The Resurrection Axioms

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Existential Closure

I shall introduce the *Resurrection Axioms*, a class of new forcing axioms, inspired by the concept of *existential closure* in model theory.

Definition

A submodel $\mathcal{M}$ of a model $\mathcal{N}$ is *existentially closed in* $\mathcal{N}$ if existential witnesses in $\mathcal{N}$ for Σ_1 formulas exist already in $\mathcal{M}$. In other words, $\mathcal{M} \prec_{\Sigma_1} \mathcal{N}$.

Examples:

- linear order $\langle \mathbb{Z}, < \rangle$ is not existentially closed in $\langle \mathbb{Q}, < \rangle$
- field $\langle \mathbb{Q}, +, \cdot, 0, 1 \rangle$ is not existentially closed in $\langle \mathbb{R}, +, \cdot, 0, 1 \rangle$

Forcing Axioms as Existential Closure

Many classical forcing axioms (such as MA,PFA, or MM) can be viewed as expressing to a degree that the universe is existentially closed.

The essence of these forcing axioms is the assertion that a certain filter, which does exist in a certain forcing extension $V[g]$, exists already in V.

$$V \subseteq V[g]$$

The universe V is *never* existentially closed in a nontrivial forcing extension $V[g]$.

But the collection

$$H_{\mathfrak{c}} = \{\text{sets of hereditary size less than } \mathfrak{c}\}$$

can be existentially closed in forcing extensions; this is exactly what certain bounded forcing axioms express.

Forcing Axioms as Existential Closure

Theorem (Stavi('80s); Bagaria '97)

Martin's Axiom MA *is equivalent to the assertion that for any c.c.c. forcing extension* $V[g]$

$$H_{\mathfrak{c}} \prec_{\Sigma_1} H_{\mathfrak{c}}^{V[g]} \quad .$$

This is equivalent to $H_{\mathfrak{c}} \prec_{\Sigma_1} V[g]$.

Theorem (Bagaria '00)

The Bounded Proper Forcing Axioms BPFA *is equivalent to the assertion that for any proper forcing extension* $V[g]$,

$$H_{\omega_2} \prec_{\Sigma_1} H_{\omega_2}^{V[g]} \quad ,$$

Again, this implies $H_{\mathfrak{c}} \prec_{\Sigma_1} V[g]$, since BPFA implies $\mathfrak{c} = \aleph_2$.

Existential Closure iff Resurrection

Theorem

The following are equivalent.

1. *The model $\mathcal{M}$ is existentially closed in $\mathcal{N}$.*
2. *$\mathcal{M}$ has Resurrection. That is, there is a model $\mathcal{N}'$ such that $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{N}'$ such that $\mathcal{M} \prec \mathcal{N}'$.*

Proof.

$(1 \to 2)$. If $\mathcal{M} \subseteq \mathcal{N}$ is existentially closed in $\mathcal{N}$, then the theory consisting of the full elementary diagram of $\mathcal{M}$ combined with the atomic diagram of $\mathcal{N}$ is consistent. A model of this theory is the desired $\mathcal{N}'$.
$(2 \to 1)$. Resurrection implies existential closure, since witnesses in $\mathcal{N}$ still exist in $\mathcal{N}'$, which is fully elementary over $\mathcal{M}$. □

The Key Point. Equivalence can break down when the class of permitted models $\mathcal{N}'$ is restricted. But resurrection remains stronger.

The Main Idea

This suggests using resurrection to formulate forcing axioms, in place of Σ_1-elementarity.

That is, we shall formulate forcing axioms by means of the resurrection concept, considering not just Σ_1 elementarity in the relevant forcing extensions

$$\forall \mathbb{Q} \ldots M \prec_{\Sigma_1} M^{V^{\mathbb{Q}}}$$

but instead asking for *full elementarity* in a further extension

$$\forall\, \mathbb{Q}\, \exists\, \dot{\mathbb{R}} \ldots M \prec M^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}$$

The Resurrection Axioms

This leads to the Resurrection Axioms.

