ALGEBRAIC APPROXIMATIONS OF FIBRATIONS IN ABELIAN VARIETIES OVER A CURVE

by

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Abstract. — For every fibration $f : X \to B$ with $X$ a compact Kähler manifold, $B$ a smooth projective curve, and a general fiber of $f$ an abelian variety, we prove that $f$ has an algebraic approximation.

1 Introduction

Let $X$ be a compact Kähler manifold. An algebraic approximation of $X$ is a deformation $\mathcal{X} \to \Delta$ of $X$ such that, up to shrinking $\Delta$, the subset parameterizing projective manifolds in this family is dense in $\Delta$. Whether a compact Kähler manifold admits an algebraic approximation is generally known as the Kodaira problem. For compact Kähler surfaces, Kodaira showed that algebraic approximations always exist [11]. But starting from dimension 4 and on, there exist compact Kähler manifolds constructed by Voisin which do not have any algebraic approximation [19].

For most compact Kähler manifolds, the existence of algebraic approximations is still unknown. Non-trivial examples of compact Kähler manifolds admitting algebraic approximations can be found in [17, 4, 10, 5, 6, 12] and the list is rather exhaustive at present. This list includes especially elliptic fibrations [12] and smooth fibrations in abelian varieties [5] (both over a projective manifold), and they therefore motivate the following question, which we will study in this text.

Question 1.1. — Let $f : X \to B$ be a fibration whose general fiber is an abelian variety. Assume that $X$ is a compact Kähler manifold and $B$ a projective manifold, does $f$ have an algebraic approximation?

In the case where $f$ is smooth, an algebraic approximation of $f$ can be realized by the so-called tautological family associated to $f$ described as follows [5]. Let $J \to B$ denote the Jacobian fibration associated to $f$ and $\mathcal{J}$ its sheaf of germs of local holomorphic sections. There is a one-to-one correspondence between the set of isomorphism classes of $J$-torsors (resp. projective $J$-torsors) and $H^1(B, \mathcal{J})$ (resp. $H^1(B, \mathcal{J}_{\text{tors}})$), and the tautological family associated to $f$ is a family of $J$-torsors

$$
\Pi : \mathcal{J} \to B \times V \to V := H^1(B, R^g f_\ast \Omega_{X/B}^{g-1}) \quad (g := \dim X - \dim B)
$$

such that \( t \in V \) parameterizes the \( J \)-torsor represented by

\[
\eta(f) + \exp(t) \in H^1(B, \mathcal{F})
\]

where \( \eta(f) \in H^1(B, \mathcal{F}) \) is the element associated to \( f \) and

\[
\exp : V \to H^1(B, \mathcal{F})
\]

is the map induced by the quotient \( R^1f_*\Omega^1_{X/B} \to \mathcal{F} \). An argument involving Hodge theory shows that \( V \) contains a dense subset parameterizing projective \( J \)-torsors [5].

When \( f : X \to B \) is an elliptic fibration, the situation is much more complicated but the construction of an algebraic approximation of \( f \) in [12] is still based on the tautological family associated to some specific elliptic fibration and Hodge theory is again involved in the proof of the density result. This raises the question of whether we can construct, more generally, for every fibration whose general fiber is an abelian variety, the analogue of tautological family and prove (by Hodge-theoretic arguments) that projective members are dense in this family, thereby answering Question 1.1.

In this text, we will focus on fibrations \( f : X \to B \) with \( \dim B = 1 \) and show that the above approach can indeed be carried out in this situation. Due to the presence of singular fibers, the proof is technically more involved than in the smooth case and the implementation of the idea relies on work of Deligne, El Zein, and Zucker on variations of Hodge structures over a curve [21, 8] and M. Saito’s compactifications of Jacobian fibrations [16]. We will first show that under the assumption that \( f \) has local sections at every point of \( B \), we can construct a tautological family associated to \( f \) (see Proposition-Definition 4.15) and prove that it is an algebraic approximation.

**Theorem 1.2.** — Let \( G \) be a finite group and \( f : X \to B \) a \( G \)-equivariant fibration with \( X \) a compact Kähler manifold, \( B \) a smooth projective curve, and a general fiber of \( f \) an abelian variety. Assume that \( f \) has local sections at every point of \( B \), then the \( G \)-equivariant tautological family

\[
\Pi : \mathcal{X} \to B \times V \to V
\]

associated to \( f \) is a \( G \)-equivariantly locally trivial (see Definition 2.1) algebraic approximation of \( f \).

Without assuming that the fibration \( f : X \to B \) has local sections everywhere in \( B \), the existence of an algebraic approximation of \( f \) will follow as a corollary of Theorem 1.2, which answers Question 1.1 in the affirmative when \( \dim B = 1 \).

