Amenable actions of the infinite permutation group — Lecture II

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Definition

The least cardinal of a generating set for a free ultrafilter on ω is denoted by $\mathfrak{u}$.

Definition

The least cardinal of a filter $\mathcal{F}$ on ω such that there is no infinite $X \subseteq \omega$ such that $X \subseteq^* A$ — in other words, $X \setminus A$ is finite — for all $A \in \mathcal{F}$ is denoted by $\mathfrak{p}$.

Theorem (Bell)

The least cardinal κ for which $MA_\kappa(\sigma$-centred) fails is equal to $\mathfrak{p}$.

The **Key Hypothesis** is the following: There is a generating set $\{G_\xi\}_{\xi\in\kappa}$ for an ultrafilter on ω such that there exist infinite $A_\xi \subseteq \omega$ satisfying:

- $A_\xi \subseteq^* G_\eta$ for each $\eta \leq \xi$
- $A_\xi \cap A_\eta$ is finite if $\xi \neq \eta$.

Lemma

If $\mathfrak{p} = \mathfrak{u}$ then the Key Hypothesis is satisfied.

To see this, let $\{G_\xi\}_{\xi \in \mathfrak{u}}$ generate an ultrafilter $\mathcal{F}$. Choose inductively for each $\xi \in \mathfrak{u}$ an infinite set $A_\xi \notin \mathcal{F}$ such that

- $A_\xi \subseteq^* G_\eta$ for each $\eta \leq \xi$
- $A_\xi \cap A_\eta$ is finite if $\xi > \eta$.

This can be done because $\{G_\xi\} \cup \{G_\eta, \omega \setminus A_\eta\}_{\eta \in \xi}$ generates a filter and $\xi < \mathfrak{p}$.

Corollary

If $\mathfrak{p} = \mathfrak{c}$ or $\mathfrak{u} = \aleph_1$ then the Key Hypothesis holds.

It will follow from the results of Lecture III that the Key Hypothesis fails in the model obtained by adding $\aleph_2$ Cohen reals to a model of CH.

Given the Key Hypothesis, it can be assumed that $A_\xi \subseteq G_\xi$ rather than just $A_\xi \subseteq^* G_\xi$ for each $\xi \in \kappa$. Let A_ξ be enumerated by $\omega \times \omega$ — in other words, $A_\xi = \{a_\xi(i,j)\}_{(i,j)\in\omega\times\omega}$. Let $\omega \setminus G_\xi$ be enumerated by $\{a_\xi(-1,j)\}_{j\in\omega}$. Now define permutations $\theta_{\xi,i}$, for $i \geq -1$, as follows:

$$\theta_{\xi,i}(x) = \begin{cases} a_\xi(i+1,j) & \text{if } x = a_\xi(i,j) \\ a_\xi(i,j) & \text{if } x = a_\xi(i+1,j) \\ x & \text{otherwise} \end{cases}$$

Note that each $\theta_{\xi,i}$ is an involution sending $A_{\xi,i}$ to $A_{\xi,i+1}$ and the $A_{\xi,i}$ are pairwise disjoint where $A_{\xi,i} = \{a_\xi(i,j)\}_{j\in\omega}$. Let $\mathbb{G}$ be the subgroup of $\mathbb{S}(\omega)$ generated by $\{\theta_{\xi,i}\}_{\xi\in\kappa, i\geq -1}$.

The first thing to notice is that each element of $\mathbb{G}$ preserves the ultrafilter $\mathcal{F}$ generated by $\{G_\xi\}_{\xi\in\kappa}$ — in other words, if $\mu_{\mathcal{F}}(X) = 1$ if and only if $X \in \mathcal{F}$ then $\mu_{\mathcal{F}}$ is an invariant mean for the natural action of $\mathbb{G}$ on ω.

Moreover, $\mu_{\mathcal{F}}$ is unique. To see this, suppose that ν is another invariant mean on ω. There must be some ξ such that $\nu(\omega \setminus G_\xi) > 0$ and let $k > \nu(\omega \setminus G_\xi)$. Recall that $A_{\xi,-1} = \omega \setminus G_\xi$ and $A_{\xi,i} = \{a_\xi(i,j)\}_{j\in\omega}$ for $i \geq 0$ and, hence, $\theta_{\xi,i}(A_\xi, i) = A_{\xi,i+1}$ when $-1 \leq i < k$. Moreover $A_{\xi,i} \cap A_{\xi,i'} = \emptyset$ if $i \neq i'$.

The invariance of ν under $\mathbb{G}$ then implies that $\nu(\bigcup_{i=-1}^{k} A_{\xi,i}) = (k+2)\nu(\omega \setminus G_\xi) > 1$ contradicting that ν is a probability measure.

The fact that $\mathbb{G}$ is amenable will be established by showing that it satisfies the much stronger property of being locally finite. The Key Hypothesis will be crucial for the argument.

Definition

If $\sigma \in \mathbb{S}(\omega)$ then the support of σ is defined to be $\{n \in \omega \mid \sigma(n) \neq n\}$. If $H \subseteq \mathbb{S}(\omega)$ then the support of H is defined to be the union of the supports of its elements.

Lemma

A finite subset $H \subseteq \mathbb{S}(\omega)$ generates a finite group H' if and only if there is a uniform bound for the size of the orbits $O_{H'}(\{n\})$ of n under H'.

To see this, note first that $|H'| \geq |O_{H'}(\{n\})|$ for each $n \in \omega$.