Definition

- The *Resurrection Axiom* RA is the assertion that for every forcing notion $\mathbb{Q}$ there is further forcing $\dot{\mathbb{R}}$ such that whenever $g * h \subseteq \mathbb{Q} * \dot{\mathbb{R}}$ is V-generic, then $H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{V[g*h]}$.
- More generally, for any class Γ of forcing notions, the *Resurrection Axiom* $\mathrm{RA}(\Gamma)$ asserts that for every $\mathbb{Q} \in \Gamma$ there is $\dot{\mathbb{R}} \in \Gamma^{V^{\mathbb{Q}}}$ such that whenever $g * h \subseteq \mathbb{Q} * \dot{\mathbb{R}}$ is V-generic, then $H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{V[g*h]}$.
- The *weak Resurrection Axiom* $\mathrm{wRA}(\Gamma)$ is the assertion that for every $\mathbb{Q} \in \Gamma$ there is $\dot{\mathbb{R}}$, such that if $g * h \subseteq \mathbb{Q} * \dot{\mathbb{R}}$ is V-generic, then $H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{V[g*h]}$.

Relation to MA and BPFA

Theorem

wRA(ccc) *implies* MA.

Proof.

If $V[g]$ is ccc, $H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{V[g*h]}$, then $H_{\mathfrak{c}} \prec_{\Sigma_1} H_{\mathfrak{c}}^{V[g]}$, so MA. □

Similarly,

1. wRA(proper) + ¬CH implies BPFA.
2. wRA(stationary-preserving) + ¬CH implies BMM.
3. wRA(Axiom A) + ¬CH implies BAAFA.
4. generalizing, wRA(Γ) implies BFA(Γ, $<\mathfrak{c}$).

One cannot omit ¬CH.

Relation to CH

Theorem

If $\mathrm{wRA}(\Gamma)$*, then every* $\mathbb{Q} \in \Gamma$ *preserves all cardinals below* $\mathfrak{c}$*.*

Proof.

If $\delta < \mathfrak{c}$ is a cardinal, then $H_{\mathfrak{c}}$ sees that δ is a cardinal, so it cannot be collapsed in $H_{\mathfrak{c}}^{V[g*h]}$. □

Corollary

1. RA *implies* CH.
2. $\mathrm{wRA}($*countably closed*$)$ *implies* $\mathfrak{c} \leq \omega_2$.
3. $\mathrm{wRA}($*proper*$)$, $\mathrm{wRA}($*semi-proper*$)$, $\mathrm{wRA}($*stationary-preserving*$)$,... *each implies* $\mathfrak{c} \leq \omega_2$

More CH implications

Theorem

If $\mathrm{wRA}(\Gamma)$*, then every* $\mathbb{Q} \in \Gamma$ *is stationary-preserving below* $\mathfrak{c}$*.*

Proof.

If $S \subseteq \delta$ is a stationary subset of some $\delta < \mathfrak{c}$, then $H_{\mathfrak{c}}$ sees that S is stationary, so it cannot be non-stationary in $H_{\mathfrak{c}}^{V[g*h]}$. □

Corollary

1. $\mathrm{wRA}($*countably distributive*$)$ *implies* CH.
2. $\mathrm{wRA}(\omega_1$*-preserving*$)$ *implies* CH.
3. $\mathrm{wRA}($*cardinal-preserving*$)$ *implies* CH

RA(proper) consistent with CH

Theorem

RA(*proper*) *is relatively consistent with* CH.

Proof.

Assume RA(proper). Let $V[G] \models \mathrm{CH}$ via $\mathbb{P} = \mathrm{Add}(\omega_1, 1)$. We claim $V[G] \models \mathrm{RA}$(proper). If $\mathbb{Q}$ is proper in $V[G]$, then $\mathbb{P} * \mathbb{Q}$ is proper in V, so there is proper $\dot{\mathbb{R}}$ with $H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{V[G*g*h]}$. It follows that

$$H_{\omega_1} \prec H_{\omega_1}^{V[G*g*h]}.$$

Force CH again to $V[G * g * h * h_2] \models \mathrm{CH}$. Observe that

$$H_{\omega_1}^{V[G]} = H_{\omega_1} \prec H_{\omega_1}^{V[G*g*h]} = H_{\omega_1}^{V[G*g*h*h_2]}.$$

This shows $V[G] \models \mathrm{RA}(\text{proper}) + \mathrm{CH}$. □

Same for RA(Axiom A), RA(semi-proper), RA(stationary-preserving),...

