Amenable actions of the infinite permutation group — Lecture III

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It will be shown in Lecture III that if the natural action of G on $\mathbb{N}$ has a *unique* invariant mean μ then this mean is defined by $\mu(A) < r$ for any rational r if and only if

$$(\exists Z \in [G]^{<\aleph_0})(\forall k \in \mathbb{N})\frac{|\{z \in Z \mid zk \in A\}|}{|Z|} < r$$

In the case of a $\{0,1\}$-valued invariant mean μ this yields that $\{A \subseteq \mathbb{N} \mid \mu(A) = 1\}$ is an ultrafilter. The preceding definition shows that if the definition of G is simple, then so is the quantifier "$\exists Z \in [G]^{<\aleph_0}$". This ultrafilter would then have to be analytic.

Recall from Lecture I that the argument establishing there are no analytic subgroups of $\mathbb{S}(\omega)$ that act with a unique mean relied on the fact that a unique mean, if it exists, has a nice definition. This will now be proved.

Definition

Let G be subgroup of $\mathbb{S}(\omega)$. A set $X \subseteq \omega$ is said to be r-thick (with respect to G) if and only if for every finite subset $H \subseteq G$ there is $n \in \omega$ such that

$$\frac{|\{h \in H \mid hn \in X\}|}{|H|} \geq r$$

Lemma (Wang)

If G is an amenable subgroup of $\mathbb{S}(\omega)$ then $X \subseteq \omega$ is r-thick if and only if there is a G-invariant mean μ on ω such that $\mu(X) \geq r$.

To see this first assume that $X \subseteq \omega$ is r-thick. Using that G is amenable — and hence satisfies the Følner condition — let $\{F_{\epsilon,H}\}_{\epsilon>0, H\in[G]^{<\aleph_0}}$ be such that

- $H \subseteq F_{\epsilon,H} \in [G]^{<\aleph_0}$
- if $\epsilon < \delta$ and $H \supseteq D$ then $F_{\epsilon,H} \supseteq F_{\delta,D}$
- $\frac{|hF_{\epsilon,H} \Delta F_{\epsilon,H}|}{|F_{\epsilon,H}|} < \epsilon$ for all $h \in H$.

Using the fact that X is r-thick choose for each $H \in [G]^{<\aleph_0}$ and $\epsilon > 0$ there is an integer $N_{\epsilon,H}$ such that

$$\frac{|\{h \in F_{\epsilon,H} \mid hN_{\epsilon,H} \in X|\}}{|F_{\epsilon,H}|} \geq r$$

Now define a measure $\mu_{\epsilon,H}$ by defining

$$\mu_{\epsilon,H}(Y) = \frac{|\{h \in F_{\epsilon,H} \mid hN_{\epsilon,H} \in Y\}|}{|F_{\epsilon,H}|}$$

and note that $\mu_{\epsilon,H}(X) \geq r$ for all H and ϵ.

Moreover, by the Følner property it follows that $\frac{\mu_{\epsilon,H}(gY)}{\mu_{\epsilon,H}(Y)} =$

$$\frac{|\{h \in F_{\epsilon,H} \mid hN_{\epsilon,H} \in gY\}|}{|\{h \in F_{\epsilon,H} \mid hN_{\epsilon,H} \in Y\}|} = \frac{|\{h \in g^{-1}F_{\epsilon,H} \mid hN_{\epsilon,H} \in Y\}|}{|\{h \in F_{\epsilon,H} \mid hN_{\epsilon,H} \in Y\}|}$$

for each $g \in H$ and since $\frac{|g^{-1}F_{\epsilon,H} \Delta F_{\epsilon,H}|}{|F_{\epsilon,H}|} < \epsilon$ it follows that

$$\lim_{\epsilon \to 0} \frac{\mu_{\epsilon,H}(gY)}{\mu_{\epsilon,H}(Y)} = 1$$

Let μ be a weak* limit of the $\mu_{\epsilon,H}$ along the net of (ϵ, H) in $(0,\infty) \times [G]^{<\aleph_0}$. This yields a G invariant measure such that $\mu(X) \geq r$.

