The Core model induction beyond $L(\mathbb{R})$

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THEOREM. Assume CH and there ex. $j : V \to M \subseteq V[G]$, s.t.

(1) $cp(j) = \omega$

(2) $(M^\omega)^{V[G]} \subseteq M$

(3) G is for a homogeneous forcing

(4) $\varepsilon =$ " $j \restriction Ord \in V$ "

Then there is $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ s.t. $L(\Gamma, \mathbb{R}) \models AD_{\mathbb{R}} + \Theta$ is reg.

1. Why prove such a thm
2. How to prove it.

1. Thm. (S.-Woodin)

$Con(AD_{\mathbb{R}} + \Theta \text{ is reg.}) \iff Con(\text{thy hypo})$

The inner model problem (since 1960's):
Construct ("canonical") models for large cardinals like superstrong or sc.
$\triangleq$ mice

Mouse

premouse $\mathcal{M} = L_\alpha[\vec{E}]$ (extenders) first-order property

mouse an iterable premouse

$\mathcal{M} = \mathcal{M}_0 \ni E_0$ $\to$ $Ult(\mathcal{M}_0, E_0) = \mathcal{M}_1 \ni E_1$ ...

$\to$ $Ult(\mathcal{M}_0, E_2) = \mathcal{M}_3$

$\to$ $Ult(\mathcal{M}_0, E_1) = \mathcal{M}_2$

what to do at $\mathcal{M}_\omega$?

I produces Ult

II chooses br. at limit steps

$\mathcal{M}$ is iterable iff II has a winning strategy

(Neeman) suppose there is a measurable cardinal above a Wlim then there is a countable mouse with a Wlim

Wlim := Woodin limit of Woodins

Core model induction is a technique for solving the inner model problem in a way that is independent of ext. assumptions.

For core model induction the background theory is AD^+
(when wanting mice with large cardinals)

"Characterize" the set of reals of a model of AD^+ via mice that exist in the model. If there are complicated sets in AD^+ "characterized" by mice, then there are complicated mice. (compl. mice = mice with large card.)

Applications: • the previously stated theorem
• (Steel) $\neg \Box_\kappa$, κ sing. $\overset{\text{str. limit}}{\Longrightarrow}$ $AD^{L(\mathbb{R})}$ "=" ω-Woodin

THEOREM. ① (Woodin) $Con(AD^{L(\mathbb{R})}) \longleftrightarrow Con(\omega\text{-Wdns})$
② (Steel, Woodin) $Con(AD_{\mathbb{R}}) \longleftrightarrow AD_{\mathbb{R}}$-hypo
= "? is a limit of Wdns and ?-strong"

Solovay hierarchy (assume AD^+)

$$\Theta = \sup\{\alpha \mid \exists f : \mathbb{R} \xrightarrow{\text{onto}} \alpha\}.$$

Define a sequence $\langle \theta_\alpha \mid \alpha \le \Omega \rangle$

• $\theta_0 = \sup\{\alpha \mid \exists f : \mathbb{R} \xrightarrow{\text{onto}} \alpha$, s.t. f is OD$\}$

• if $\theta_\alpha < \Theta$ then fix $A \subseteq \mathbb{R}$ s.t. $w(A) = \theta_\alpha$
„continuous preimage of something"

and let

$$\theta_{\alpha+1} = \sup\{\alpha \mid \exists f : \mathbb{R} \xrightarrow{\text{onto}} \alpha \wedge f \text{ is } OD(A)\}$$

($\theta_{\alpha+1}$ is ind of A)

$$\theta_\lambda = \sup_{\alpha<\lambda} \theta_\alpha.$$

Ω : the least s.t. $\theta_\Omega = \Theta$.

$$AD^+ + \theta_0 = \Theta \le_{con} AD^+ + \theta_1 = \Theta$$
$$\le_{con} \cdots AD^+ + \theta_\omega = \Theta$$

(Woodin) (AD^+) $AD_{\mathbb{R}} \longleftrightarrow \Omega$ is limit

$AD_{\mathbb{R}} + \theta_\Omega$ reg. $\longleftrightarrow \Theta = \Omega \wedge \Omega$ is reg.