**Corollary 1.3.** — Let \( f : X \to B \) be a fibration with \( X \) a compact Kähler manifold, \( B \) a smooth projective curve, and a general fiber of \( f \) an abelian variety. Then \( f \) has a locally trivial algebraic approximation.

Our original motivation of this work comes from the long-standing Kodaira problem in dimension 3 which we will study in [13]. Roughly speaking, given a non-uniruled compact Kähler threefold \( X \) of algebraic dimension \( a(X) = 1 \), the algebraic reduction of \( X \) is bimeromorphic to a fibration \( f : X' \to B \) such that either \( f \) is isotrivial or a general fiber of \( f \) is an abelian surface [9]. In the latter case, the existence of
algebraic approximations of \(X\) will be a consequence of Corollary 1.3. The reader is referred to [13] for the detail.

In order to explain the basic idea of the proof of Theorem 1.2 without dealing with technical difficulties, we will first recall (following [5]) in Section 3 the construction of the tautological family \(\varPi\) associated to a smooth torus fibration \(f\), together with a sketch of proof of the fact that \(\varPi\) is an algebraic approximation under the assumption that the base and the fibers of \(f\) are projective [5, Theorem 1.2]. We will then construct in Section 4 the tautological families associated to fibrations as in Theorem 1.2, and prove Theorem 1.2 and Corollary 1.3 in Section 5.

2 Preliminaries and general results

2.1 Basic notions and terminologies

In this text, a fibration is a proper holomorphic surjective map \(f : X \to B\) whose general fiber is irreducible. The fiber \(f^{-1}(b)\) of \(f\) over \(b \in B\) will often be denoted by \(X_b\). A local section of a fibration \(f\) at a point \(b \in B\) is a section of \(f^{-1}(U) \to U\) for some (Euclidean) neighborhood \(U \subset B\) of \(b\). A multi-section of \(f\) is a subvariety \(\Sigma \subset X\) such that \(f\big|_\Sigma\) is surjective and finite.

A deformation of a complex variety \(X\) is a surjective and flat morphism \(\varPi : \mathcal{X} \to \Delta\) containing \(X\) as a fiber. Let \(f : X \to B\) be a holomorphic map. A deformation of \(f\) is a family of maps \(f_t : \mathcal{X}_t \to B\) containing \(f\) as a member. Namely, it is a composition

\[
\varPi : \mathcal{X} \xrightarrow{\eta} B \times \Delta \xrightarrow{\text{Pr}_2} \Delta
\]

such that \(\varPi\) is a deformation of \(X\) and \(\eta, \mathcal{X}_a : \mathcal{X}_a \to B\) equals \(f\) for some \(a \in \Delta\). Note that in the definition, the target of \(f_t\) does not deform in the family.

Let \(G\) be a group and \(X\) a complex variety endowed with a \(G\)-action. We say that a deformation \(\varPi : \mathcal{X} \to \Delta\) of \(X\) preserves the \(G\)-action, or \(\varPi\) is a \(G\)-equivariant deformation of \(X\), if there exists a \(G\)-action on \(\mathcal{X}\) extending the given \(G\)-action on \(X\) such that \(\varPi\) in \(G\)-invariant. Similarly, let \(f : X \to B\) be a \(G\)-equivariant map. We say that a deformation \(\varPi : \mathcal{X} \xrightarrow{\eta} B \times \Delta \to \Delta\) of \(f\) preserves the \(G\)-action (or is \(G\)-equivariant) if there exists a \(G\)-action on \(\mathcal{X}\) extending the \(G\)-action on \(f\) such that \(\eta\) is \(G\)-equivariant (the \(G\)-action on \(B \times \Delta\) being the pullback of the given \(G\)-action on \(B\)) and \(\varPi : \mathcal{X} \to \Delta\) is \(G\)-invariant.

Definition 2.1 (Locally trivial deformations). —

i) A deformation \(\varPi : \mathcal{X} \xrightarrow{\eta} B \times \Delta \to \Delta\) of \(f : X \to B\) is called locally trivial (over \(B\)) if there exists an open cover \(\{U_i\}\) of \(B\) such that \(q^{-1}(U_i \times \Delta) \simeq f^{-1}(U_i) \times \Delta\) over \(\Delta\).

ii) In i), let \(G\) be a group and \(f : X \to B\) a \(G\)-equivariant map. We say that \(\varPi\) is \(G\)-equivariantly locally trivial (over \(B\)) if \(\varPi\) preserves the \(G\)-action and the isomorphisms \(\eta^{-1}(U_i \times \Delta) \simeq f^{-1}(U_i) \times \Delta\) above are \(G\)-equivariant for some \(G\)-invariant open cover \(\{U_i\}\) of \(B\).
An obvious property about locally trivial deformations is that the quotient of a $G$-equivariantly locally trivial deformation is still locally trivial.

