A NOTE ON NON-DOMINATING ULTRAFILTERS

VERA FISCHER AND BERNHARD IRRGANG

ABSTRACT. We show that if $\text{cov}(\mathcal{M}) = \kappa$, where κ is a regular cardinal such that $\forall \lambda < \kappa(2^\lambda \leq \kappa)$, then for every unbounded directed family $\mathcal{H}$ of size κ there is an ultrafilter $\mathcal{U}_\mathcal{H}$ such that the relativized Mathias forcing $\mathbb{M}(\mathcal{U}_\mathcal{H})$ preserves the unboundedness of $\mathcal{H}$. This improves a result of M. Canjar (see [4, Theorem 10]). We discuss two instances of generic ultrafilters for which the relativized Mathias forcing preserves the unboundedness of certain unbounded families of size $< \mathfrak{c}$.

1. INTRODUCTION

Recall that Mathias forcing $\mathbb{M}$ consists of pairs (u, A) where u is a finite subset of ω, $A \in [\omega]^\omega$ and $\max u < \min A$. The extension relation $\leq_\mathbb{M}$ is defined as follows: $(u_2, A_2) \leq (u_1, A_1)$ if u_2 is an end-extension of u_1, $A_2 \subseteq A_1$ and $u_2 \backslash u_1 \subseteq A_1$. Whenever $\mathcal{U}$ is a filter on ω, the relativized Mathias forcing $\mathbb{M}(\mathcal{U})$ is the suborder of $\mathbb{M}$ consisting of all conditions (u, A) such that $A \in \mathcal{U}$. It is well known if $\mathcal{U}$ is a selective ultrafilter the relativized Mathias poset $\mathbb{M}(\mathcal{U})$ adds a dominating real. In [4] M. Canjar gives a characterization of the ultrafilters for which the relativized Mathias poset does not add a dominating real. Namely, if $\mathcal{U}$ is an ultrafilter such that $\mathbb{M}(\mathcal{U})$ is weakly bounding (i.e. preserves the ground model reals as an unbounded family) then $\mathcal{U}$ is a P-point with no rapid predecessors in the Rudin-Keisler order.

In [4] it is shown that if $\mathfrak{d} = \mathfrak{c}$, then there is an ultrafilter $\mathcal{U}$ for which $\mathbb{M}(\mathcal{U})$ is weakly bounding. In this paper we show that given any regular cardinal κ such that $\forall \lambda < \kappa(2^\lambda \leq \kappa)$, the weaker hypothesis $\text{cov}(\mathcal{M}) = \kappa$, implies the existence of ultrafilters $\mathcal{U}$ for which $\mathbb{M}(\mathcal{U})$ is weakly bounding. Furthermore, we show that under this hypothesis, if $\mathcal{H} \subseteq {}^\omega\omega$ is an unbounded directed family of size κ then there is an ultrafilter $\mathcal{U}_\mathcal{H}$ which preserves the unboundedness of $\mathcal{H}$. Thus in a sense our result improves Canjar's result, since the existence of such ultrafilters allows one to preserve the unboundedness of a fixed unbounded family along certain finite support iterations. In section 3 we discuss the

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generic existence of ultrafilters for which the relativized Mathias forcing preserves the unboundedness of unbounded families of size $< \mathfrak{c}$.

2. Non-dominating ultrafilters

Under CH, there are known methods with which one can associate to a given unbounded family of size $\mathfrak{c}$ an ultrafilter which preserves the unboundedness of the family. In [7, Proposition 5.1], C. Laflamme shows that CH implies the existence of a maximal almost disjoint family $\mathcal{A}$ such that the dual filter $\mathcal{F}(\mathcal{A})$ is not contained in any K_σ-filter. Then using the techniques of [2, Theorem 3.1], one can extend $\mathcal{F}(\mathcal{A})$ to an ultrafilter $\mathcal{U}$ such that $\mathbb{M}(\mathcal{U})$ does not add a dominating real. Furthermore one can associate such an ultrafilter with every unbounded directed family of cardinality $\mathfrak{c} = \aleph_1$.

