Some Remarks on the Tree Property in a Choiceless Context

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We begin by presenting some basic definitions. Throughout, κ will be an uncountable regular cardinal.

Definition 1: A *κ-tree* is a tree of height κ, all of whose levels have cardinality less than κ.

Definition 2: κ satisfies the *tree property* if every κ-tree has a branch of length κ.

Definition 3: *The Singular Cardinals Hypothesis (SCH)* holds at a singular cardinal κ if κ is a strong limit cardinal and $2^\kappa = \kappa^+$.

A counterexample to Definition 2 is called a *κ-Aronszjan tree*. Also, note that for our purposes, all κ-trees will be of cardinality κ and will have base set $\kappa \times \kappa$. This means that every κ-tree may be coded by a set of ordinals.

We now briefly review some of what is known about the tree property in ZFC.

- κ is weakly compact iff κ is strongly inaccessible and satisfies the tree property.

- (Aronszajn) The tree property fails at $\aleph_1$, i.e., an $\aleph_1$-Aronszajn tree exists.

- (Silver 1971, Mitchell 1972/1973) The tree property at the successor of a regular cardinal greater than $\aleph_1$ is equiconsistent with a weakly compact cardinal.

- (Abraham 1983) Relative to the existence of a supercompact cardinal with a weakly compact cardinal above it, it is consistent for $2^{\aleph_0} = \aleph_2$ and for $\aleph_2$ and $\aleph_3$ both to satisfy the tree property.

- (Shelah 1996/Magidor and Shelah 1996) The successor of a singular limit of strongly compact cardinals satisfies the tree property. Further, relative to a huge cardinal with ω many supercompact cardinals above it, it is consistent for SCH to hold at $\aleph_\omega$ and for $\aleph_{\omega+1}$ to satisfy the tree property.

- (Cummings and Foreman 1998) Relative to the existence of ω many supercompact cardinals, it is consistent for $2^{\aleph_n} = \aleph_{n+2}$ for every $n < \omega$ and for every $\aleph_n$ for $1 < n < \omega$ to satisfy the tree property.

- (Schindler 1999) If both $\aleph_2$ and $\aleph_3$ satisfy the tree property, then there is an inner model with a strong cardinal.

- (Foreman, Magidor, and Schindler 2001) If $\aleph_n$ has the tree property for all $1 < n < \omega$ and $\aleph_\omega$ is a strong limit cardinal, then for all $X \in H_{\aleph_\omega}$ and all $n < \omega$, $M_n^\sharp(X)$ exists.

- (Neeman 2008) Relative to the existence of ω many supercompact cardinals, it is consistent for there to be a singular strong limit cardinal $\kappa > \aleph_\omega$ of cofinality ω such that SCH fails at κ (i.e., $2^\kappa > \kappa^+$) and κ^+ satisfies the tree property.

The above results raise the following questions:

Question 1: Is it possible to extend the Cummings-Foreman result to all successor cardinals, i.e., is it possible to get a model of ZFC in which every successor cardinal satisfies the tree property?

Question 2: Is it possible to transfer Neeman's result down to $\aleph_\omega$, i.e., is it possible to obtain a model of ZFC in which SCH fails at $\aleph_\omega$ yet $\aleph_{\omega+1}$ satisfies the tree property?

Unfortunately, an answer to both of these questions in ZFC is unknown. However, it is possible to provide non-AC answers to each question. Specifically, we have the following two theorems.

Theorem 1 *(**AA**) Con(ZFC + There is a proper class of supercompact cardinals) $\Longrightarrow$ Con(ZF + DC + Every successor cardinal is regular + Every limit cardinal is singular + Every successor cardinal satisfies the tree property).*

Theorem 2 *(**AA**) Con(ZFC + There exist ω many supercompact cardinals) $\Longrightarrow$ Con(ZF + $\neg$AC$_\omega$ + $2^{\aleph_n} = \aleph_{n+1}$ for every $n < \omega$ + There is an injection from $\aleph_{\omega+2}$ into $\wp(\aleph_\omega)$ + $\aleph_{\omega+1}$ satisfies the tree property).*

We remark that in Theorem 1, $\aleph_1$ satisfies the tree property. This contrasts with the situation in ZFC, where $\aleph_1$ carries an Aronszajn tree. Further, Theorem 1 represents an improvement over an earlier model in which every successor cardinal satisfied the tree property, but in which AC_ω failed and which was constructed from hypotheses in consistency strength between a supercompact limit of supercompact cardinals and an almost huge cardinal. Finally, in Theorem 2, there is nothing special about $\aleph_{\omega+2}$. It is also possible to get an injection from larger cardinals into $\wp(\aleph_\omega)$.

