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Aim: Get large cardinal strength from „All cardinals $> \omega$ are singular" ($=:\varphi$) (consistent by work of Moti Gitik)

We will use:

$\boxed{\text{Core model thy}}$ $\boxed{\text{descriptive set thy}}$

core model induction $+ \ \varphi \rightsquigarrow$ lower bounds

(why not just core model thy?)

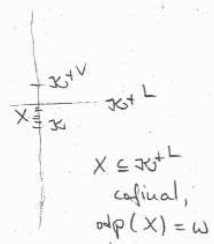
· Why does φ give $0^{\#}$? (J. covering Lemma)

No inner model with choice can compute $\varkappa^+$ correctly, for all $\varkappa$.

$L \subset HOD[X]$.

Jensens Cov. Lemma yields Either ~~$\exists Y \supseteq X, \bar{\bar{Y}} \leq \aleph_1$~~, $Y \in L$

or else $0^{\#}$ ex



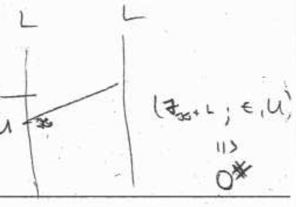
Properties of L used in this proof:

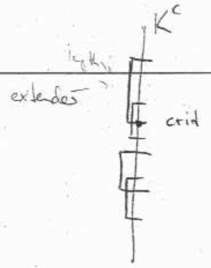
- $L^{HOD} = L^{HOD[X]}$ where X is gen. / HOD forcing absoluteness of L
- weak covering for L, i.e., if $\varkappa$ is singular cardinal, then $\varkappa^{+L} = \varkappa^{+}$ (?)

Now we want to produce a model K which has these properties, but which also has $0^{\#}$ in it, etc.

How do we produce such a K?

First produce K^c, a preliminary version of K. c stands for "certified"



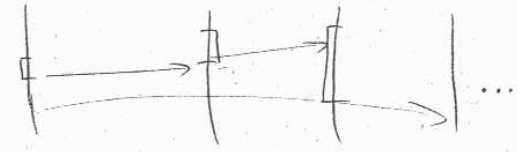


K^c is not forcing absolute, also K^c does not seem to satisfy weak covering (in general)

Using K^c, we want to take a hull $K \cong X \prec K^c$.

In order to do that, we need the full iterability of K^c, we need:

$ult_0 = K^c \quad ult_1 \quad ult_2 \quad ult_3$



yields tree structure

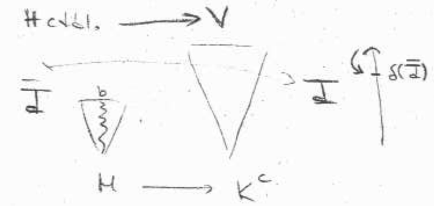
„one can always pick the right branch, if one did not do a mistake before"

K^c is "weakly iterable": if $M \to K^c$, M countable, then there is an iteration strategy for M w.r.t. countable iteration trees

We want: K^c is fully iterable

If (*) V is closed under Q-structures, then (1) $\Rightarrow$ (2).

$Q \trianglelefteq M_b^{\mathcal{I}}$
$Q \models$ "$\delta(\bar{\mathcal{I}})$ is not woodin"

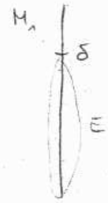


However, (*) is sometimes false, and in fact K^c need <u>not</u> be fully iterable.