To check the other direction suppose that $X \subseteq \omega$ and that μ is a mean such that $\mu(X) \geq r$. Then let $\psi : \ell_\infty \to \mathbb{R}$ be the linear function defined by Lebesgue integration with respect to μ. Then for any finite $H \subseteq G$ by linearity and G-invariance of ψ it follows that

$$\psi\left(\sum_{h\in H} \chi_{h^{-1}X}\right) = \sum_{h\in H} \psi(\chi_{h^{-1}X})) = |H|\mu(X) \geq |H|r$$

By the positivity of ψ this means that there must be at least one $n \in \omega$ such that $\sum_{h\in H} \chi_{h^{-1}X}(n) \geq |H|r$. In other words, $|\{h \in H \mid hn \in X\}| \geq |H|r$ as required.

Definition

For any group G acting on ω define a function m_G on the power set of ω by $m_G(X) = \sup(\{r \in \mathbb{R} \mid X \text{ is } r\text{-thick}\})$.

Corollary

If G is an amenable group acting on ω then m_G is a finitely additive probability measure if and only the action of G on ω has a unique invariant mean.

Note that the preceding lemma yields the following alternate definition of m_G:

$$m_G(X) = \sup(\{r \in \mathbb{R} \mid (\exists\mu)\ \mu \text{ is an invariant mean and } \mu(X) = r\})$$

and if there is a unique invariant mean μ this yields that $m_G(X) = \mu(X)$. Hence m_G is an invariant probability measure.

For the other direction, suppose that m_G is an invariant mean. From the definition of m_G it follows that if μ is an other invariant mean then

$$\mu(X) \leq \sup(\{\mu(X) \mid \mu \text{ is an invariant mean}\}) = m_G(X)$$

for every X. But if $\mu(X) \lneq m_G(X)$ for some X then $\mu(\omega \setminus X) \leq m_G(\omega \setminus X)$ and hence $\mu(\omega) = \mu(X) + \mu(\omega \setminus X) \lneq m_G(X) + m_G(\omega \setminus X) = 1$.

Foreman showed that in the model obtained by adding $\aleph_2$ Cohen reals to a model of CH that there is no locally finite subgroup of $\mathbb{S}(\omega)$ that acts on ω with a unique invariant mean. An analysis of his argument will show that he actually proved the following.

Theorem

Let $\mathbb{P} = \prod_{\xi\in\omega_2} \mathbb{P}_\xi$ be a finite support product of ccc partial orders. If $G \subseteq \prod_{\xi\in\omega_2} \mathbb{P}_\xi$ is generic over V then in $V[G]$ the following holds: There is no subgroup $G \subseteq \mathbb{S}(\omega)$ acting with a unique invariant mean on ω such that for any finite set $H \subseteq G$ there is a recursive function $F_H : \omega \to \omega$ such that the orbit of each n under the subgroup generated by H has cardinality bounded by $F_H(n)$.

Note that if G is locally finite then F_H is a constant function for each H. "Recursive" is actually weaker than needed since it will be shown that F_H can not be chosen from V.

The support of $\mathbb{P}$ adds $\aleph_2$ Cohen reals; but, for notational convenience, assume that each $\mathbb{P}_\xi$ has exactly two maximal elements, 0_ξ and 1_ξ, and let $c_\xi \subseteq \omega$ be defined by $n \in c_\xi$ if and only if $1_{\xi+n} \in G$.

Now assume that $\mathbb{G}$ is a $\mathbb{P}$ name for a subgroup $G \subseteq \mathbb{S}(\omega)$ acting with a unique mean on ω such that for any finite set $H \subseteq G$ there is a recursive function $F_H : \omega \to \omega$ such that for each n the orbit of n under the subgroup generated by H has cardinality bounded by $F_H(n)$. It must be that the unique mean is

$$m_{\mathbb{G}} = \sup(\{r \in \mathbb{R} \mid X \text{ is } r\text{-thick}\})$$