We now sketch the proofs of Theorems 1 and 2. For Theorem 1, suppose $V \vDash$ "ZFC + There is a proper class of supercompact cardinals". Without loss of generality, we assume that each supercompact cardinal κ has been made indestructible under κ-directed closed forcing, and that there is no inaccessible limit of supercompact cardinals.

Let $K = \{\omega\} \cup \{\kappa \mid \kappa$ is either a supercompact cardinal or the successor of a limit of supercompact cardinals$\}$. Assume that $\langle \kappa_i \mid i \in \mathsf{Ord} \rangle$ enumerates K in increasing order. For each $i \in \mathsf{Ord}$, let $\mathbb{P}_i = \mathsf{Coll}(\kappa_i, {<}\kappa_{i+1})$, i.e., $\mathbb{P}_i$ is the Lévy collapse of all cardinals in the open interval (κ_i, κ_{i+1}) to κ_i. Let $\mathbb{P} = \prod_{i \in \mathsf{Ord}} \mathbb{P}_i$ be the countable support proper class product, and let G be V-generic over $\mathbb{P}$.

$V[G]$, being a model of AC, is not our desired choiceless inner model N witnessing the conclusions of Theorem 1. In order to define N, we first note that by the Product Lemma, for $i \in \mathsf{Ord}$, G_i, the projection of G onto $\mathbb{P}_i$, is V-generic over $\mathbb{P}_i$. Next, let $\mathcal{F} = \prod_{i \in \mathsf{Ord}} (\kappa_i, \kappa_{i+1})$ be the countable support product of the open intervals (κ_i, κ_{i+1}). For each $f \in \mathcal{F}$, $f = \langle \alpha_i \mid i < \omega \rangle$, define $G \restriction f = \prod_{i < \omega} (G_i \restriction \alpha_i)$. In other words, every f is a countable sequence of ordinals each of whose elements is a member of a unique interval of the form $(\kappa_{j(i)}, \kappa_{j(i)+1})$, and every $G_i \restriction \alpha_i$ collapses each cardinal in the open interval $(\kappa_{j(i)}, \alpha_i)$ to $\kappa_{j(i)}$. N can now be intuitively described as the least model of ZF extending V which contains, for each $f \in \mathcal{F}$, the set $G \restriction f$.

It can be shown that $N \models$ "ZF + DC + Every successor cardinal is regular + Every limit cardinal is singular". Our sketch of the proof of Theorem 1 is therefore completed by the following lemma.

Lemma 1: $N \models$ "Every successor cardinal satisfies the tree property".

Sketch of proof: Suppose $N \models$ "κ is a successor cardinal and $\mathfrak{T}$ is a κ-tree". By the construction of N, κ must either be a ground model supercompact cardinal, or a ground model successor of a singular limit of supercompact cardinals. In either situation, since $\mathfrak{T}$ may be coded by a set of ordinals, we can assume that $\mathfrak{T} \in V[G_1 \times G_2]$. Here, G_2 is V-generic over a partial ordering of the form $\mathrm{Coll}(\kappa, <\lambda)$ for some cardinal λ, and G_1 is V-generic over a countable product of Lévy collapses based on cardinals less than κ.

If κ is a ground model successor of a singular limit of supercompact cardinals, then because each ground model supercompact cardinal is indestructible and each Lévy collapse is appropriately directed closed and of small enough cardinality, κ is in $V[G_1 \times G_2]$ a successor of a singular limit of supercompact cardinals. Thus, by Shelah's theorem, κ satisfies the tree property in $V[G_1 \times G_2]$. This means that in $V[G_1 \times G_2] \subseteq N$, there is a branch of length κ through $\mathfrak{T}$. If, however, κ is a ground model supercompact cardinal, then by indestructibility, κ remains supercompact in $V[G_2]$.

Since under these circumstances, G_1 is generic over a partial ordering having cardinality less than κ, by the Lévy-Solovay results, κ remains supercompact in $V[G_2 \times G_1] = V[G_1 \times G_2]$. Because κ is supercompact in $V[G_1 \times G_2]$, κ is weakly compact in this model as well and hence satisfies the tree property in $V[G_1 \times G_2]$. As before, this means that in $V[G_1 \times G_2] \subseteq N$, there is a branch of length κ through $\mathfrak{T}$. Therefore, in either situation, $N \vDash$ "κ satisfies the tree property". This completes the proof sketch of both Lemma 1 and Theorem 1. $\square$

Turning to our sketch of the proof of Theorem 2, suppose $V' \vDash$ "ZFC + There exist ω many supercompact cardinals". Without loss of generality, we assume that V' has been generically extended to Neeman's model V. In particular, we may assume that $V \vDash$ "ZFC + κ is a limit of ω many strongly inaccessible cardinals $\langle \kappa_i \mid i < \omega \rangle$ (where $\kappa_0 = \omega$) + $2^\kappa = \kappa^{++}$ + κ^+ satisfies the tree property".

