two model that appear different but are quite different:

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(1) $V \models GCH$, $\varkappa$ strong (or superst compact)

$\cap$

V_1 : is a forcing ext. of V s.t. no new bounded subsets of $\varkappa$ are added, $cf\ \varkappa = \omega$, $2^\varkappa = \varkappa^{++}$

Extender based Prikry forcing

$\varkappa \quad\quad \alpha \quad\quad \varkappa^{++}$

extender, i.e. each α gives a measure from $j: V \to M$, $crit(j) = \varkappa$, $M \supseteq V_{\varkappa+1}$

$\mathcal{U}_\alpha := \{X \subseteq \varkappa \mid \alpha \in j(X)\}$

We obtain $\langle \mathcal{U}_\alpha \mid \varkappa \le \alpha < \varkappa^{++} \rangle$ where $\mathcal{U}_\varkappa$ is normal u.f. over $\varkappa$

For each $\mathcal{U}_\alpha$ there will be a Prikry sequence. No new bounded subsets of $\varkappa$ will be added, $cf\ \varkappa = \omega$, $2^\varkappa = \varkappa^{++}$.

$\cap$ Denote by $\langle \varkappa_n \mid n < \omega \rangle$ the Prikry sequence for $\mathcal{U}_\varkappa$.

V_2 : Force over V_1 and add $\varkappa^{++}$-many Cohen reals.

In V_2 $2^{\aleph_0} = 2^\varkappa = \varkappa^{++}$, GCH above $\varkappa$.

(2) Again $V \models GCH$

V_1' : $V[\langle \varkappa_n \mid n < \omega \rangle]$ ordinary Prikry forcing, i.e. $V_1' \models GCH$, $cf\ \varkappa = \omega$

$\cap$

V_2' : Force $\varkappa^{++}$-many Cohen reals. We obtain GCH above $\varkappa$.

Shelah: $\varkappa$ is a singular, define

$pp(\varkappa) := \sup \{ cof(\prod a / D) \mid a \subseteq \varkappa$ cofinal, $|a| = cf\ \varkappa$, a consists of reg. card., D is Uf. over a, which includes all co-bounded subsets of $a \}$

Calculate it in V_2:

Prikry sequ. for

In V_1: We have $\langle \varkappa_n \mid n<\omega\rangle$, $\langle \varkappa_n^{++} \mid n<\omega\rangle$

$$cf\left(\prod_{n<\omega} \varkappa_n^{++} \Big/ \begin{smallmatrix}\text{co. bounded}\\ \hat{=}\ \text{co finite}\end{smallmatrix}\right) = \varkappa^{++}$$

Let for each $\alpha < \varkappa^{++}$ $f_\alpha : \omega \to \varkappa$ denote the generic Prikry sequ. for U_α $\forall n\ \varkappa_n \le f_\alpha(n) < \varkappa_n^{++}$

(1) $\alpha<\beta \Rightarrow f_\alpha < f_\beta$ mod finite

(2) for every $g \in \prod_{n<\omega} \varkappa_n^{++}$ there is $\alpha < \varkappa^{++}$ s.t. $g < f_\alpha$ mod finite

The sequ. $\langle f_\alpha \mid \alpha < \varkappa^{++}\rangle$ is a scale in $\prod_{n<\omega} \varkappa_n^{++} /_{\text{co finite}}$.

In V_2: Let $h \in \prod_{n<\omega} \varkappa_n^{++}$ in V_2. Let $\underset{\sim}{h}$ be a name for h.
There is $h' \in V_1$ s.t. $\forall n$ $h(n) < h'(n) < f_\alpha(n)$ for some α.
$\in \prod_{n<\omega} \varkappa_n^{++}$

Thus $(pp(\varkappa))^{V_2} = \varkappa^{++}$.