By symmetry, there is no harm in assuming that $m_{\mathbb{G}}(c_\xi) < 1$ for $\aleph_2$ of the ξ. In other words, $\aleph_2$ of the c_ξ are not 1-thick and hence there are finite $H_\xi \subseteq \mathbb{G}$ such that for all $n \in \omega$

$$H_\xi n \not\subseteq c_\xi$$

Now let S_ξ be a countable subset of ω_2 such that c_ξ and H_ξ have $\prod_{\eta \in S_\xi} \mathbb{P}_\eta$ names. Let R be a countable set and $\xi \neq \eta$ be such that $\{\xi + j\}_{j \in \omega} \subseteq S_\xi \setminus R$ and $\{\eta + j\}_{j \in \omega} \subseteq S_\eta \setminus R$. Let $G_R \subseteq \prod_{\rho \in R} \mathbb{P}_\rho$ be generic over V. Let $H_\xi / G_R = H'_\xi$ and $H_\eta / G_R = H'_\eta$ be names in $V[G_R]$.

Let $\mathbb{Q}_\xi = \prod_{\alpha \in S_\xi \setminus R} \mathbb{P}_\alpha$ and $\mathbb{Q}_\eta = \prod_{\alpha \in S_\eta \setminus R} \mathbb{P}_\alpha$ and $\mathbb{Q} = \prod_{\rho \in \omega_2 \setminus R} \mathbb{P}_\rho$

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In $V[G_R]$ choose a condition $q \in \mathbb{Q}$ such that

$$q \Vdash_{\mathbb{Q}} \text{“}F_{H'_\xi \cup H'_\eta} = \check{F}\text{”}$$

Claim

For $p \leq q$ the set of $n \in \omega$ such that

$$|\{m \in \omega \mid p \restriction S_\xi \nVdash_{\mathbb{Q}_\xi} \text{“}m \notin \langle H'_\xi \rangle n\text{”}\}| < \aleph_0$$

is finite where $\langle H'_\xi \rangle$ is the subgroup generated by H'_ξ. Same for η.

To see this let S be the support of p and $S^* = \{j \mid \xi + j \in S\}$ and suppose, heading towards a contradiction, that

$$Z \subseteq \{n \in \omega \mid |\{m \in \omega \mid p \nVdash_{\mathbb{Q}} \text{“}m \notin \langle H'_\xi \rangle n\text{”}\}| < \aleph_0\}$$

is such that $|Z| > \sum_{j \in S^*} F(j)$.

Let $Y = \left\{ m \in \omega \;\middle|\; (\exists n \in Z) p \not\Vdash_{\mathbb{Q}_\xi} \text{“} m \notin \langle H'_\xi \rangle n \text{”} \right\}$ and note that Y is finite. Let $p' \geq p$ be such that $p'(\xi + k) = 1_{\xi+k}$ for each $k \in Y \setminus S^*$. Note that p' and q are compatible.

Let q' extend both q and p' such that $q' \Vdash_{\mathbb{Q}}$ “$\langle H'_\xi \rangle S^* = \check{W}$” and note that $|W| \leq \sum_{j \in S^*} F(j) < |Z|$. Let $z \in Z \setminus W$ and note that, since $q \Vdash_{\mathbb{Q}}$ “$\langle H'_\xi \rangle$ is a group”, it follows that $q \Vdash_{\mathbb{Q}}$ “$\langle H'_\xi \rangle z \cap S^* = \emptyset$”.

But since $z \in Z$ it follows that if $q' \Vdash_{\mathbb{Q}}$ “$m \in H'_\xi z$” then $p \not\Vdash_{\mathbb{Q}_\xi}$ “$m \notin \langle H'_\xi \rangle z$” and hence $m \in Y \subseteq c_\xi$. In other words, $q' \Vdash_{\mathbb{Q}}$ “$H'_\xi z \subseteq c_\xi$” and this contradicts the choice of H_ξ using the fact that $m_{\mathbb{G}}(c_\xi) < 1$.