For the other direction suppose that $M \geq |O_{H'}(\{n\})|$ for each $n \in \omega$ and let $\{W_n\}_{n\in\omega}$ list all the distinct orbits of H'. Let $D_m = \{i \in \omega \mid |W_i| = m\}$ and for each $i \in D_m$ let $W_i = \{w_{i,j}\}_{j\in m}$. For $h \in H$ let

$$\Phi_h : D_m \to \mathbb{S}(m)$$

be defined by $\Phi_h(i)(j) = k$ if and only if $h(w_{i,j}) = w_{i,k}$. Note that there is a partition

$$D_m = \bigcup \left\{ D_m^\phi \mid \phi : H \to \mathbb{S}(m) \right\}$$

such that $\Phi_h(j) = \phi(h)$ for each h and $j \in D_m^\phi$.

Let H_m^ϕ be the subgroup of $\mathbb{S}(m)$ generated by $\{\phi(h) \mid h \in H\}$. It suffices to observe that H' is isomorphic to

$$\prod_{m \leq J} \prod_{\phi : H \to \mathbb{S}(m)} H_m^\phi$$

and that this is a finite group.

Lemma

Let θ be an involution that sends A to B and B to A and fixes all else. If $H \subseteq \mathbb{S}(\omega)$ is a subgroup whose support is almost disjoint from A then $H \cup \{\theta\}$ generates a finite group.

To begin note that there is no loss of generality in assuming that the support of H is actually disjoint from A. To see this let Z be the orbit under H of the intersection of A with the support of H. This is finite and $H = H_Z \times H_{\omega \setminus Z}$ where $H_X = \{h \restriction X \mid h \in H\}$. The support of $H_{\omega\setminus Z}$ is actually disjoint from A. Since H_Z consists only of permutations with finite support it follows that the group generated by $H \cup \{\theta\}$ is finite if and only if the group generated by $H_{\omega\setminus Z} \cup \{\theta\}$ is finite.

The argument will rely on showing that there is a uniform bound for $|O_{H\cup\{\theta\}}(\{n\})|$. To see that this is the case, it suffices to show that for any such n, if f is in the group generated by $H \cup \{\theta\}$ then $f(n)$ is equal to one of:

- $\theta h\theta(n)$
- $h\theta(n)$
- $\theta h(n)$
- or $h(n)$

where $h \in H$. The reason this suffices is that there is a uniform bound for $O_H(\{n\})$ and adding θ will, at most, increase the size by a factor of 4.

An arbitrary element f of the group generated by $H \cup \{\theta\}$ looks like

$$f = \theta^{n_0} h_0 \theta^{n_1} h_1 \ldots \theta^{n_k} h_k \theta^{n_{k+1}}$$

and, keeping in mind that θ is an involution, one of the following alternatives holds

$$f = \theta h_0 \theta h_1 \ldots \theta h_k \theta$$

$$f = \theta h_0 \theta h_1 \ldots \theta h_k$$

$$f = h_0 \theta h_1 \ldots \theta h_k \theta$$

$$f = h_0 \theta h_1 \ldots \theta h_k$$

All that needs to be checked now is that if h_0 and h_1 belong to H then...

… $h_0\theta h_1(n)$ is equal to one of the following:

- $h_0\theta(n)$
- $h_0 h_1(n)$
- $\theta h_1(n)$.

To see this, first suppose that $h_1(n) = n$. Then $h_0\theta h_1(n) = h_0\theta(n)$.

Otherwise, it must be that $h_1(n)$ is in the support of H and there are two possibilities The first is that $h_1(n)$ is not in the support of θ. In this case $h_0\theta h_1(n) = h_0 h_1(n)$. Otherwise, because the support of θ is $A \cup B$ and A is disjoint from the support of H it must be that. $h_1(n) \in B$. Since θ sends B to A it follows that $\theta(h_1(n)) \in A$ and hence is not in the support of h_0. Therefore $h_0\theta h_1(n) = \theta h_1(n)$.

To finish the proof that $\mathbb{G}$ is locally finite it suffices to show that any finite set of generators of $\mathbb{G}$ generates a finite group. Recall that the Key Hypothesis provided a generating set $\{G_\xi\}_{\xi\in\kappa}$ for an ultrafilter on ω and infinite $A_\xi \subseteq \omega$ satisfying:

- $A_\xi \subseteq^* G_\eta$ for each $\eta \leq \xi$
- $A_\xi \cap A_\eta$ is finite if $\xi \neq \eta$.

and that this yields involutions $\theta_{\xi,-1}$ from $\omega \setminus G_\xi$ to a subset of A_ξ and $\theta_{\xi,i} : A_{\xi,i} \to A_{\xi,i+1}$ such that the $A_{\xi,i}$ are pairwise disjoint. Proceed by induction on the size of $H \subseteq \mathbb{G}$ to show that H generates a finite set. When $|H| = 1$ use that the generators are involutions.

Supposing the result is true for H of size n let $|H| = n+1$ and let ξ be the largest ordinal such that there is some j such that $\theta_{\xi,j} \in H$. Let J be the largest integer such that $\theta_{\xi,J} \in H$. It follows that $A_{\xi,J+1}$ is almost disjoint from the support of the group generated by $H \setminus \{\theta_{\xi,J}\}$. Since $\theta_{\xi,J}$ is an involution the preceding lemma implies that the group generated by $\theta_{\xi,J}$ and $H \setminus \{\theta_{\xi,J}\}$ is finite.

Question

Is it possible to strengthen locally finite to some other condition?

Abelian is not possible. Rosenblatt and Talagrand showed that no nilpotent subgroup of $\mathbb{S}(\omega)$ acts with a unique mean and this was improved by Krasa to show that there is no solvable subgroup of $\mathbb{S}(\omega)$ acts with a unique mean.

No countable subgroup of $\mathbb{S}(\omega)$ acts with a unique mean and assuming MA there is no such group of cardinality less than $2^{\aleph_0}$.