{ X_N (\aleph_{m_n}) \mid$ some $m_n < m_{n+1} < \omega \}$ is free with respect to F. In particular $FSP(\aleph_\omega)$.

Question How strong is AFSP?

$\leadsto$ Not known to be consistent rel. to large cardinals.

THM (ZFC). Suppose $\forall F : \aleph_\omega^{<\omega} \to \aleph_\omega \ \exists N \prec H_\theta$ N IA of type $\aleph_x$ (for some $x < \omega$) with $F \in N$, s.t. some ω-subset of $\{ X_N(\aleph_m) \mid k < m < \omega \}$ is free for F. $\sim$?

Then for arbitrarily large $m > k$

$$\{ \alpha < \omega_m \mid o^{K^{\text{mitchell}}}(\alpha) \geq \omega_k \} \text{ is stationary.}$$

Conclusion AFSP $\Rightarrow$ This property.

- This should have stronger (AC) consequences
- Question: Is it possible to have AFSP "at" $\aleph_\omega$, without AC, assuming an I.H of a single measurable ?