Using the notion of logarithmic measures, S. Shelah obtains a modification of the Mathias poset which is almost ${}^\omega\omega$-bounding and thus in particular does not add a dominating real. Recall also that countable support iterations of proper almost ${}^\omega\omega$-bounding posets is weakly bounding (see [8]).

Definition 2.1 (S. Shelah, [8])**.** A function $h : [s]^{<\omega} \to \omega$, where $s \subseteq \omega$ is a *logarithmic measure* if $\forall a \in [s]^{<\omega}$, $\forall a_0, a_1$ such that $a = a_0 \cup a_1$, there is $i \in \{0,1\}$ such that $h(a_i) \geq h(a) - 1$ unless $h(a) = 0$. If s is a finite set and h a logarithmic measure on s, the pair $x = (s,h)$ is *a finite logarithmic measure.*

Shelah's poset Q (see [5, Definition 3.8]) consists of all pairs $p = (u,T)$ where u is a finite subset of ω and $T = \langle (s_i,h_i)\rangle_{i\in\omega}$ is an infinite sequence of finite logarithmic measures such that $\max u < \min s_0$, $\max s_i < \min s_{i+1}$ for all $i \in \omega$ and $\langle h_i(s_i)\rangle_{i\in\omega}$ is unbounded. The sequence T is called the *pure part* of p also *pure condition* and is identified with the pair $(\emptyset, T)$. Let $\mathrm{int}(T) = \bigcup_{i\in\omega} s_i$. Note that if (u,T) is a condition in Q, then $(u, \mathrm{int}(T))$ is a condition in the Mathias poset $\mathbb{M}$. The extension relation $\leq_Q$ is defined as follows: $(u_2,T_2) \leq_Q (u_1,T_1)$ if

1. $(u_2, \mathrm{int}(T_2)) \leq_{\mathbb{M}} (u_1, \mathrm{int}(T_1))$
2. Let $T_\ell = \langle (s_i^\ell, h_i^\ell)\rangle_{i\in\omega}$, $\ell \in \{0,1\}$. Then $\exists \langle B_i\rangle_{i\in\omega} \subseteq [\omega]^{<\omega}$ such that $\max u_2 < \min s_j^1$ for $j = \min B_0$ and for all $i \in \omega$, $\max B_i < \min B_{i+1}$, $s_i^2 \subseteq \bigcup_{j\in B_i} s_j^1$ and if $e \subseteq s_i^2$ is such that $h_i^2(e) > 0$, then there is $j \in B_i$ for which $h_j^1(e \cap s_j^1) > 0$.

Remark 2.2. For the purposes of this note, it is sufficient to know that if $(u_2,T_2) \leq_Q (u_1,T_1)$ then $(u_2, \mathrm{int}(T_2)) \leq_{\mathbb{M}} (u_1, \mathrm{int}(T_1))$. However for completeness we have stated the entire definition of $\leq_Q$.

Definition 2.3 ([5, Definition 3.9])**.** Let C be a centered family of pure conditions in Q. Then $Q(C)$ is the the suborder of Q consisting of all $(u, R) \in Q$ such that $T \leq_Q R$ for some $T \in C$.

Lemma 2.4. *Let C be a centered family of pure conditions in Q. Then $Q(C)$ is densely embedded in $\mathbb{M}(\mathcal{F}_C)$ where*

$$\mathcal{F}_C = \{X \in [\omega]^\omega : \exists T \in C(int(T) \subseteq X)\}.$$

