Notes on $\aleph_1$-dense sets isomorphism*

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Abstract

An exposition of the consistency of Baumgartner Axiom, following the approach taken by Shelah to prove Baumgartner's result. For details and much more see Abraham, Rubin, Shelah: On the consistency of some partition theorems for continuous colorings, and the structure of $\aleph_1$-dense real order types, APAL 29 (1985) pp 123–206,

1 The nearness axiom

A non-empty $A \subseteq \Re$ is said to be $\aleph_1$ dense if it has no first nor last member and between any two members of A there are exactly $\aleph_1$ members of A. Let K denote the collection of all $\aleph_1$-dense (order-types of) subsets of $\Re$. The nearness axioms (NA) says that if $A, B \in K$ then $NA(A, B)$ holds, which means that there is some $C \in K$ such that $C \preceq A$ and $C \preceq B$. We will show in this section how to obtain the consistency of NA. The structure of the proof is standard. Assuming CH in the ground model we iterate c.c.c posets with finite support iteration along ω_2, each of size $\aleph_1$ and taking care of all possible A and B's. So the main point is in the following theorem.

Theorem 1.1 *Assume CH, and let A and B be two $\aleph_1$ dense subsets of $\Re$. Then there is a c.c.c poset of cardinality $\aleph_1$ which makes $NA(A, B)$.*

We may assume that A and B are disjoint: although not strictly necessary, this assumption somewhat simplifies the picture.

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We may try for our poset all finite (partial) functions $p : A \to B$ that are order-preserving ($x_1, x_2 \in \mathrm{dom}(p)$ and $x_1 < x_2$ implies that $f(x_1) < f(x_2)$). But this is clearly not c.c.c, and in fact it immediately collapses the continuum. So the basic idea (of Baumgartner) is to limit the possibilities for $p(x)$ to a countable set. To express this limitation in a convenient way, we prefer to have $A, B \subseteq \omega_1$ and to have an ordering relation $<_R$ over ω_1 which reflects the real ordering (namely $(\omega_1, <_R)$ is isomorphic to $\Re$ (assuming CH) or just to a set of reals of size $\aleph_1$ that contains the two given sets. Let $M \supseteq \omega_1$ be an $\in$ structure that contains all relevant information (for example let it be $H(\aleph_1)$ with $A, B, <_R$ as predicates over ω_1). Then let $M_\alpha \prec M$ be countable and increasing elementary substructures, and define $C = \{\delta \in \omega_1 \mid \delta = M_\delta \cap \omega_1\}$ as the resulting club set. Let $\langle \delta_i \mid i < \omega_1 \rangle$ be an increasing and continuous enumeration of C (and it is convenient to start with $\delta_0 = 0$). The ordinal interval $E_i = [\delta_i, \delta_{i+1})$ is called the i-th slice of C. As a limitation on a condition p we may require that for any x in its domain $p(x)$ must remain in the same slice. This will prevent an obvious counterexample to the c.c.c since there is only a countable set of possibilities for $p(x)$, but it may be problematic if for example there is an order reversing map f from A to B and there are uncountably many slices containing some x and $f(x)$. So the idea is that a condition should separate x and $p(x)$ but not too much. For example, a limitation that works is for a point in slice E_i to move to the next slice E_{i+1}.

We define now P as the set of all finite functions $p : A \to B$ that are $<_R$ order preserving and satisfy the following: for every slice E_i, the intersection $E_i \cap (\mathrm{dom}(p) \cup \mathrm{range}(p))$ contains at most one point, and if $x \in E_i \cap \mathrm{dom}(p)$ then $p(x) \in E_{i+1}$.

So there is a member of C between x and $p(x)$, but the distance between these two points is not too big. Since $E_i \cap A$ is dense in A and $E_i \cap B$ is dense in B (by elementarity of the models M_δ), each condition can be extended on any slice, and the generic function is order preserving and its domain intersects every other slice and is thence uncountable.