$\mathrm{RA}(\mathrm{ccc})$ implies $\mathfrak{c}$ is enormous

But:

Theorem

$\mathrm{RA}(\mathrm{ccc})$ *implies* $\mathfrak{c}$ *is a weakly inaccessible cardinal, and a limit of such cardinals, and so on.*

Proof.

It implies MA, so $\mathfrak{c}$ is regular. The continuum cannot be a successor cardinal, since if $\mathfrak{c} = \delta^+$, then let $\mathbb{Q}$ add δ^{++} many Cohen reals. If $\dot{\mathbb{R}}$ is further ccc forcing and

$$H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{V[g*h]},$$

then on the left side, $H_{\mathfrak{c}}$ thinks δ is the largest cardinal, but the right side does not. Similar for limit of inaccessibles, and so on. □

RA(cardinal-preserving) is inconsistent

Theorem

RA(*cardinal-preserving*) *is inconsistent.*

Proof.

We observed wRA(cardinal-preserving) implies CH. But, the same argument as for $\mathrm{RA}(\mathrm{ccc})$, but now for cardinal-preserving forcing $\dot{\mathbb{R}}$ shows that $\mathfrak{c}$ cannot be a successor cardinal. □

How strong is Resurrection?

Uplifting cardinals

Definition

A regular cardinal κ is *uplifting* if $H_\kappa \prec H_\gamma$ for unboundedly many regular cardinals γ.

It follows that κ, γ are all inaccessible.
Thus: κ is uplifting iff κ is inaccessible and $V_\kappa \prec V_\gamma$ for unboundedly many inaccessible γ.

Observation

- *If κ is uplifting, then κ is uplifting in L.*
- *If κ is Mahlo, then V_κ has a proper class of uplifting cardinals.*
- *If κ is uplifting, then κ is Σ_2-reflecting, and a limit of Σ_2-reflecting cardinals*

Thus: Σ_2-reflecting $<$ uplifting $<$ Mahlo.

RA implies $\mathfrak{c}^V$ is uplifting in L

Theorem

RA *implies that* $\mathfrak{c}^V = \aleph_1^V$ *is uplifting in* L*.*

Proof.

Let $\kappa = \mathfrak{c} = \aleph_1$, which is regular in L. To see that κ is uplifting in L, fix any $\alpha > \kappa$, and let $\mathbb{Q}$ collapse α to $\aleph_0$. By RA there is $\dot{\mathbb{R}}$ such that

$$H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{V[g*h]}.$$

Let $\gamma = \mathfrak{c}^{V[g*h]}$, which is a cardinal above α. Restricting to constructible sets shows that $H_\kappa^L = (H_\kappa \cap L) \prec (H_\gamma^{V[g*h]} \cap L) = H_\gamma^L$. The cardinal $\gamma = \aleph_1^{V[g*h]}$ is regular in L. □

Resurrection implies $\mathfrak{c}^V$ uplifting in L

Similarly, we obtain:

Theorem

Many instances of $\mathrm{wRA}(\Gamma)$ *imply that* $\mathfrak{c}^V$ *is uplifting in* L*.*

1. RA *implies* $\mathfrak{c}^V$ *is uplifting in* L*.*
2. $\mathrm{RA}(\mathrm{ccc})$*, implies that* $\mathfrak{c}^V$ *is uplifting in* L*.*
3. $\mathrm{wRA}($*countably closed*$) + \neg\mathrm{CH}$ *implies* $\mathfrak{c}^V$ *uplifting in* L*.*
4. *Hence,* $\mathrm{wRA}($*proper*$) + \neg\mathrm{CH}$ *(etc.) imply* $\mathfrak{c}^V$ *is uplifting in* L*.*

