Research statement

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My interest in mathematics concerns compact scattered spaces obtained by the methods of the infinite combinatorics and their applications in Banach space theory. By scattered space I mean a space in which every non-empty subset has an isolated point. I am interested only in such compact and Haussdorff, thus normal, spaces. Some examples of such spaces are: Ψ-spaces, the Kunen space and the Ciesielski-Pol space.

It is easy to see that every compact scattered space is also zero-dimensional. Therefore every such space corresponds to an Boolean algebra by the Stone duality. It turns out [6] that the class of Boolean algebras obtained this way is exactly the class of superatomic algebras defined earlier by Mostowski and Tarski [2], that is algebras in which every non-zero element in every subalgebra has an atom below it.

Metrizable compact scattered spaces have clear classification due to Mazurkiewicz and Sierpiński [1], which says that every compact scattered metrizable space is homeomorphic to a countable successor ordinal with ordinal topology. Therefore every countable superatomic algebra is isomorphic to the algebra of clopen sets of some countable successor ordinal. But uncountable superatomic algebras make a much larger class and classification of that class is still beyond the scope of mathematical research.

Nevertheless one can define some important parameters of an superatomic algebra, such as height or width of an algebra [6]. Intuitively the height parameter measures when the process of taking iterated Cantor-Bendixon derivative of a the Stone space dual to the algebra will stabilize (it will stabilize on $\varnothing$ if and only if the space was scattered) and the width determines maximum cardinality of the set of isolated points in the spaces obtained in this process.

There are some open problems concerning these parameters. For example it is not known if there exists an superatomic Boolean algebra of countable width and height ω_3. Also there is not much research concerning the continuous maps between compact scattered spaces. For example the question whether exists (under ZFC) a compact scattered space K of uncountable height and countable width without non-trivial continuous maps, where by trivial map I mean a map $f\colon K \to K$, for which exists a countable subset $A \subseteq K$, such that f restricted to $K \setminus f^{-1}[A]$ is identity, is also open.

We can also consider Banach spaces of real-valued continuous functions on a compact scattered space. It is well known [7] that the dual space of $C(K)$, where K is compact and Haussdorff, is the space $M(K)$ of signed Radon measures on K. But when K is scattered every $\mu \in M(K)$ is of form $\mu(x) = \sum_{n=1}^{\infty} a_n x(t_n)$ for some $t_1, \ldots, t_n \in K$ and $\sum |a_n| < \infty$ – see [4].

Banach spaces of form $C(K)$, where K is compact and scattered, play also significant role in the Banach space theory as examples and counterexamples. For example in 1930 J. Schreier proving that dual space of $C(\omega+1)$ is isomorphic with the l_1 space, and the same is true for $C(\omega^\omega+1)$, but these spaces are not isomorphic, settled the problem if there exist non-isomorphic Banach spaces with isomorphic dual spaces. It was [3] proved (under CH) that if K is Kunen space (constructed under CH compact scattered space) than K^n is hereditarily separable for each $n \in \omega$, but $C(K)$ has no uncountable biorthogonal system. It is possible that spaces of form $C(K)$ where K is compact and scattered can give more examples or counterexamples for open problems.

My current work focuses on studying known results concerning this theme and advanced topics in forcing, combinatorial set theory and Banach space theory. I am a Ph.D. student at Faculty of Mathematics, Informatics and Mechanics of Warsaw University. My advisor is Prof. Piotr Koszmider (Mathematical Institute of Polish Academy of Science and Lodz Technical University).

References

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