It follows for $p \in P$ that if $\mathrm{dom}(p) = \{\xi_0, \ldots, \xi_{n-1}\}$ is an enumeration in increasing ordinal order, then $\xi_0 < p(\xi_0) < \xi_1 < p(\xi_1) < \cdots < p(\xi_{n-1})$, and these ordinals are C separated. So, any condition in P of size n is a member of the space ω_1^{2n}.

Lemma 1.2 *P satisfies the c.c.c.*

Proof. Suppose $D = \{p_\alpha \mid \alpha < \omega_1\} \subset P$ is given. Say $U(p_\alpha) = \mathrm{dom}(p_\alpha) \cup \mathrm{range}(p_\alpha)$ is the universe of p_α. We may assume that $U(p_\alpha)$, form a Δ system with an empty core, and that for $\alpha < \beta$ an ordinal in C separates $U(p_\alpha)$ from $U(p_\beta)$. (Note that without the limitation on the conditions of P we would not be able to claim that the core of the Delta system can be safely removed with the remaining part being a condition.)

Now the duplication method can work, but some notations will be needed for a detailed presentation.

In the following, we take the rationals to be a subset of ω_1, and so every rational interval $(q_1, q_2) = \{\xi \mid q_1 <_R \xi <_r q_2\}$ is a subset of ω_1. Relation $<_R$ induces a partial ordering on these intervals: $X <_R Y$ iff for all $x \in X$ and $y \in Y$ $x <_R y$ holds.

An open "envelop" is a sequence of pairwise disjoint rational intervals $\overline{I} = (I_0, \ldots, I_{k-1})$. Given a sequence of ordinals $\overline{\xi} = (\xi_0, \ldots, \xi_{k-1})$ we say that $\overline{I}$ "covers" $\overline{\xi}$ when $\xi_i \in I_i$ for all i. An open envelop b refines an open envelop a if a and b have the same length and each each interval of a contains an interval from b (a unique one).

We now turn to the proof of the c.c.c. Let $U(p_\alpha) = (\xi^\alpha_0, \ldots, \xi^\alpha_{2n-1})$ be enumerated in increasing ordinal order. The index set of the domain of p_α is the set of even indices, and we have $p_\alpha(\xi^\alpha_{2i}) = \xi^\alpha_{2i+1}$. In the first step we find for every $\alpha < \omega_1$ an open envelop that covers $U(p_\alpha)$. By renaming our antichain D we may assume that a single open envelop a covers all p_α's. Say $a = I_0, \ldots, I_{2n-1}$ where $\xi^\alpha_i \in I_i$. The following remark will be used later on:

$$\text{The function which takes } I_{2i} \text{ to } I_{2i+1} \text{ is order preserving.} \tag{1}$$

Let $D_0 \subset D$ be countable and dense in D. (Recall that conditions in P are represented as members of ω_1^{2n}.) So for every $p_\beta \in D$ and envelop b that covers p_β there is some $p_\alpha \in D_0$ that is covered by b. There is $\delta_0 < \omega_1$ so that $D_0 \in M_{\delta_0}$.

Let $\beta < \omega_1$ be so that p_β is above δ_0. We are going to define by descending induction on $k < 2n$ envelops b_k and b'_k of length $2n - k$ and are of the form:

$$b_k = (X_\ell \mid k \leq \ell < 2n\},$$

$$b'_k = (X'_\ell \mid k \leq \ell < 2n\},$$

so that the following hold.

1. $X_\ell, X'_\ell \subset I_\ell$, and $X_\ell <_R X'_\ell$ for every $k \leq \ell < 2n$.

2. There are two sequences $(\zeta_k, \dots, \zeta_{2n-1}) \in X_k \times \cdots X_{2n-1}$ and $(\zeta'_k, \dots, \zeta_{2n-1}) \in X'_k \times \cdots X'_{2n-1}$ so that the following two ordinal sequences are in the closure of D_0:
$$(\xi_0^\beta, \dots, \xi_{k-1}^\beta)^\frown(\zeta_k, \dots, \zeta_{2n-1})$$
and
$$(\xi_0^\beta, \dots, \xi_{k-1}^\beta)^\frown(\zeta'_k, \dots, \zeta'_{2n-1})$$