Proof. It is sufficient to observe that the mapping

$$i : (a, T) \mapsto (a, \mathrm{int}(T))$$

from $Q(C)$ to $\mathbb{M}(\mathcal{F}_C)$ is a dense embedding. Indeed, it is clear that i is order preserving. Let $(a, X) \in \mathbb{M}(\mathcal{F}_C)$. Then by definition there is $T \in C$ such that $\mathrm{int}(T) \subseteq X$ and so in particular $\max a < \min \mathrm{int}(T)$. Therefore (a, T) is a condition in $Q(C)$ such that $(a, \mathrm{int}(T)) \leq (a, X)$. It remains to show that i preserves incompatability. Let (a, T) and (b, R) be incompatible conditions in $Q(C)$. By definition of $Q(C)$ there are T_0, R_0 in C such that $T_0 \leq T$, $R_0 \leq R$. However C is centered family and so there is a pure condition Z in C which is a common extension of T_0, R_0. Then Z is a common extension of T, R. *Case 1.* If a is not an end-extension of b and b is not an end-extension of a, then clearly $(a, \mathrm{int}(T))$ and $(b, \mathrm{int}(R))$ are incompatible. *Case 2.* Suppose w.l.o.g. that a end-extends b. If $a \backslash b \subseteq \mathrm{int}(R)$ then (a, Z) is a common extension of (a, T) and (b, R), which is a contradiction. Therefore $a \backslash b \not\subseteq \mathrm{int}(R)$ and so the conditions $(a, \mathrm{int}(T))$ and $(b, \mathrm{int}(R))$ are incompatible. $\square$

By [5, Lemma 6.2], if $\mathrm{cov}(\mathcal{M}) = \kappa$ for some regular cardinal κ such that $\forall \lambda < \kappa(2^\lambda \leq \kappa)$ and $\mathcal{H} \subseteq {}^\omega\omega$ is an unbounded, directed family of size κ then there is a centered family C such that $Q(C)$ preserves the unboundedness of $\mathcal{H}$ and adds a real which is not split by the ground model reals. Applying Lemma 2.4 we obtain the following.

Theorem 2.5. *Let κ be a regular cardinal such that $\forall \lambda < \kappa(2^\lambda \leq \kappa)$ and let $cov(\mathcal{M}) = \kappa$. Then there is an ultrafilter $\mathcal{U}$ such that $\mathbb{M}(\mathcal{U})$ is weakly bounding. Furthermore if $\mathcal{H} \subseteq {}^\omega\omega$ is an unbounded directed family of size κ then there is an ultrafilter $\mathcal{U}_\mathcal{H}$ such that $\mathbb{M}(\mathcal{U}_\mathcal{H})$ preserves the unboundedness of $\mathcal{H}$.*

Proof. To obtain the first part of the claim consider a dominating directed family of size κ, which exists since $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d} = \kappa$. Let $\mathcal{H}$ be an unbounded directed family of size κ and let $C = C_\mathcal{H}$ be the associated centered family constructed in [5, Lemma 6.2]. By Lemma 2.4 $Q(C)$ is densely embedded in $\mathbb{M}(\mathcal{U})$, where

$$\mathcal{U} = \mathcal{F}_C = \{X \in [\omega]^\omega : \exists T \in C(\mathrm{int}(T) \subseteq X)\}.$$

Therefore $Q(C)$ and $\mathbb{M}(\mathcal{U})$ are forcing equivalent and so $\mathbb{M}(\mathcal{U})$ preserves the unboundedness of $\mathcal{H}$. It remains to observe that $\mathcal{U}$ is an ultrafilter. For this consider an arbitrary $A \in [\omega]^\omega$.

Note that the centered family $C = \bigcup_{\alpha<\omega_2} C_\alpha$, where $\sigma = \langle C_\alpha \rangle_{\alpha<\omega_2}$ is an inductively defined sequence of centered families such that for all $\alpha < \beta$, $C_\alpha \subseteq Q(\mathcal{C}_\beta)$. Let $\{A_{\beta+1}\}_{\beta<\kappa}$ be the fixed enumeration of $[\omega]^\omega$ from the proof of [5, Lemma 6.2]. Let T_α, C'_α be the pure condition and centered family respectively, defined at stage α in the inductive definition of σ from the same proof. Then $A = A_\alpha$ for some $\alpha = \beta + 1 < \kappa$ and so by construction $\mathrm{int}(T_\alpha) \subseteq A_\alpha$ or $\mathrm{int}(T_\alpha) \subseteq A^c_\alpha$. However C_α is defined to be equal to $\{R_\alpha \wedge T\}_{T\in C'_\alpha}$ where R_α is some generic pure extension of T_α and for all $T \in C'_\alpha$, the pure condition $R_\alpha \wedge T$ is a carefully chosen subsequence of R_α (see [5, Corollary 3.18]). Therefore for every $X \in C_\alpha$, $\mathrm{int}(X) \subseteq A$ or $\mathrm{int}(X) \subseteq A^c$ and so A or A^c is an element of $\mathcal{U}$. □

