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Goal: ZF + "the first ρ-many unctbl cardinals are singular of cof. ω"

Symmetric models

- $\mathbb{P}$ p.o., $\quad G$ autom. group of $\mathbb{P}$, $\mathcal{F}$ a normal filter / G, i.e., $K \in \mathcal{F}$, $a \in G \Rightarrow aKa^{-1} \in \mathcal{F}$
  aut of $\mathbb{P} \to$ aut of $V^{\mathbb{P}}$
- a name $z \in V^{\mathbb{P}}$ is symmetric if
  $$\text{sym } z = \{a \in G \mid az = z\} \in \mathcal{F}$$
- HS: the class of hereditarily symm. names
- A symm. model
  $$V(G)^{\mathcal{F}} := \{z^G \mid z \in HS\},$$
  G $\mathbb{P}$-generic
- For $E \subseteq \mathbb{P}$ define
  $$\text{fix } E := \{a \in G \mid \forall p \in E \; ap = p\}.$$

  A set $I \in \wp(\mathbb{P})$ is a G-symmetry generator if it's closed under fin unions & $\forall a \in G \; \forall E \in I$ $\exists E' \in I$ s.t. $a.\text{fix } E.a^{-1} \supseteq \text{fix } E'$.

  makes sure that $\mathcal{F}_I$: generated by $\{\text{fix } E \mid E \in I\}$ is a normal filter / G

  $E \in I$ is a support for a $z \in HS$ if $\text{fix } E \supseteq \text{sym } z$

- Let $\langle \varkappa_\xi \mid \xi < \rho \rangle$ increasing seq of strongly compacts with limit η & with no regular limits in it (i.e. for every β $\langle \varkappa_\xi \mid \xi < \beta \rangle$ has singular limit)

  Call Reg^η the regular cardinals in η, $> \omega$.

(1) An $\alpha \in Reg^\eta$ is said to be <u>of type 1</u> if there is a largest strongly compact $\varkappa_\xi < \alpha$.

- if $\varkappa_\xi > \omega$
  Let H_α be a fine ultrafilter over $\wp_{\varkappa_\xi}(\alpha)$ & $h_\alpha : \wp_{\varkappa_\xi}(\alpha) \twoheadrightarrow \alpha$. Define
  $$\Phi_\alpha := \{ X \in \alpha \mid h^{-1}{}''X \in H_\alpha \}$$
  a $\varkappa_\xi$-complete ultrafilter$/\alpha$.

- if $\varkappa_\xi = \omega$ then $\Phi_\alpha := \{X \in \alpha \mid |X| = \alpha\}$

(2) Define $cf'\alpha = \alpha \quad \alpha \in Reg^\eta$ <u>of type 2</u> if there is no

largest $\kappa_\xi < \alpha$. Let β be the largest singular limit of κ_ξ's below α. Define $cf'\alpha := cf\,\beta < \beta$. Fix $\langle \kappa^\alpha_\nu \mid \nu < cf'\alpha \rangle$ a sequence with limit β. Let $\Phi_{\alpha,\nu}$ be the κ^α_ν-complete u.f. $/\alpha$ that comes from some $H_{\alpha,\nu}, h_{\alpha,\nu}$.

<u>The forcing:</u>

<u>α of type 1</u> : $\mathbb{P}_\alpha \ni T \subseteq {}^{<\omega}\alpha$ s.that

T is a Φ_α-tree, i.e.

1 - T consists of injective sequences

2 - T has a trunk tr_T $\quad \forall t \in T \quad tr_T \trianglerighteq t$ or $t \trianglerighteq tr_T$

3 - $T \neq \emptyset$

- for every $t \in T$ ($t \trianglerighteq tr_T$) $\quad suc_T(t) = \{\beta \in \alpha \mid t^\frown\langle\beta\rangle \in T\} \in \Phi_\alpha$

$T \leq S :\Leftrightarrow T \subseteq S$

<u>α of type 2</u> : Look at $\mathbb{P}_{cf'\alpha}$. For every $T \in \mathbb{P}_{cf'\alpha}$ define $\mathbb{P}_\alpha$: $X_T \ni S :\Leftrightarrow$ 1-3 hold &