We end the inductive construction with open envelops $b = (X_0, \dots, X_{2n-1})$ and $b' = (X'_0, \dots, X'_{2n-1})$. Let f_b be the function that takes interval X_{2i} to X_{2i+1}, and $f_{b'}$ be the function that takes X'_{2i} to X'_{2i+1}. Then condition 1 and the fact that these envelops refine a together with the remark at (1) imply that $f_b \cup f_{b'}$ is also an order preserving function. But by property 2, b covers a member of the closure of D_0 and hence a member of D_0. And likewise b' covers a condition in D_0, and hence these two conditions of D_0 are compatible.

Turning our attention to the inductive construction now, suppose that b_{k+1} and b'_{k+1} are defined and we have to define X_k and X'_k so that conditions 1 and 2 hold. Pick δ in C such that
$$\xi_0^\beta, \dots, \xi_{k-1}^\beta < \delta \leq \xi_k^\beta.$$

Let $\varphi(\zeta)$ be the formula with free ordinal variable ζ which says that the following two sets have a non-empty intersection with the closure of D_0
$$\{\xi_0^\beta\} \times \cdots \times \{\xi_{k-1}^\beta\} \times \{\zeta\} \times X_{k+1} \times \cdots \times X_{2n-1}. \tag{2}$$
and
$$\{\xi_0^\beta\} \times \cdots \times \{\xi_{k-1}^\beta\} \times \{\zeta\} \times X'_{k+1} \times \cdots \times X'_{2n-1}. \tag{3}$$

Then all parameters of $\varphi(\zeta)$ are in M_δ (the rational intervals are certainly there), and $\varphi[\xi_k^\beta]$ holds. Since $\xi_k^\beta \geq \delta$ and $\delta = M_\delta \cap \omega_1$, the set of ζ for which $\varphi(\zeta)$ hold is uncountable (by the following lemma), and in particular we can find $\zeta <_R \zeta'$ that are both in this set. Thus we can find disjoint open rational intervals $X_k <_R X'_k$ that contain ζ and ζ' respectively and that are contained in the corresponding interval I_k of a.

It follows now that both of (2) and (3) hold when we replace $\{\zeta\}$ with X_k (or with X'_k). Yet, for the induction step we only need to know that (2) holds when $\{\zeta\}$ is replaced with X_k, and that (3) holds when $\{\zeta\}$ is replaced with X_k. That is, we get b_k and b'_k as required.

The following simple and yet useful lemma was employed: If M is a countable elementary substructure say of $H(\aleph_2)$, and X is any set in M. If there exists $x \in X \setminus M$, then X is uncountable. Proof: if X were countable it would have an enumeration in M and in this case we would have $X \subset M$.

2 Baumgartner's Axiom

Baumgartner's axiom BA says that every two $\aleph_1$-dense sets of reals with no end-points are order isomorphic.

Let A, B be two $\aleph_1$-dense subsets of $\mathbb{R}$ (no endpoints and $\aleph_1$ members between any two). For notational simplicity we may assume that A and B are disjoint (any interval contains a copy of the reals). Assuming CH we want to define a c.c.c poset which introduces a generic isomorphism from A onto B. This would be enough. Define first a club set $C \subset \omega_1$ as in the previous section. Recall that in that section we forced a partial function from A into B and hence we could require that a condition touches a single member of a slice $[\delta_i, \delta_{i+1})$ of C. Here, however, the isomorphism has to be defined over all of A and its range must be all of B and this complication requires a new idea. Suppose we take as our restriction on a condition p the demand that if x is in a slice then $p(x)$ is in an adjacent slice. This will not work and the problem in proving the c.c.c can already be seen with conditions of size two as follows. Say p_α has domain $x_1^\alpha <_R x_2^\alpha$ in the αth slice, and $p_\alpha(x_i^\alpha) = y_i^\alpha$ is in the next slice. The duplication technique may not work here. For when we try to duplicate the y's part we will have to duplicate a pair of y's rather than a single one. We may get $y_1 <_R y_2$ and $y'_1 <_R y'_2$ but so that (for example) $y_1 <_R y'_1 <_R y'_2 <_R y_2$. But when we try to make a corresponding duplication of the x's, we may only have a pattern $x_1 <_R x_2 <_R x'_1 <_R x'_2$, which will not allow us to continue. The solution is in demanding stricter restrictions on our conditions, which we now describe.