Proper Lottery Preparation

The *proper lottery preparation* $\mathbb{P}$ of a cardinal κ, relative to the function $f:\kappa\to\kappa$, is the countable support κ-iteration, which forces at stages $\beta\in\mathrm{dom}(f)$ with the lottery sum

$$\dot{\mathbb{Q}}_\beta = \oplus\{\,\mathbb{Q}\in H_{f(\beta)^+}^{V[G_\beta]} \mid \mathbb{Q} \text{ proper}\,\}.$$

- At stage $\beta\in\mathrm{dom}(f)$, the generic filter selects a $\mathbb{Q}$, and forces with it.
- If $\mathrm{ran}(f)$ is unbounded and κ inaccessible, then $\mathbb{P}$ forces $\mathfrak{c}=\aleph_2=\kappa$.
- $\mathbb{P}$ works best when f exhibits certain *fast growth behavior* relative to κ

Suppose that $f:\kappa\to\kappa$ has the *fast-growing uplifting Menas property*: for every ordinal β there is inaccessible $\gamma>\beta$ with $\langle V_\kappa, f\rangle\prec\langle V_\gamma, f^*\rangle$ for which $f^*(\kappa)\geq\beta$.

Proper Lottery Preparation is flexible

The proper lottery preparation can be used for various different large cardinals, such as:

Theorem

The proper lottery preparation of

- *a strongly unfoldable cardinal* κ *forces* $\mathrm{PFA}(\mathfrak{c}$*-proper) (J. '07)*
- *a* Σ^2_1*-indescribable cardinal forces* $\mathrm{PFA}(\mathfrak{c}$*-linked)* *(Neeman+Schimmerling '08)*
- *a strongly unfoldable cardinal* κ *forces* $\mathrm{PFA}_{\mathfrak{c}} + \mathrm{PFA}(\aleph_2$*-covering*$) + \mathrm{PFA}(\aleph_3$*-covering*$)$ *(Hamkins & J., '09)*
- *a supercompact cardinal* κ *forces* PFA

What about the proper lottery preparation of an uplifting cardinal?

Uplifting to $\mathrm{RA}(\text{proper})$

Theorem

If κ is uplifting, then the proper lottery preparation forces $\mathrm{RA}(\text{proper})$ with $\mathfrak{c} = \aleph_2 = \kappa$.

Proof.

The proper lottery preparation $G \subseteq \mathbb{P}$ forces $\mathfrak{c} = \aleph_2 = \kappa$ in $V[G]$. Suppose $\mathbb{Q}$ proper in $V[G]$. Find $\langle V_\kappa, f\rangle \prec \langle V_\gamma, f^*\rangle$ with $\mathbb{Q}$ proper in $V_\gamma[G]$ and $|\mathrm{trcl}(\dot{\mathbb{Q}})| < f^*(\kappa)$. Note $\mathbb{P} \subseteq V_\kappa$ is definable, so we get corresponding $\mathbb{P}^* \subseteq V_\gamma$. Opt for $\mathbb{Q}$ at stage κ in $\mathbb{P}^*$, so $\mathbb{P}^* \cong \mathbb{P} * \mathbb{Q} * \dot{\mathbb{R}}$. Lift $H_\kappa \prec H_\gamma$ to

$$H_{\mathfrak{c}}^{V[G]} = H_\kappa[G] \prec H_\gamma[G * g * h] = H_{\mathfrak{c}}^{V[G*g*h]},$$

which verifies $\mathrm{RA}(\text{proper})$ in $V[G]$. □

More lottery preparations

Theorem

If κ is uplifting, then:

1. *The semi-proper lottery preparation forces* RA(*semi-proper*) *with* $\mathfrak{c} = \aleph_2 = \kappa$.
2. *The axiom A lottery preparation forces* RA(*Axiom A*) *with* $\mathfrak{c} = \aleph_2 = \kappa$.
3. *The unrestricted lottery preparation forces* RA *with* $\mathfrak{c} = \aleph_1 = \kappa$.