Calculate it in V_2':

Let $a \subseteq \varkappa$, $|a| = \aleph_0$, a cof. in $\varkappa$, $a \subseteq Reg$. There is $b \in V_1'$, $b \supseteq a$, $|b| = \aleph_0$, $b \subseteq Reg$. In $V_1' = \prod b /_{\text{cofinite}}$.
By induction it is possible to construct $\langle g_\alpha \mid \alpha < \varkappa^+\rangle$ a scale in $\prod b$. (?)
Thus $(pp(\varkappa))^{V_2'} = \varkappa^+$.

Now let $\varkappa_0 < \varkappa_1$ be singular s.t. $pp(\varkappa_0) = pp(\varkappa_1) = \varkappa_1^{++}$.
One has to start with $\varkappa_1$, since otherwise the measurability of $\varkappa_1$ would be destroyed.

Shelah - G: $\varkappa_0$ strong up to $\varkappa_1^{++}$, $\varkappa_1$ strong up to $\varkappa_1^{++}$
(then the forcing won't destroy the strong compactness (?) of $\varkappa_0$)

he did not write this and maybe he meant something completely different

GCH, $\varkappa_1$ is strong up to $\varkappa_1^{++}$, $\varkappa_0$ is a sing. of cof ω, it is a limit of an increasing sequ. $\langle \varkappa_{0n} \mid n<\omega\rangle$ s.t. for each $n<\omega$ $\varkappa_{0n}$ is λ_{0n}-strong, where $\lambda_{0n} < \varkappa_{0n+1}$, λ_{0n} Mahlo.
Change $cf\,\varkappa_1$ to ω and make $2^{\varkappa_1} = \varkappa_1^{++}$, no new bounded subsets of $\varkappa_1$ added. Let $\langle \varkappa_{1n} \mid n<\omega\rangle$ be the Prikry sequence for the normal measure of the extender over $\varkappa_1$: $\langle \varkappa_{1n}^{++} \mid n<\omega\rangle$

$$cf\left(\prod_{n<\omega} \varkappa_{1n}^{++} /_{\text{cofinite}}\right) = \varkappa_1^{++}.$$

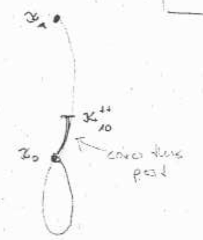
2 MG

We wish Connections: $(\kappa_0, \kappa_{10}^{++}] \longrightarrow (\kappa_{00}, \lambda_{00}]$

$(\kappa_0, \kappa_{11}^{++}] \longrightarrow (\kappa_{01}, \lambda_{01}]$

$\vdots$

$(\kappa_0, \kappa_{1n}^{++}] \longrightarrow (\kappa_{0n}, \lambda_{0n}]$



(We want to control everything between κ_0 and κ_1 by the λ's.)

So that somehow one obtains $cf\left(\prod_{n<\omega} \lambda_{0n} / \text{cofinite}\right) = \kappa_1^+$

$\langle \kappa_n \mid n<\omega \rangle$ incr. sequ. of card. of cof ω s.t. $\kappa = \bigcup_{n<\omega} \kappa_n$

$pp(\kappa_n) = \kappa^+$ for all $n<\omega$. [Magidor - G.]

Assume that for each $n<\omega$, $\kappa_n = \bigcup_{m<\omega} \kappa_{nm}$ s.t. $\overbrace{\kappa_{n0}}^{\kappa_{n-1}<} < \kappa_{n1}$ $< \dots < \kappa_{nm} < \kappa_{n(m+1)} < \dots < \kappa_n$, for each $m<\omega$ κ_{nm} is λ_{nm}-strong for some Mahlo $\lambda_{nm} < \kappa_{n(m+1)}$.

$(\kappa_0, \lambda_{10}] \longmapsto (\kappa_{00}, \lambda_{00}]$

$(\kappa_0, \lambda_{11}] \longmapsto (\kappa_{01}, \lambda_{01}]$

$(\kappa_1, \lambda_{20}] \longmapsto (\kappa_{11}, \lambda_{11}]$

$(\kappa_0, \lambda_{12}] \longmapsto (\kappa_{02}, \lambda_{02}]$

$(\kappa_1, \lambda_{21}] \longmapsto (\kappa_{12}, \lambda_{12}]$

$(\kappa_2, \lambda_{30}] \longmapsto (\kappa_{22}, \lambda_{22}]$

$\vdots$

Conditions: $\langle a, A, f \rangle$

a: function that arranges the mapping, i.e. order preserving fct. of small cof.