Recall $CH + j : V \longrightarrow M \subseteq V[G]$
• $(M^\omega)^{V[G]} \subseteq M$
• G comes from hom. forcing
• $cp(j) = \omega_1$
• $j \upharpoonright Ord \in V$

Let $S := AD_{\mathbb{R}} + \Theta$ is reg.

Let T be the hypo

Assume not, Let $\Gamma = \{ A \subseteq \mathbb{R} \mid L(A, \mathbb{R}) \models AD^+ \}$

1. show that $L(\Gamma, \mathbb{R}) \models AD^+$
2. Show there is $A \subseteq \mathbb{R}$ s.t. $L(A, \mathbb{R}) \models AD^+ \wedge A \notin \Gamma$

Woodin's idea

Show that there is a mouse $\mathcal{M}$ s.t. $L(\Gamma, \mathbb{R}) \models$ "$\mathcal{M}$ is not iterable", but $\mathcal{M}$ is iterable. Let A be the wde (?) set of the strategy of $\mathcal{M}$. shows that $L(A, \mathbb{R}) \models AD^+$ (just like in Ralf's talk)

How to add this real M?

Woodin's idea

In some cases $HOD \restriction \Theta$ of models of AD^+ is a "mouse". Let $\mathcal{H}$ "=" $HOD^{L(\Gamma, \mathbb{R})} \restriction \Theta$.

Look for a strategy for $\mathcal{H}$.

In M $\mathcal{H}$ is countable ($|\mathcal{H}| = \omega_1^V$ and ω_1^V is countable in M)

$\mathcal{H} = L_p(\mathcal{H})$.

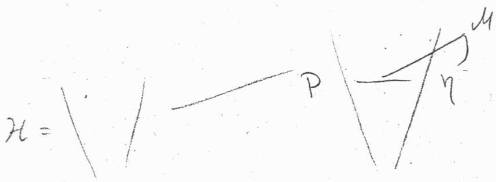
Use $j \restriction \mathcal{H} : \mathcal{H} \to j(\mathcal{H})$ to define a strategy for $\mathcal{H}$.

$$\begin{tikzcd} \mathcal{H} \arrow[r, "j"] \arrow[d, "\mathcal{F}"', "b"] & j(\mathcal{H}) \\ \mathcal{M}_b \arrow[ur, "\sigma"'] & \end{tikzcd}$$

i_b

Let Σ be this strategy. Show that $j(\mathcal{H})$ is a Σ-it. of $\mathcal{H}$ $\qquad \mathcal{H} < \cdots j(\mathcal{H})$

(show that Σ is f. pr.)



then $\mathcal{M} \in P$

Pull back there is (P, Λ) in V s.t. $\mathcal{H}$ is a Λ-it. of P and P is countable.

Thus $\Lambda \notin \Gamma$

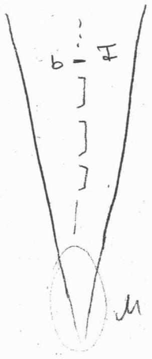
Θ $= \mathcal{H}$ $\Rightarrow \Lambda \notin \Gamma$

$\{ Q \mid Q$ is a Λ-it of $P \}$

the last step is show that if $A \subseteq \mathbb{R}$ codes Λ then $L(A, \mathbb{R}) \models AD^+$

<u>Hybrid mice</u> $\quad \mathcal{M}, \Sigma$ strategy

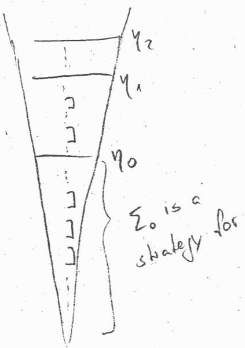
Mouse $L_\alpha[\vec{E}]$



$\mathcal{N}$ is a Σ-premouse if $\mathcal{N} = L_\alpha[\vec{E}, \Sigma]$ $[\mathcal{M}]$

P is a hybrid-mouse if there ex. Σ s.t. P is a Σ-mouse.