To arrive at a contradiction construct, using the claim, a sequence, $\{(p_i, p'_i, m_i, m'_i)\}_{i\in\omega}$ such that

- $p_i \in \mathbb{Q}_\xi$ and $p'_i \in \mathbb{Q}_\eta$
- $p_{i+1} \leq p_i \leq q \restriction S_\xi$ and $p'_{i+1} \leq p'_i \leq q \restriction S_\eta$
- $p_i \Vdash_{\mathbb{Q}_\xi}$ "$m'_i \in \langle H'_\xi \rangle m_i$"
- $p'_i \Vdash_{\mathbb{Q}_\eta}$ "$m_{i+1} \in \langle H'_\eta \rangle m'_i$"
- all the m_i and m'_i are distinct.

To carry out the induction it will be assumed as an additional induction hypothesis that

- $X_i = \left\{ m \in \omega \;\middle|\; p_{i-1} \not\Vdash_{\mathbb{Q}_\xi} \text{"}m \notin \langle H'_\xi \rangle m_i\text{"} \right\}$ is infinite
- $X'_i = \{ m \in \omega \mid p'_{i-1} \not\Vdash_{\mathbb{Q}_\eta} \text{"}m \notin \langle H'_\eta \rangle m'_i\text{"} \}$ is infinite.

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To begin the induction choose m_0 using the claim such that $X_0 = \left\{ m \in \omega \;\middle|\; q \restriction S_\xi \not\Vdash_{\mathbb{Q}_\xi} \text{“} m \notin \langle H'_\xi \rangle m_0 \text{”} \right\}$ is infinite and let $p_{-1} = q \restriction S_\xi$ and $p'_{-1} = q \restriction S_\eta$.

Given that X_i is infinite, it is possible to use the claim to choose $m'_i \in X_i$ such that

$$X'_i = \left\{ m \in \omega \;\middle|\; p'_{i-1} \not\Vdash_{\mathbb{Q}_\eta} \text{“} m \notin \langle H'_\xi \rangle m_i \text{”} \right\}$$

is infinite. It is then possible to find $p_i \leq p_{i-1} \in \mathbb{Q}_\xi$ such that $p_i \Vdash_{\mathbb{Q}_\xi} \text{“} m'_i \in \langle H'_\xi \rangle m_i \text{”}$.

Next, choose $m_{i+1} \in X'_i$ such that $X_{i+1} = \left\{ m \in \omega \;\middle|\; p_i \not\Vdash_{\mathbb{Q}_\xi} \text{“} m \notin \langle H'_\xi \rangle m_{i+1} \text{”} \right\}$ is infinite. Then choose $p'_i \leq p'_{i-1}$ such that $p'_i \Vdash_{\mathbb{Q}_\eta} \text{“} m_{i+1} \in \langle H'_\xi \rangle m'_i \text{”}$ as required.

Let $k > F(m_0)/2$ and note that

$$q \cup p_k \cup p'_k \Vdash_{\mathbb{Q}} \text{“}\bigcup_{i \in k} \{m_i, m'_i\} \subseteq \langle H'_\xi \cup H\eta \rangle m_0\text{”}$$

But

$$q \cup p_k \cup p'_k \Vdash_{\mathbb{Q}} \text{“}|\langle H'_\xi \cup H\eta \rangle m_0| < F(m_0) < 2k\text{”}$$

while $|\bigcup_{i \in k} \{m_i, m'_i\}| = 2k$.

Corollary

Adding $\aleph_2$ Cohen reals to any model of set theory yields a model where no locally finite subgroup of $\mathbb{S}(\omega)$ acts with a unique mean.

Corollary

Let $\mathbb{P}$ be a ccc poset for getting a model of Martin's Axiom. Then the finite support product of $\aleph_2$ copies of $\mathbb{P}$ forces that no locally finite subgroup of $\mathbb{S}(\omega)$ acts with a unique mean.

An example of a non-locally finite subgroup $G \subseteq \mathbb{S}(\omega)$ which, nevertheless, satisfies the property that for any finite set $H \subseteq G$ there is a recursive function $F_H : \omega \to \omega$ such that for each n the orbit of n under the subgroup generated by H has cardinality bounded by $F_H(n)$ is easy to construct. Let $\{A_n\}_{n\in\omega}$ partition ω into finite sets such that $\lim_{n\to\infty} |A_n| = \infty$. Let G consist of all permutations θ such that $\theta \restriction A_n \in \mathbb{S}(A_n)$ for all n.