Infinitary Combinatorics without the Axiom of Choice
Consistency Strengths of Choiceless Failures of SCH

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Outline

The ICWAC Project

- Study strong combinatorial principles like Chang's Conjecture, Rowbottom Cardinals, $\neg\mathrm{SCH}$, ... without assuming AC
- Consistency strengths go down without AC and become amenable to forcing and inner model arguments for relatively small large cardinals
- Equiconsistencies are possible in several cases
- (Also combinatorics under AD)
- Joint DFG-NWO project with Benedikt Löwe and Arthur Apter

Cardinals without AC

- $\kappa = \lambda^+$
  $\leftrightarrow\ \forall\gamma < \kappa\ \exists f : \gamma \to \lambda$ injective
  $\leftrightarrow\ (\mathrm{AC!!})\ \exists F : \kappa \times \kappa \to \lambda\ \forall\gamma < \kappa\ F(*, \nu) : \gamma \to \lambda$ injective
- Under AC, $\kappa = \lambda^+$ is not Ramsey:
  define a partition $P : \kappa^3 \to 2$ by $P(\alpha, \beta, \gamma) = 1$ iff
  $F(\alpha, \gamma) < F(\beta, \gamma)$, for $\alpha < \beta < \gamma < \kappa$

Cardinals without AC

To get strong combinatorial properties at accessible cardinals

- arrange $\forall\gamma < \kappa\ \exists f : \gamma \to \lambda$ injective
  without $\exists F : \kappa \times \kappa \to \lambda\ \forall\gamma < \kappa\ F(*, \nu) : \gamma \to \lambda$ injective
- use symmetric submodels N of forcing extensions
- make N a limit of models $M_i \vDash \mathrm{ZFC}$:
  $N \cap \mathcal{P}(\mathrm{Ord}) = \bigcup_i (M_i \cap \mathcal{P}(\mathrm{Ord}))$
- let every M_i be a “small” forcing extension of the ground model V

Example: Chang's Conjecture

- Let $\kappa \to (\omega_2)_2^{<\omega}$
- Levy collapse κ to ω_3: $V[G] \models \kappa = \omega_3$
- Let N be a submodel of $V[G]$ spanned by $V[G \restriction i]$ for $i < \kappa$
- $N \cap \mathcal{P}(\mathrm{Ord}) = \bigcup_i (V[G \restriction i] \cap \mathcal{P}(\mathrm{Ord}))$
- $V[G \restriction i]$ is a small forcing extension relative to κ
- $V[G \restriction i] \models \kappa \to (\omega_2)_2^{<\omega}$
- $N \models \kappa \to (\omega_2)_2^{<\omega}$
- $N \models$ Chang's Conjecture for (ω_3, ω_2)
- Chang's Conjecture for (ω_3, ω_2) is equiconsistent with $\exists \kappa \quad \kappa \to (\omega_2)_2^{<\omega}$

Outline

Choiceless Failures of SCH

Theorem

For a fixed $\alpha \geq 2$, the following theories are equiconsistent:

$$ZFC + \exists\kappa[\kappa \text{ is measurable}]$$

and

$$ZF + \neg AC + GCH \text{ holds below } \aleph_\omega +$$

$$\text{There is a surjective } f : [\aleph_\omega]^\omega \to \aleph_{\omega+\alpha}.$$

Choiceless Failures of SCH

Theorem

For a fixed $n < \omega$, $n \geq 1$, the following theories are equiconsistent:

$$ZFC + \exists\kappa[(\mathrm{cof}(\kappa) = \omega)$$

$$\wedge(\forall i < \omega)(\forall\lambda < \kappa)(\exists\delta < \kappa)[(\delta > \lambda) \wedge (o(\delta) \geq \delta^{+i})]]$$

and

$$ZF + \neg AC + GCH \text{ holds below } \aleph_\omega +$$

$$\text{There is an injective } f : \aleph_{\omega_n} \rightarrow [\aleph_\omega]^\omega.$$

Outline

Parallel Prikry forcing

Fix a normal measure $\mathcal{U}$ on κ and a set $Z \subseteq \mathrm{Ord}$.
$p = (s_\alpha, A_\alpha)_{\alpha \in Z}$ is a condition in $\mathbb{P}$ iff

1. $\forall \alpha \in Z[(s_\alpha \in [\kappa]^{<\omega}) \wedge (A_\alpha \in \mathcal{U}) \wedge (\max(s_\alpha) < \min(A_\alpha))]$
2. $\mathrm{dom}(p) := \{\alpha \in Z \mid A_\alpha \neq \kappa\}$ is finite.

Write (s_α, A_α) instead of $(s_\alpha, A_\alpha)_{\alpha \in Z}$.