Σ: strategy for some $\mathcal{M}$

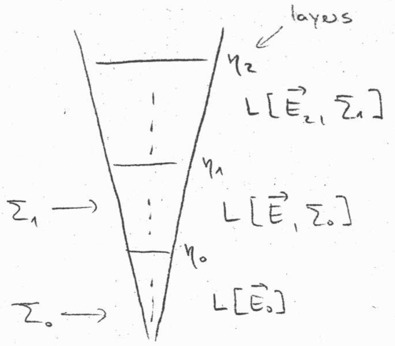


Def. Suppose $\mathcal{M}$ is a mouse and Σ is an it. st. for $\mathcal{M}$. Σ has branch condensation if

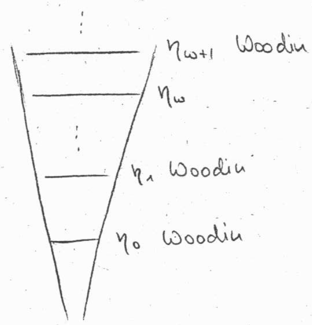
$$\begin{array}{ccc} & \xrightarrow{\mathcal{I} \quad \varepsilon(\mathcal{I}) = b} & \mathcal{M}_b^{\mathcal{I}} \\ \mathcal{M} & & \uparrow \\ & \xrightarrow{\mathcal{U}, c} & \mathcal{M}_c^{\mathcal{U}} \end{array}$$

then $c = \Sigma(\mathcal{U})$.

<u>Rigidly layered hybrid mice</u>



no new bounded subsets of η's



<u>hod mice</u>

Layers = Woodins or limits of Woodins

<u>Thm</u> (Woodin) (AD)

If $\Theta_{\alpha+1}$ exists then $\Theta_{\alpha+1}$ is Woodin in HOD.

<u>Theorem</u> (AD^+) Assume there is no $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ s.t. $L(\Gamma, \mathbb{R}) \models AD_{\mathbb{R}} + \Theta$ is reg.

Then $HOD \restriction \Theta$ is a hod premouse.

Mouse set conjecture

Conjecture (Steel & Woodin)

Assume AD^+ + there is no mouse with a superstrong.

$\forall x, y \in \mathbb{R}$, $x \in OD(y) \iff$ there is y-mouse $\mathcal{M}$ s.t. $x \in \mathcal{M}$.

Thm AD^+ + no model of $AD_{\mathbb{R}} + \Theta$ reg $\Rightarrow$ MSC

Thm AD^+ + largest Suslin isn't a member of Sol. seq. Then MSC

No special place

Thm (Kleene) $x \in \Delta^1_1(y) \iff x \in L_{\omega_1^{CK}(y)}[y]$

(Schoenfield) $x \in \Delta^1_2(y)$ in a countable ordinal $\iff x \in L[y]$

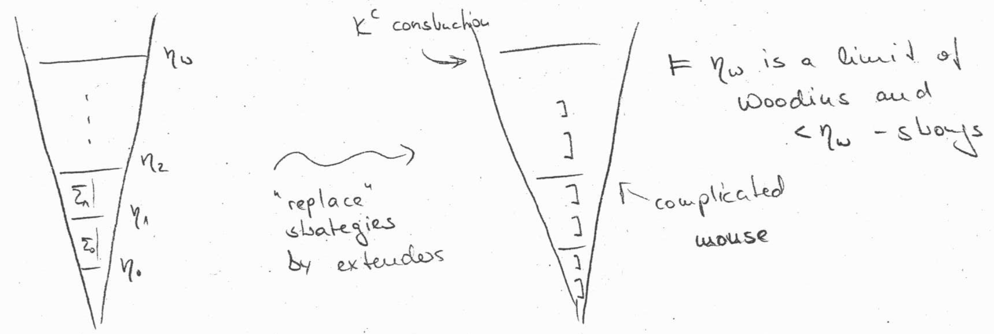
(Steel & Woodin) $x \in \Delta^1_{2n+2}(y)$ in a countable ord. $\iff x \in M^{\#}_{2n}(y)$