Absoluteness argument would yield $b \in M$

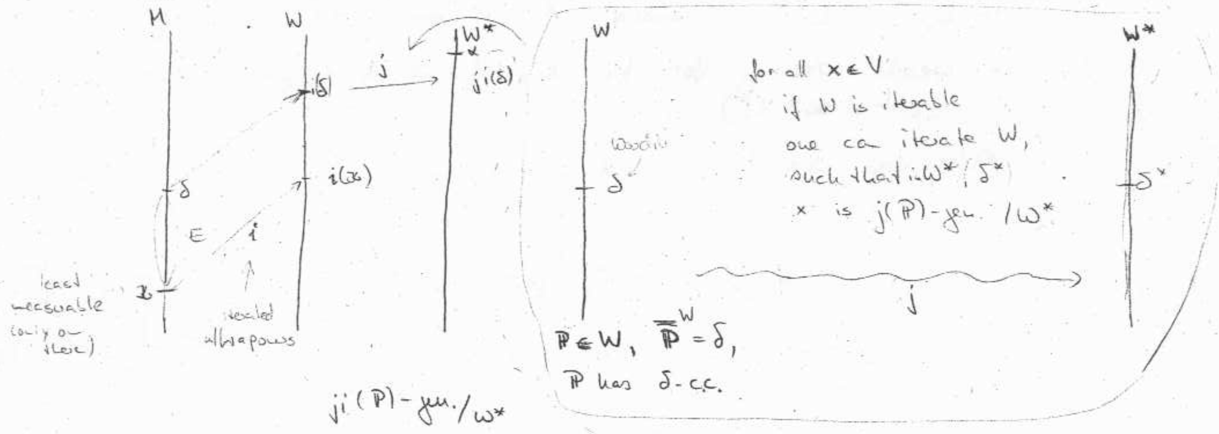
Look at M_1:

$M_1 \models$ "δ is Woodin"
$\| $
$L[E]$, where $E \subseteq \delta$



We prove, that M_1 (is not iterable) does not know, how to iterate itself:

Suppose $M_1 \models$ "I'm fully iterable".



$ji(\mathbb{P})$-gen.$/_{W^*}$

$W^*[E] = L[E]$

$\alpha = ji(\delta^{+M_1}) = ji(\delta)^{+W^*[E]} = ji(\delta)^{+M_1}$

$\Rightarrow ji'' \delta^{+M_1}$ is cof in $ji(\delta)^{+M_1}$

(nonsense for large enough δ)

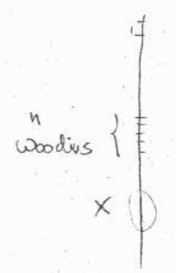
Core model induction:

we showed: $\varphi \Rightarrow 0^\#$ ex. in fact $X^\#$ ex. for all X

First ω induction steps: Use φ to show:

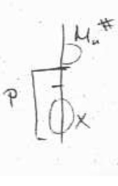
Assume $M_n^\#(X)$ ex. f.a. X. Show $M_{n+1}^\#(X)$ ex. f.a. X.

$L^{M_n^\#}(H_\vartheta^{HOD})$, look at $K^c(X)^{L^{M_n^\#}(H_\vartheta^{HOD})} =: W$



Possibilities:

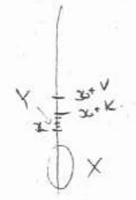
- W has a Woodin cardinal



so we're finished

- W is fully iterable

isolate (using full iterability) $K(X)^{L^{M_n^*}(H_\vartheta^{HOD})}$



$L^{M_n^*}(H_\vartheta^{HOD})[Y]$ ... (?)

Construct K + use φ to get $\mathcal{Y}$.

So $\forall n \forall X$ $M_n^*(X)$ ex. $\Rightarrow$ Proj. Det. =: PD

PD = $J_2(\mathbb{R}) \models AD$

$J_1(\mathbb{R}) = V_{\omega+1}$
$J_2(\mathbb{R})$ = rud. cl. of $J_1(\mathbb{R}) \cup \{J_1(\mathbb{R})\}$.
So $\mathcal{P}(\mathbb{R}) \cap J_2(\mathbb{R})$ = the proj. sets

Now we want to prove inductively

$J_\alpha(\mathbb{R}) \models AD$ f.a. α, using φ.

If α is the least one with

$J_\alpha(\mathbb{R}) \models \neg AD$, $\alpha = \beta + 1$, some β

In "$\neg AD$" is Σ_1, so $J_\alpha(\mathbb{R}) \models$ a Σ_1-statement which is false in all $J_\gamma(\mathbb{R})$, $\gamma < \alpha$

<u>Def.</u> $[\alpha, \beta]$ is a Σ_1-gap iff $J_\alpha(\mathbb{R}) \prec^{\mathbb{R}}_{\Sigma_1} J_\beta(\mathbb{R})$, and there is no $\bar{\alpha} < \alpha$ s.t. $J_{\bar{\alpha}}(\mathbb{R}) \prec^{\mathbb{R}}_{\Sigma_1} J_\alpha(\mathbb{R})$. There's no $\bar{\beta} > \beta$ $J_\beta(\mathbb{R}) \prec^{\mathbb{R}}_{\Sigma_1} J_{\bar{\beta}}(\mathbb{R})$.