We need two simple combinatorial observations which we state first.

Lemma 2.1 *There is a graph (ω, E) on ω (with undirected set of edges E) so that 1) there are no cycles, and 2) every node n is connected to infinitely many other nodes.*

Proof. For example take the tree $\omega^{<\omega}$ of all finite sequences of natural numbers, and let E be the set of all $\{\sigma, \sigma^\frown(n)$ where $\sigma \in \omega^{<\omega}$ and $n \in \omega$. That is, every finite sequence is connected to its immediate successors and to its immediate predecessor (if it is not the empty sequence). By definition a cycle is a sequence $v_0, \ldots, v_k$ of nodes where $k > 2$, $v_0 = v_k$ but $v_m \neq v_n$ for indexes below k, and $\{v_i, v_{i+1}\} \in E$ for every $i < k$. There are no cycles since once you go up on the tree you cannot go down. Now, as the tree is countable the graph can be made on ω and the lemma follows.

Here is another simple observation.

Lemma 2.2 *Let (V, E) be an arbitrary cycle-free graph, and suppose that $g : E \to \{=, \neq\}$ is a function assigning to every edge one of the $=$ and $\neq$ tokens. Then there exists a function $h : V \to \{0, 1\}$ on the vertices so that for every edge $e = \{v_1, v_2\}$ $g(e)$ is "$=$" if and only if $h(v_1) = h(v_2)$.*

Now we return to the definition of the poset P. Let $\langle \delta_i \mid i \in \omega_1\}$ be an increasing and continuous enumeration of the club set C, starting with $\delta_0 = 0$. For any ordinal α the interval $B_\alpha = [\delta_{\omega\alpha}, \delta_{\omega\alpha+\omega})$ is called the αth block, and the subinterval $S_{\alpha,m} = [\delta_{\omega\alpha+m}, \delta_{\omega\alpha+m+1})$ is the mth slice of that block. We define now a graph on the set of all slices with edges E that satisfy the properties of Lemma 2.1 on each block (no cycles and surely no self connecting edges, and every node is connected to an infinite number of nodes). (No edge in E connects slices in different blocks.)

Now we define our poset $P_{A,B}$ as the collection of all finite $<_R$ order preserving functions $p : A \to B$ (here $A, B \subset \omega_1$) such that for every $\xi \in \mathrm{dom}(p)$ ξ and $p(\xi)$ are in the same block, and if S, S' are the slices containing ξ and $p(\xi)$ then $\{S, S'\} \in E$. Moreover, this edge $\{S, S'\}$ is unique. Namely if ξ' is in $\mathrm{dom}(p)$ ($\xi' \neq \xi$) and $\xi' \in S \cup S'$, then $p(\xi) \notin S \cup S'$. (In other words, if $\{S, S'\} \in E$ then there is at most one $\xi \in S \cap \mathrm{dom}(p)$ with $p(\xi) \in S'$.) The ordering of $P_{A,B}$ is extension.

It can be checked that for every $\xi \in A$ and $\xi' \in B$ the set of conditions with these ordinals in their domain and range is dense. For this, use the fact that the graph has an infinite number of neighbors of every node (and the observation that every slice is dense in both A and B).

We will prove now that $P_{A,B}$ satisfies the c.c.c. So let $D = \{p_\alpha \mid \alpha \in \omega_1\}$ be a set of conditions of size $\aleph_1$. As usual form a Δ system of $U(p_\alpha) = \mathrm{dom}(p_\alpha) \cup \mathrm{range}(p_\alpha)$ enumerated in increasing ordinal order $\xi_0^\alpha, \ldots, \xi_{2n-1}^\alpha$ (where n is the cardinality of any condition in D) and remove if necessary its core. Thus we may assume that D is already a Δ system with an empty core.