A: measure 1 set (measure arises from a and extenders considered to the range of a)

f: color function

Consider for κ_n $\underbrace{\mathcal{P} \setminus \kappa_n}_{\kappa_n^+\text{-closed}} * \underbrace{(\mathcal{P} \restriction \kappa_n)}_{\kappa_n^{++}\text{-c.c.}}$

$\kappa^+ = cf \prod_{n<\omega} \lambda_{\ell n} / \text{cofinite}$ for every $\ell<\omega$

(one has GCH below κ_{00})

Shelah Weak Hypothesis:

For each cardinal δ

$$|\{\theta < \delta \mid pp(\theta) \geq \delta\}| \leq \aleph_0 .$$

pcf-conjecture:

Let A be a set of regular card. s.t. $|A| < \min A$, then $|pcf(A)| = |A|$, where $pcf(A) = \{ cf(\prod A/_D) \mid D \text{ is u.f. over } A\}$

$\overset{\text{loc. theorem}}{\Longrightarrow}$ they are related

Pick a sequence $g_\alpha : \omega \longrightarrow \omega$ $(\alpha < \omega_1)$ s.t.

(1) $\langle g_\alpha(n) \mid n < \omega \rangle$ incr.

(2) for every $\alpha < \beta < \omega_1$ there is $n(\alpha,\beta) < \omega$ s.t.

$$\forall n \geq n(\alpha,\beta) \quad g_\alpha(n) \geq \sum_{m=0}^{n} g_\beta(m)$$

Let $\langle \varkappa_\alpha \mid \alpha < \omega_1 \rangle$ be an increasing sequence of sing. cardinals of cof. ω. For each $\overbrace{\text{succ.}}^{} \ \overbrace{\alpha < \omega_1}^{\text{or } \alpha = 0}$ let $\langle \varkappa_{\alpha n} \mid n < \omega \rangle$ be an incr. sequ. of strong enough cardinals

$$\varkappa_{\alpha-1} < \varkappa_{\alpha 0} < \ldots < \varkappa_{\alpha n} < \ldots < \varkappa_\alpha .$$

Fix blocks $\langle \varkappa_{\alpha,n,m,i} \mid m < g_\alpha(n),\ i \leq \omega_1 \rangle$.

Fix α a succ. $< \omega_1$ or $\alpha = 0$

$$g_\alpha(0) = 1.$$

For every $\beta > \alpha$, $\beta < \omega_1$, succ.

$$(\varkappa_{\beta-1}, \underbrace{\varkappa_{\beta,0,0,\omega_1}}_{\varkappa_{\beta 0 0 \omega_1}}] \longmapsto (\varkappa_{\alpha 0 0 \beta}, \varkappa_{\alpha 0 0 \omega_1}]$$

Question: $g_\alpha(1) = ?$

Let β be succ. above α, compare $g_\alpha(1)$ with $g_\beta(0) + g_\beta(1)$

Let $r = g_\alpha(1) - g_\beta(0) - g_\beta(1)$.

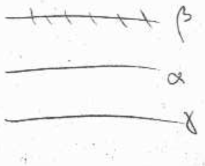
For $r < 0$ do nothing.

If $r \geq 0$ then make the connection

$$(\varkappa_{\beta-1}, \varkappa_{\beta,1,1,\omega_1}] \longmapsto (\varkappa_{\alpha 1 0 \beta}, \varkappa_{\alpha 1 0 \omega_1}]$$

$\alpha < \omega_1$, $\alpha = 0$ or succ,

$$pcf\{\varkappa^+_{\alpha,n,m,i} \mid n < \omega, m < g_\alpha(n)\}$$



$$= \{\varkappa^+_{\beta,n,m,i} \mid n < \omega,\ m < g_\alpha(n),\ \alpha < \beta < i\} \cup \{\varkappa_i^+\}$$