The gaps partition the ordinals.
The first place where AD fails begins a gap.

THEOREM (<u>Kechris - Woodin</u>?). $AD^{L(\mathbb{R})}$ follows from: $L(\mathbb{R}) \models$ every set which has scale is determined

DEF. α is critical iff there is a set $A \subseteq \mathbb{R}$ which has scales in $J_{\alpha+1}(\mathbb{R})$ but not in $J_\alpha(\mathbb{R})$.

By the K.-W.-Thm., it suff. to show: If α is critical $\underbrace{\ }_{+\, J_\alpha(\mathbb{R}) \models AD \, + \, \varphi}$, then $J_{\alpha+1}(\mathbb{R}) \models AD$

Critical ordinals:

use certain mice, obtain structures closed under Q-operator

<u>First Case:</u> α is $\mathbb{R}$-inadmissible and
(a) $cf(\alpha) = \omega_1$,
(b) $cf(\alpha) = \omega$ or
(c) α is successor ordinal, but the previous gap is not strong

<u>Second Case:</u> "end of the gap case"

(a) α ends a proper weak gap, $[\bar{\alpha}, \alpha]$ or
(b) α is a succ. ordinal and the previous gap is strong

Thm. (Neeman): Let W be an inner model, $W \models \delta$ is Woodin, $\overline{\overline{\mathcal{P}(\delta) \cap W}} = \aleph_0$. Let $A \subseteq \mathbb{R}$. Supp. $\exists\, T \in W^{Col(\omega,\delta)}$ and $\exists$ it. strat. Σ for W s.t. whenever $i: W \to W^*$ is according to Σ, then for all g $Col(\omega, i(\delta))$-gen. / W^*,

$$A \cap W^*[g] = i(T)^g.$$

Then A is determined.

The argument in Case 2(a):

Starting point: We have a ctble N

for $A \in J_\alpha(\mathbb{R})$, there is $T \in N^{Col(\omega,\delta)}$ "capturing" A

N

δ

closed under $C_{\Sigma_n^{J_{\bar{\alpha}}(\mathbb{R})}}$ $C_{\Sigma_n J_{\bar{\alpha}}(\mathbb{R})}$

$[\bar{\alpha}, \alpha]$ $\quad$ $\rho_n(J_\alpha(\mathbb{R})) = \mathbb{R}$

$B \in \Sigma_n^{J_\alpha(\mathbb{R})} \iff B = \bigcup_{n\in\omega} A_n$, $A_n \in J_\alpha(\mathbb{R})$

Go for a mouse $\mathcal{M}$ s.t.

$\mathcal{M}$

δ

$\mathcal{N}$

- $\mathcal{N} \in \mathcal{M}$, $\mathcal{M} \models \mathcal{N}$ ctbl., $\mathcal{M}$ can see $(T_{A_n} \mid n < \omega)$
- $\mathcal{M} \models \delta$ is Woodin
- $\mathcal{M}$ can see $\Sigma \cap \mathcal{M}$, as given by $\dot{\Sigma}^{\mathcal{M}} = \Sigma \cap \mathcal{M}$
- in fact $\mathcal{M}$ is iterable via Γ s.t. if $j: \mathcal{M} \to \mathcal{M}^*$ is acc. to Γ, then $j(\dot{\Sigma})^{\mathcal{M}^*} = \Sigma \cap \mathcal{M}^*$
- in fact, if $j: \mathcal{M} \to \mathcal{M}^*$ is (like) above and g is $Col(\omega, j(\delta))$-gen / $\mathcal{M}^*$, then from $j(\dot{\Sigma})^{\mathcal{M}^*}$ you can „easily" read of $\Sigma \cap \mathcal{M}^*[g]$.

Claim: $\mathcal{M}$ is a term capturing $B = \bigcup A_n$.

(„hybrid mouse")