Pick pairwise disjoint rational intervals $a = (I_0, \ldots, I_{2n-1})$ with I_k containing ξ_k^α. We may assume that this envelop a is fixed and does not depend on α.

Moreover, we may assume that there is a fixed set $DM \subset \{0, \ldots, 2n-1\}$ of size n and a fixed function $f : DM \to \{0, \ldots, 2n-1\} \setminus DM$ so that $\mathrm{dom}(p_\alpha) = \{\xi_k \mid k \in DM\}$ and $p_\alpha(\xi_k^\alpha) = \xi_{f(k)}^\alpha$ for all $\alpha < \omega_1$. If follows that the function $f_a : \{I_k \mid k \in DM\} \to \{I_j \mid j \notin DM\}$ defined by $f_a(I_k) = I_{f(k)}$ is order preserving.

Although a condition may stretch over several blocks, we assume for notational simplicity that every condition p_α lives on a single block.

Let $D_0 \subset D$ be countable and dense in D. Our aim is to find compatible envelops $J = (J_0, \ldots, J_{2n-1})$ and $J' = (J'_0, \ldots, J'_{2n-1})$ such that

1. $J_i, J'_i \subset I_i$ and $J_i \cap J'_i = \emptyset$.
2. If $f(i) = j$, then $J_i <_R J'_i$ iff $J_j <_R J'_j$.
3. Both J and J' cover members of D_0.

This will finish the proof, for if $p_\alpha, p_{\alpha'}$ are covered by J and (respectively) J', then $p_\alpha \cup p_{\alpha'}$ is order-preserving, and since there is a member of C in between this union is a condition.

Suppose that $D_0 \in M_{\delta_0}$. Let β be so that p_β lies above δ_0. By assumption p_β lives on a single block B_τ, and we let r be the number of slices occupied by p_β. We write $(\xi_0^\beta, \ldots, \xi_{2n-1}^\beta) = \sigma_1 ^\frown \sigma_2 \frown \cdots \sigma_r$ as a concatenation of r sequences where σ_s is the sequence of elements of $U(p_\beta)$ that lie in its s-th slice. Let $n(s)$ be its length. So $\sigma_1 = (\xi_0^\beta, \ldots, \xi_{n(1)-1}^\beta)$ is the sequence of elements of p_β in its first slice, $\sigma_2 = (\xi_{n(1)}^\beta, \ldots, \xi_{n(1)+n(2)-1}^\beta)$ in the second (higher) slice and generally if we let $s_i = n(1) + \cdots n(i-1)$, then $\sigma_i = (\xi_{s_i}^\beta, \ldots, \xi_{s_i+n(i)-1}^\beta)$. For simplicity of expression, we say that an index ℓ is in σ_i when $\xi_\ell^\beta \in \sigma_i$ (i.e. $s_i \leq \ell < s_i + n(i)$).

Recall that $a = (I_0, \ldots, I_{2n-1})$ and so we have a corresponding concatenation $a = \overline{I}_1 \frown \ldots \frown \overline{I}_r)$, where $\overline{I}_i = (I_{s_i}, \ldots, I_{s_i+n(i)-1})$ is the sequence of open rational intervals that covers σ_i. Recall that we have a cycle-free graph E with the slices as vertices, and the poset is defined with respect to this graph. Since each sequence σ_i corresponds to a slice, we can think of the graph as if its vertices are the indexes $\{1, \ldots, r\}$. Then for $i < j$

$$\begin{aligned}&\{i,j\} \in E \text{ holds whenever there are indices } \ell = \ell(i,j) \text{ in } \sigma_i \\ &\text{and } \ell' = \ell'(i,j) \text{ in } \sigma_j \text{ so that } f(\ell) = \ell' \text{ or } f(\ell') = \ell.\end{aligned} \tag{4}$$

By downward induction on $i = r, \ldots, 1$ we shall follow the sequences $\sigma_r, \ldots, \sigma_1$ and define for every $i = r, \ldots, 1$ two envelops $\overline{J}_i^0 = (J^0_{s_i}, \ldots, J^0_{s_i+n(i)-1})$ and $\overline{J}_i^1 = (J^1_{s_i}, \ldots, J^1_{s_i+n(i)-1})$ so that:

1. For every index $s_i \leq k < s_i + n(i)$, J^0_k and J^1_k are disjoint subsets of I_k.
2. For every function $v \in 2^{\{i,i+1,\ldots,r\}}$, the cartesian product

$$\{\xi_0^\beta\} \times \ldots, \times \{\xi^\beta_{s_i-1}\} \times \prod \overline{J}_i^{v(i)} \times \cdots \times \prod \overline{J}_r^{v(r)}$$

   has a non-empty intersection with the closure of D_0. (For a sequence of intervals $\overline{V} = (V_0, \ldots, V_{m-1})$ we write $\prod \overline{V}$ for the cartesian product $V_0 \times \cdots \times V_{m-1}$.)

When done, we shall define a function g on the set of edges which will allow us to apply Lemma 2.2 as follows. Suppose there are indices $\ell = \ell(i,j)$ in σ_i and $\ell' = \ell'(i,j)$ in σ_j as defined above in 4. Then $\{i,j\}$ is an edge. Now there are two cases in the definition of $g(i,j) \in \{=, \neq\}$. If the pairs (J^0_ℓ, J^1_ℓ) and $(J^0_{\ell'}, J^1_{\ell'})$ is order preserving, then $g(i,j)$ is "=" and otherwise it is "$\neq$". Then we have a function $v \in 2^{\{1,\ldots,r\}}$ as in the lemma, and we let v' be the complementary function. Then it follows that the interval sequences $\overline{J} = \overline{J}_1^{v(1)} \frown \ldots, \overline{J}_r^{v(r)}$ and $\overline{J'} = \overline{J}_1^{v'(1)} \frown \ldots, \overline{J}_r^{v'(r)}$ are compatible. That is, their union form an envelop. Yet each of $\overline{J}$ and $\overline{J'}$ contains a member of D_0, and so these two members are compatible in P.

Returning to the definition of the sequences, suppose that $\overline{J}_r^q, \ldots, \overline{J}_{i+1}^q$ are defined for $q = 0, 1$ and we want to define $\overline{J}_i^0$ and $\overline{J}_i^1$. Consider the following formula $\varphi(\zeta_{s_i}, \ldots, \zeta_{s_{i+1}-1})$ with parameters $\xi_0^\beta, \ldots, \xi^\beta_{s_i-1}$ and J^0_ℓ, J^1_ℓ for $s_{i+1} \leq \ell < 2n$ which is the conjunction of $(\zeta_{s_i}, \ldots, \zeta_{s_{i+1}-1}) \in \prod \overline{I}_i$ with

> For every $2^{\{i+1,\ldots,r\}}$ the product $\{\xi_0^\beta\} \times \cdots \times \{\xi^\beta_{s_i-1}\} \times \{\zeta_{s_i}\} \times \cdots \times \{\zeta_{s_{i+1}-1}\} \times \prod J_{i+1}^{v(i+1)} \times \cdots \times \prod J_r^{v(r)}$ has a nonempty intersection with the closure of D_0.

Then $\varphi[\xi^\beta_{s_i}, \ldots, \xi^\beta_{s_{i+1}-1}]$ holds. By a duplication argument which is now familiar, we can get two sequences with disjoint ranges that satisfy φ. Then by separating their points with pairwise disjoint rational intervals we get J_i^0 and J_i^1 as required.

3 Baumgartner Axiom with a larger continuum

In the model obtained for BA we have that $2^{\aleph_0} = \aleph_2$. The iteration is of length ω_2 and we used this fact in order to ensure that CH holds at each stage of the iteration. Why CH was needed? In order to find a club set $C \subset \omega_1$ which is thiner than any club definable from a real. But we can get clubs by using Jensen's forcing P_{Jensen} which introduces a generic club that is almost included in any ground model club. It was left as an exercise to find a model in which BA holds and the continuum is above $\aleph_2$.