$x \in HOD \Rightarrow x$ is in a mouse

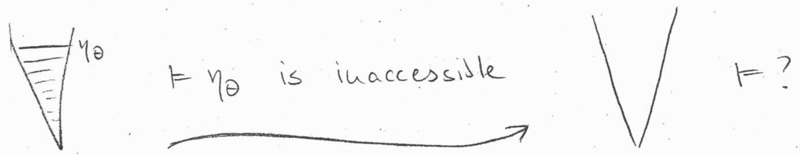
$\Rightarrow \mathbb{R}^{HOD} = \bigcup\{\mathcal{M} \mid \rho_\omega(\mathcal{M}) = \omega$, $\mathcal{M}$ is sound, $\mathcal{M}$ is a mouse$\}$

$\Rightarrow HOD \models CH$

The Woodin Cardinals of hod mice



What is the large cardinal of $AD_{\mathbb{R}} + \Theta$ reg. ?



Thm (Woodin)

Con(a Woodin limit of Woodins)

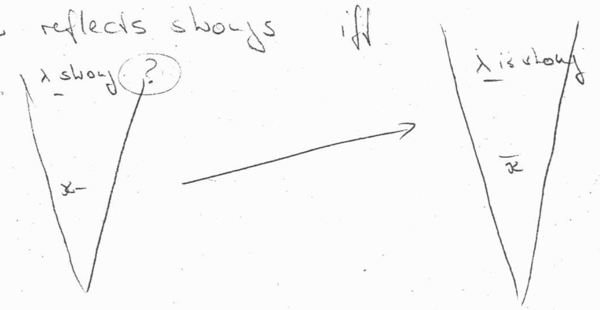
$\Rightarrow$ Con(there are $A, B \subseteq \mathbb{R}$ s.t. $L(A, \mathbb{R}) \models AD^+ \wedge L(B, \mathbb{R}) \models AD^+$ but $L(A, B, \mathbb{R}) \not\models AD^+$)

(A, B aren't Wadge comp.)

(*) <u>Thm</u> Con(There are div. models of AD^+)

$\Rightarrow$ Con($AD_{\mathbb{R}}$ + Θ is reg.)

Say κ reflects strongs iff



<u>Thm</u> Con(λ is a reg. limit of Woodins and $<\lambda$-strongs reflecting strongs)

$\rightarrow$ Con($AD_{\mathbb{R}}$ + Θ is reg)

<u>Open</u> Con(All cardinals are singular)

$\Rightarrow$ Con ($AD_{\mathbb{R}}$ + Θ reg.)

Sketch of Proof of (*) :

$L(A, \mathbb{R}) \models AD^+$, $L(B, \mathbb{R}) \models AD^+$

$L(A, B, \mathbb{R}) \not\models AD^+$

Want : $AD_{\mathbb{R}}$ + Θ reg.

Assume not. $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R}) \cap L(B, \mathbb{R})$

$\Gamma \neq \emptyset$.

$L(\Gamma, \mathbb{R}) \models AD^+$ (Woodin, $AD_{\mathbb{R}}$)

Let (P, Σ) and (Q, Λ) s.t.

$HOD^{L(P,\mathbb{R})} \restriction \Theta^{L(P,\mathbb{R})}$ a Σ-it. of P and a Λ-it. of Q and

$(P, \Sigma) \in L(A, \mathbb{R})$ and $(Q, \Lambda) \in L(B, \mathbb{R})$

In V compare $(P, \Sigma) \longrightarrow (R, \Psi) \in L(A, \mathbb{R}) \cap L(B, \mathbb{R}) \in \Gamma$

$(Q, \Lambda) \nearrow$

$\rightarrow \leftarrow$