3. Preserving small unbounded families

There is very little known about models in which $\mathfrak{c} \geq \aleph_2$ and there is an ultrafilter which preserves the unboundedness of a given unbounded family of size $< \mathfrak{c}$. Let $\mathbb{C}(\kappa)$ denote the poset for adding κ-many Cohen reals and let V denote the ground model.

Theorem 3.1. *Assume CH. There is a countably closed, $\aleph_2$-c.c. poset $\mathbb{P}$ which adds a $\mathbb{C}(\omega_2)$-name for an ultrafilter $\mathcal{U}$ such that in $V^{\mathbb{P}\times\mathbb{C}(\omega_2)}$ the forcing notion $\mathbb{M}(\mathcal{U})$ preserves the unboundedness of all families of Cohen reals of size ω_1.*

Proof. Let $\mathbb{P}$ be the poset defined in [6, Definition 16] and let C be the $\mathbb{C}(\omega_2)$-name for the centered family of pure condition added by $\mathbb{P}$. In $V^{\mathbb{P}\times\mathbb{Q}(\omega_2)}$ by [6, Theorem 1], the poset $Q(C)$ preserves the unboundedness of all families of Cohen reals of cardinality ω_1. Furthermore by Lemma 2.4 $Q(C)$ is densely embedded in $\mathbb{M}(\mathcal{U})$ where $\mathcal{U} = \{X \in [\omega]^\omega : \exists T \in C(\mathrm{int}(T) \subseteq X)\}$. It remains to observe that $\mathcal{U}$ is an ultrafilter (see [6, Lemma 7 and Theorem 1]). □

Theorem 3.2 (Brendle, Fischer [3])**.** *Assume GCH. Let $\kappa < \lambda$ be regular uncountable cardinals. Let $V_1 = V^{\mathbb{C}(\kappa)}$ and let $\mathcal{C}$ be the family of Cohen reals. Then there is a ccc generic extension V_2 of V_1 such that $V_2 = \mathfrak{c} = \lambda$ and in V_2 there is an ultrafilter $\mathcal{U}$ which preserves the unboundedness of $\mathcal{C}$.*

Proof. Let $\mu = \lambda + 1$ and let $\mathbb{P}'_{\kappa,\mu}$ be a forcing notion defined as $\mathbb{P}_{\kappa,\mu}$ from [3, Section 4], with the only difference that $\mathbb{P}'_{\alpha,0} = \mathbb{C}(\alpha)$ for all

$\alpha \leq \kappa$. Then $V_2 = V^{\mathbb{P}'_{\kappa,\lambda}}$ is the desired generic extension (following the notation of [3], let $\mathcal{U} = \mathcal{U}_{\kappa,\lambda}$). □

The method used in [3], referred to as *matrix-iteration*, first appears in [1], where assuming GCH with any regular cardinal λ one associates generic extensions $V_1 \subseteq V_2$ such that $V_1 = V^{\mathbb{C}(\omega_1)}$ and $V_2 \vDash (\mathfrak{c} = \lambda)$ is a ccc extension of V_1. If $\mathcal{C}$ is the family of the ω_1 Cohen reals added over the ground model V, then in V_2 there is an ultrafilter for which the relativized Mathias forcing preserves the unboundedness of $\mathcal{C}$.

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Kurt Gödel Research Center, University of Vienna, Währinger Strasse 25, A-1020 Vienna, Austria

E-mail address: vfischer@logic.univie.ac.at

Mathematisches Institut, University of Bonn, Endenicher Allee 60, D-53115 Bonn, Germany

E-mail address: irrgang@math.uni-bonn.de