The partial order on $\mathbb{P}$

Conditions $p' = (s'_\alpha, A'_\alpha)$ and $p = (s_\alpha, A_\alpha)$ in $\mathbb{P}$ are partially ordered by $p' \leq p$ iff there is an integer $n < \omega$ such that

1. $\forall\alpha \in \mathrm{dom}(p)[(\mathrm{otp}(s'_\alpha \setminus s_\alpha) = n) \wedge (s'_\alpha \setminus s_\alpha \subseteq A_\alpha)]$.
2. $(\forall\alpha, \beta \in \mathrm{dom}(p))(\forall\xi \in s'_\alpha \setminus s_\alpha)(\forall\zeta \in s_\beta)[\xi > \zeta]$.
3. $(\forall\alpha < \beta \in \mathrm{dom}(p))(\forall i < n)[(s'_\alpha \setminus s_\alpha)[i] < (s'_\beta \setminus s_\beta)[i]]$ ($s[i]$ is the i-th element of the monotone enumeration of the set s)
4. $(\forall\alpha, \beta \in \mathrm{dom}(p))(\forall i < n)[(i + 1 < n) \implies ((s'_\alpha \setminus s_\alpha)[i] < (s'_\beta \setminus s_\beta)[i + 1])]$.
5. $\forall\alpha \in \mathrm{dom}(p)[A'_\alpha \subseteq A_\alpha]$.

The partial order on $\mathbb{P}$

1. The stems s_α are extended into the corresponding reservoir sets A_α in a systematic fashion.
2. The extension points are chosen greater than all of the previous stem points.
3. There are the same number of new points at all indices in $\mathrm{dom}(p)$, and these are chosen in layers which are strictly ascending.
4. Reservoirs may be thinned out, and new stems outside the old domain may be grown.

Properties of $\mathbb{P}$

Let G be $\mathbb{P}$-generic over V. G adjoins a system $(C_\alpha \mid \alpha \in Z)$

$$C_\alpha = \bigcup \{s_\alpha \mid (s_\beta, A_\beta)_{\beta \in Z} \in G\}.$$

Lemma

a) *Let $\gamma \in Z$. Then C_γ is a Prikry sequence for $\mathcal{U}$, i.e.,*

$$\forall X \in \wp(\kappa) \cap V[(X \in \mathcal{U}) \Longleftrightarrow (C_\gamma \setminus X \text{ is finite})].$$

b) *Let $\gamma, \delta \in Z$, $\gamma < \delta$. Then $C_\gamma \cap C_\delta$ is finite, and $C_\gamma \Delta C_\delta$ is infinite.*

Properties of $\mathbb{P}$

Lemma

$(\mathbb{P}, \leq)$ *satisfies the* κ^+*-chain condition.*

The symmetric extension

Define

$$N = \mathrm{HOD}^{V[G]}(\bigcup_{\alpha \in Z} \tilde{C}_\alpha \cup \{(\tilde{C}_\alpha \mid \alpha \in Z)\}),$$

where $\tilde{C}_\alpha = \{C \in \wp(\kappa) \mid C \Delta C_\alpha \text{ is finite}\}$. N is the class of sets which are hereditarily definable in the generic extension from finitely many parameters from the class $\mathrm{Ord} \cup \{C_\alpha \mid \alpha \in Z\} \cup \{(\tilde{C}_\alpha \mid \alpha \in Z)\}$.

The powerset of κ is large

Lemma

In N, there is a surjection $f : [\kappa]^\omega \to Z$.

Proof.

Define f using the parameter $(\tilde{C}_\alpha \mid \alpha \in Z)$ by

$$X \mapsto \begin{cases} \text{The unique } \alpha \in Z \text{ such that } X \in \tilde{C}_\alpha, \text{ if that exists,} \\ 0, \text{ otherwise.} \end{cases}$$

□

Finite support approximations

Lemma

Let G be $\mathbb{P}_Z$-generic for V, where $\mathrm{card}(Z) < \omega$. Then $V[G]$ is an extension of V by Prikry forcing $\mathbb{P}_1$. Therefore, by the properties of standard Prikry forcing, $V[G]$ has the same bounded subsets as V.

Lemma

Let G be $\mathbb{P}$-generic, with $C_\alpha = (\dot{C}_\alpha)^G$ for $\alpha \in Z$ and $D = \dot{D}^G$. Let $X \in V[G]$ be defined by

$$X = \{\zeta \in \mathrm{Ord} \mid V[G] \vDash \varphi(\zeta, \vec{\xi}, C_{\alpha_0}, \ldots, C_{\alpha_{n-1}}, D)\}$$

where $\alpha_0, \ldots, \alpha_{n-1} \in Z$. Then $X \in V[G \restriction \{\alpha_0, \ldots, \alpha_{n-1}\}]$.