We define the partial ordering $\mathbb{P}$ used in the proof of Theorem 2. For each $i < \omega$, let $\mathbb{P}_i = \text{Coll}(\kappa_i, {<}\kappa_{i+1})$. Let $\mathbb{P} = \prod_{i<\omega} \mathbb{P}_i$ be the full support product, and let G be V-generic over $\mathbb{P}$. Once again, $V[G]$, being a model of AC, is not our desired choiceless inner model N witnessing the conclusions of Theorem 2. In order to define N, as before, we note that G_i, the projection of G onto $\mathbb{P}_i$, is V-generic over $\mathbb{P}_i$. Next, for $n < \omega$, define $G^n = \prod_{i \le n} G_i$. N can now be intuitively described as the least model of ZF extending V which contains, for each $n < \omega$, the set G^n.

It can be shown that $N \vDash$ "ZF + $\neg\text{AC}_\omega$ + GCH holds below $\kappa = \aleph_\omega$ + There is an injection from $\aleph_{\omega+2}$ into $\wp(\aleph_\omega)$". Our sketch of the proof of Theorem 2 will be completed by the following two lemmas.

Lemma 2: Suppose $V \vDash$ "λ is a regular cardinal satisfying the tree property + $\mathbb{Q}$ is a partial ordering such that $|\mathbb{Q}| < \lambda$". Then $V^{\mathbb{Q}} \vDash$ "λ is a regular cardinal satisfying the tree property".

Sketch of proof: Standard arguments show that $V^{\mathbb{Q}} \vDash$ "λ is a regular cardinal". To see that $V^{\mathbb{Q}} \vDash$ "λ satisfies the tree property", suppose that $p \Vdash$ "$\dot{\mathfrak{T}}$ is a λ-tree". There must be some q extending p such that for λ many pairs $\langle \alpha, \beta \rangle$, $q \Vdash$ "$\langle \alpha, \beta \rangle \in \dot{\mathfrak{T}}$". Otherwise, by the regularity of λ, there is a set $A \in V$, $|A| < \lambda$ such that $p \Vdash$ "$\dot{\mathfrak{T}} \subseteq A$".

For such a q, define the set $\mathfrak{T}^* \in V$ by $\langle \alpha, \beta \rangle \in \mathfrak{T}^*$ iff $q \Vdash$ "$\langle \alpha, \beta \rangle \in \dot{\mathfrak{T}}$". Since $q \Vdash$ "$\dot{\mathfrak{T}}$ is a λ-tree", $\mathfrak{T}^*$ is a λ-tree in V. Because $V \vDash$ "λ satisfies the tree property", $V \vDash$ "There is some branch b^* through $\mathfrak{T}^*$ having length λ". But then $q \Vdash$ "$\breve{b}^*$ generates a branch $\dot{b}$ through $\dot{\mathfrak{T}}$ having length λ". $\square$

Lemma 3: $N \vDash$ "$\kappa^+ = \aleph_{\omega+1}$ satisfies the tree property".

Sketch of proof: Suppose $N \models$ "$\mathfrak{T}$ is a κ^+-tree". Because $\mathfrak{T}$ may be coded by a set of ordinals, $\mathfrak{T} \in V[G^n]$ for some $n < \omega$. Since G^n is V-generic over a partial ordering having cardinality less than κ^+, by Lemma 2, $V[G^n] \models$ "κ^+ satisfies the tree property". As $V[G^n] \models$ "$\mathfrak{T}$ is a κ^+-tree", it follows that in $V[G^n] \subseteq N$, there is a branch of length κ^+ through $\mathfrak{T}$. Thus, $N \models$ "$\kappa^+ = \aleph_{\omega+1}$ satisfies the tree property". This completes the proof sketch of both Lemma 3 and Theorem 2. $\square$

We conclude by asking the following questions:

- Is it possible to establish analogues of Theorems 1 and 2 in a model of ZFC (thereby completely answering Questions 1 and 2)?

- Is it possible to establish a version of Theorem 2 with a surjective failure of SCH?

- Is it possible to establish a version of Theorem 2 in which some of the Axiom of Choice is true?

- What is the exact consistency strength of each of the patterns involving the tree property mentioned previously?