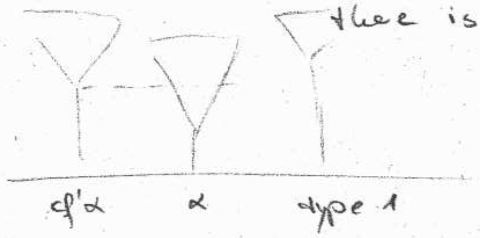
$\forall s \in S \;\; s \trianglerighteq tr_T$ &
for every $t \in T$
with $\mathrm{dom}\, s \in \mathrm{dom}\, t$
$Suc_S(s) \in \Phi_{\alpha, t(\mathrm{dom}\, s)}$
& $\mathrm{dom}\, tr_S \in \mathrm{dom}\, tr_T$

set $\mathbb{P}_\alpha := \bigcup_{T \in \mathbb{P}_{cf'\alpha}} X_T$

same ordering

$$\mathbb{P} \subseteq \prod^{fin}_{\alpha \in Reg^\eta} \mathbb{P}_\alpha \quad \text{s.t.} \quad \vec{T} = \langle T_\alpha \mid \alpha \in \mathrm{dom}\,\vec{T} \rangle \in \mathbb{P}$$

iff whenever $\alpha \in \mathrm{dom}\,\vec{T}$ is of type 2 there is $cf'\alpha \in \mathrm{dom}\,\vec{T}$ & $T_\alpha \in X_{T_{cf'\alpha}}$



For every $\alpha \in Reg^\eta$. Let G_α be the Group of permutations of α that only move finitely many things.

$$G := \prod^{fin}_{\alpha \in Reg^\eta} G_\alpha$$

$a \in G$ is of the form $a = \langle a_\alpha \mid \alpha \in Reg^\eta \rangle$, $a_\alpha \in G_\alpha$

but there is a set dom(a) finite s.t. if $\alpha \notin \text{dom}\, a$
$\rightarrow a_\alpha = id_\alpha$

$a_\alpha T_\alpha = \{ \{ (n, a_\alpha \gamma) \mid (n, \gamma) \in t \} \mid t \in T_\alpha \}$

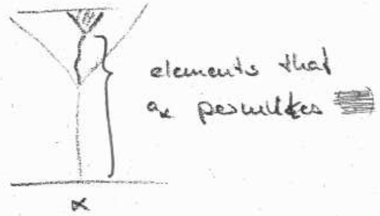
$\in G_\alpha \quad \in \mathbb{P}_\alpha$

$\alpha \vec{T} = \langle a_\alpha T_\alpha \mid \alpha \in \text{dom}\, \vec{T} \rangle$

$\vec{T} = \langle T_\alpha \mid \alpha \in \text{dom}\, \vec{T} \rangle$

($\alpha \vec{T} \notin \mathbb{P}$ is possible)

Define $\forall a \in G \quad \mathbb{P}^a \subseteq \mathbb{P} \quad \forall \vec{T} \in \mathbb{P}^a \quad a\vec{T} \in \mathbb{P}$.



* $\mathbb{P}^a$ is dense subset of $\mathbb{P}$
* $a: \mathbb{P}^a \rightarrow \mathbb{P}^a$ is an autom.
* we can extend a to an autom. of $\mathbb{P}$

- G has been extended to an autom group of $\mathbb{P}$
- For $e \subseteq Reg^\eta$ finite & closed und cf'

$E_e = \{ \vec{T} \upharpoonright e \mid \vec{T} \in \mathbb{P} \}$

$\mathbb{P} \upharpoonright e$

$I = \{ E_e \mid e - \text{———} \}$

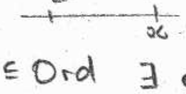
$\downarrow$

$\mathcal{F}_I$ generated by $\{ \text{fix}\, E_e ; \; e \overset{\text{fin.}}{\underset{\text{cf}'}{\subseteq}} Reg^\eta \}$

$\vec{T} \in \mathbb{P}^a$ iff

- $\forall \alpha \in \text{dom}\, \vec{T}, \; h_{T_\alpha} = h_{T_{cf'\alpha}}$
- $\forall \alpha \in \text{dom}\, \vec{T}, \; \text{rng}\, h_{T_\alpha} \subseteq \{ \beta \in \alpha \mid \beta \text{ is moved by } a_\alpha \}$

Thus:
- all $\alpha \in Reg^\eta$ are sing with cof ω
- „fake" ones :



- $\forall X \in V(G) \quad X \subseteq Ord \quad \exists e \overset{\text{fin}}{\underset{\text{cf}'}{\subseteq}} Reg^\eta \quad X \in V[G \upharpoonright e]$

- sing. ✓
- Prikry lemma: For every φ formula, $z_1, \dots, z_n \in HS$ with support $\mathbb{P} \restriction e$ and every $\vec{T} \in \mathbb{P} \restriction e$, there is a $\vec{S} \leq \vec{T}$, dom $\vec{S}$ = dom $\vec{T}$.
  For every $\alpha \in$ dom $\vec{T} \setminus \varkappa_1$ $h_{T_\alpha} = h_{S_\alpha}$.
  s.t. $\vec{S}$ decides $\varphi(z_1, \dots, z_n)$

... all cardinals in (ω, η) are almost Ramsey