We may assume that $V \models \mathrm{GCH}$.
Define $(\mathbb{P}, \leq) = (\mathbb{P}_Z, \leq)$ with $Z = \kappa^{+\beta}$. Let $V[G]$ be a generic extension of V by $\mathbb{P}$ with Prikry sequences $(C_\alpha)_{\alpha<\kappa^{+\beta}}$.
Let

$$N = \mathrm{HOD}^{V[G]}(\{C_\alpha \mid \alpha < \kappa^{+\beta}\} \cup \{(\tilde{C}_\alpha \mid \alpha < \kappa^{+\beta})\}).$$

Every set of ordinals in N is of the form

$$X = \{\zeta \in \mathrm{Ord} \mid V[G] \models \varphi(\zeta, \vec{\xi}, C_{\alpha_0}, \ldots, C_{\alpha_{n-1}}, (\tilde{C}_\alpha \mid \alpha < \kappa^{+\beta}))\}$$

Then

$$X \in V[G \restriction \{\alpha_0, \ldots, \alpha_{n-1}\}].$$

Finite support parallel Prikry forcing does not add bounded subsets of κ. So κ is a singular cardinal in N, and $N \models$ "GCH holds below κ".
There is a surjection $f : [\kappa]^\omega \to (\kappa^{+\beta})^V$ in N.
By the κ^+-cc, $(\kappa^{+\beta})^V = (\kappa^{+\beta})^N$. So f yields a choiceless, surjective failure of SCH.

Collapsing to $\aleph_\omega$

Let $\kappa_0, \kappa_1, \ldots$ be a Prikry sequence in N for the cardinal κ. Extend N generically by collapsing each κ_{n+1} to κ_n^{++}. Then κ becomes $\aleph_\omega$ without destroying GCH below κ. So SCH can fail at $\aleph_\omega$.

Outline

The lower bound

Theorem

Assume that SCH fails in a surjective way in a model V of ZF. Then there is an inner model of ZFC with a measurable cardinal.

Using the Dodd-Jensen Core Model K

Let κ be a singular cardinal such that $(\forall\nu < \kappa)[2^\nu < \kappa]$, and let $f : [\kappa]^{\mathrm{cof}(\kappa)} \to \kappa^{++}$ be a surjection. Let $\lambda = \mathrm{cof}(\kappa) + \aleph_2$. Assume that there were no inner model of ZFC with a measurable cardinal. For $Y \subseteq \mathrm{Ord}$, take $g_Y : \mathrm{otp}(Y) \leftrightarrow Y$ to be the uniquely defined order preserving map.

Consider $X \in [\kappa]^{\mathrm{cof}(\kappa)}$. By the Dodd-Jensen covering theorem (in $\mathrm{HOD}[X]$), there is $Y \in K$, $X \subseteq Y \subseteq \kappa$, $\mathrm{otp}(Y) < \lambda$. Let $Z = g_Y^{-1}[X] \in \wp(\lambda)$. Then

$$X = g_Y[Z] \text{ for some } Y \in \wp(\kappa) \cap K \text{ and } Z \in \wp(\lambda).$$

Since GCH holds in K, take a surjective $k : \kappa^+ \to \wp(\kappa) \cap K$. Since $2^\lambda < \kappa$, take a surjective $h : \kappa \to \wp(\lambda)$. By (4), the map

$$(\gamma, \eta) \mapsto f(g_{k(\gamma)}[h(\eta)])$$

is a surjection from $\kappa^+ \times \kappa$ onto κ^{++}.

Outline

Injective failures

Theorem

The following theories are equiconsistent:

$$ZFC + \exists\kappa[o(\kappa) = \kappa^{++} + \omega_2]$$

and

$$ZF + \neg AC + GCH \text{ holds below } \aleph_{\omega_2}$$

$$+ \text{ There is an injective } f : \aleph_{\omega_2+2} \to [\aleph_{\omega_2}]^{\omega_2}.$$

Injective failures

Theorem

a) *If the theory*

$$ZFC + \exists\kappa[o(\kappa) = \kappa^{++} + \omega_1]$$

is consistent, then so is the theory

$$ZF + \neg AC + GCH \text{ below } \aleph_{\omega_1} + \text{ there is injective } f : \aleph_{\omega_1+2} \longrightarrow [\aleph_{\omega_1}]^{\omega_1}.$$

b) *If the theory*

$$ZF + \neg AC + GCH \text{ below } \aleph_{\omega_1} + \text{ there is injective } f : \aleph_{\omega_1+2} \longrightarrow [\aleph_{\omega_1}]^{\omega_1}$$

is consistent, then so is the theory

$$ZFC + \exists\kappa[o(\kappa) = \kappa^{++}].$$

Questions

- Can one achieve equiconsistencies in all cases?
- Can one lift the equiconsistency for the surjective failure to uncountable cofinalities?