We use revised countable support in 1), countable support in 2), and finite support in 3).

The lottery preparation doesn't work with c.c.c. forcing, since the lottery sum of c.c.c. forcing is not c.c.c.

Solution: adapt the original Laver preparation.

Uplifting Laver functions

Theorem

Every uplifting cardinal κ has a definable ordinal-anticipating Laver function.

That is, a function $f : \kappa \to \kappa$, such that for every ordinal β, there are arbitrarily large inaccessible γ with $\langle V_\kappa, f\rangle \prec \langle V_\gamma, f^*\rangle$ and $f^*(\kappa) = \beta$.

Proof.

If $\delta < \kappa$ is not uplifting, let $f(\delta)$ be the order type of all inaccessible $\gamma < \kappa$ such that $V_\delta \prec V_\gamma$. This defines f. Fix β. Let θ be β^{th} inaccessible with $V_\kappa \prec V_\theta$. So $V_\theta \models \beta$ many, so $f^*(\kappa) = \beta$, as desired. □

Corollary

If $V = L$, then every uplifting κ has a definable Laver function.

That is, a function $\ell : \kappa \to V_\kappa$, such that for any set x, there are arbitrarily large inaccessible γ with $\langle V_\kappa, \ell\rangle \prec \langle V_\gamma, \ell^*\rangle$ and $\ell^*(\kappa) = x$.

The world's smallest Laver preparation

Theorem

The finite support c.c.c. Laver preparation of an uplifting cardinal κ forces $\mathrm{RA}(\mathrm{ccc})$ with $\kappa = \mathfrak{c}$.

If κ is uplifting and ℓ is an uplifting Laver function, then let $\mathbb{P}$ be the finite support κ-iteration, using $\mathbb{Q}_\beta = \ell(\beta)$, if this is c.c.c. in $V[G_\beta]$.

The Laver function ℓ hands us exactly the desired c.c.c. poset at stage κ, and the iteration is thus c.c.c. As before, we lift $H_\kappa \prec H_\gamma$ to

$$H_{\mathfrak{c}}^{V[G]} = H_\kappa[G] \prec H_\kappa[G * g * h] = H_{\mathfrak{c}}^{V[G*g*h]},$$

which shows $\mathrm{RA}(\mathrm{ccc})$ in $V[G]$.

Equiconsistency Strength of RA

Theorem

The following are equiconsistent over ZFC*:*

1. *The Resurrection Axiom* RA.
2. $\mathrm{RA}($*proper*$) + \neg\mathrm{CH}$.
3. $\mathrm{RA}($*semi-proper*$) + \neg\mathrm{CH}$.
4. $\mathrm{RA}(\mathrm{ccc})$.
5. $\mathrm{wRA}($*countably closed*$) + \neg\mathrm{CH}$.
6. $\mathrm{wRA}($*proper*$) + \neg\mathrm{CH}$.
7. $\mathrm{wRA}($*semi-proper*$) + \neg\mathrm{CH}$.
8. $\mathrm{wRA}($*stationary-preserving*$) + \neg\mathrm{CH}$.
9. $\mathrm{wRA}(\omega_1$*-preserving*$) + \neg\mathrm{CH}$.
10. *There is an uplifting cardinal.*

Restricted Resurrection and Boldface Resurrection

- *Restricted Resurrection:* merely require that

$$H_{\mathfrak{c}} \prec_{\Sigma_n} H_{\mathfrak{c}}^{V[g*h]}$$

for some fixed $n \in \mathbb{N}$.

- *Boldface Resurrection:* require for every $A \subseteq \mathfrak{c}$ that

$$\langle H_{\mathfrak{c}}, A \rangle \prec \langle H_{\mathfrak{c}}^{V[g*h]}, A^* \rangle$$

for some $A^* \subseteq \mathfrak{c}^{V[g*h]}$

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