**Lemma 2.2.** — If $\Pi : \mathcal{X} \rightarrow B \times \Delta \rightarrow \Delta$ is a $G$-equivariantly locally trivial deformation of a $G$-equivariant fibration $f : X \rightarrow B$ where $G$ is a finite group, then the quotient $\Pi' : \mathcal{X}/G \rightarrow (B/G) \times \Delta \rightarrow \Delta$ of $\Pi$ by $G$ is a locally trivial deformation of $f' : X/G \rightarrow B/G$.

**Proof.** — By assumption, there exists a $G$-invariant open cover $\{U_i\}$ of $B$ such that $q^{-1}(U_i \times \Delta)$ is $G$-equivariantly isomorphic to $f^{-1}(U_i) \times \Delta$ over $\Delta$. Therefore if $U'_i \subset B/G$ denotes the image of $U_i$ in $B/G$, then $\{U'_i\}$ is an open cover of $B/G$ and we have

$$q'^{-1}(U'_i \times \Delta) = q^{-1}(U_i \times \Delta)/G = (f^{-1}(U_i) \times \Delta)/G = f'^{-1}(U'_i) \times \Delta,$$

where, by abuse of notation, $q^{-1}(U_i \times \Delta)/G$ denotes the image of $q^{-1}(U_i \times \Delta)$ in $\mathcal{X}/G$ and same for $(f^{-1}(U_i) \times \Delta)/G$. □

### 2.2 Campana’s criterion

Let $X$ be a complex variety. We say that $X$ is *algebraically connected* if a general pair of points $x, y \in X$ is contained in a compact connected (but not necessarily irreducible) curve of $X$. We have the following criterion due to Campana for a variety to be Moishezon in terms of algebraic connectedness.

**Theorem 2.3 (Campana [3, Corollaire on p.212]).** — Let $X$ be a compact complex variety bimeromorphic to a compact Kähler manifold. Then $X$ is Moishezon if and only if $X$ is algebraically connected.

Together with Moishezon’s criterion, Theorem 2.3 implies that a compact complex manifold $X$ is projective if and only if $X$ is Kähler and algebraically connected.

Since we will mainly deal with fibrations $f : X \rightarrow B$ with $\dim B = 1$, here is a variant of Campana’s criterion in this particular situation.

**Corollary 2.4 (Special case of Campana’s criterion).** — Let $f : X \rightarrow B$ be a fibration from a compact Kähler manifold to a smooth projective curve. Assume that a general fiber of $f$ is projective, then $X$ is projective if and only if $f$ has a multi-section.

### 3 Smooth torus fibrations and their tautological families

The heuristic proving Theorem 1.2 is originated from the smooth case [5], although the basic idea can be traced back to Kodaira [11] in his studies of deformations of elliptic surfaces. So in this section, we will recall the construction of the tautological families associated to smooth torus fibrations and explain how (assuming the fibers and the bases of the fibrations are projective) these families are proven to be algebraic approximations following [5].
Let $f : X \to B$ be a smooth torus fibration of relative dimension $g$ and $J \to B$ the Jacobian fibration associated to $f$. The sheaf $\mathcal{H}_{f/B}$ of germs of holomorphic sections of $J \to B$ lies in the exact sequence
\[
0 \to \mathcal{H} \to \mathcal{E}_{f/B} \overset{\exp}{\to} \mathcal{H}_{f/B} \to 0
\]  
(3.1)
where $\mathcal{H} := R^{2g-1}f_*\mathcal{O}$ and
\[
\mathcal{E}_{f/B} := (\mathcal{H} \otimes \mathcal{O}_B)/R^{2g-1}f_*\Omega^g_{X/B} = R^g f_*\Omega^g_{X/B}.
\]
Every morphism $\phi : B' \to B$ induces a map $\mathcal{H}_{f/B} \to \phi_*\mathcal{H}_{f/B}$ by pulling back sections.

The fibration $f$ is a $J$-torsor and to each isomorphism class of $J$-torsors, we can associate in a biunivocal way an element $\eta(f) \in H^1(B, \mathcal{H}_{f/B})$ satisfying the property that $\eta(f)$ is torsion if and only if $f$ has a multi-section [5, Proposition 2.2]. Moreover, if
\[\exp : H^1(B, \mathcal{E}_{f/B}) \to H^1(B, \mathcal{H}_{f/B})\]
denotes the morphism induced by $\exp : \mathcal{E}_{f/B} \to \mathcal{H}_{f/B}$, then there exists a family
\[
\Pi : \mathcal{X} \xrightarrow{\theta} B \times V \to V := H^1(B, \mathcal{E}_{f/B})
\]  
(3.2)
of $J$-torsors such that $t \in V$ parameterizes the $J$-torsor which corresponds to
\[\eta(f) + \exp(t) \in H^1(B, \mathcal{H}_{f/B}).\]
The family $\Pi$ is called the *tautological family* associated to $f$.

Concretely, $\Pi$ is constructed as follows. Let $\text{pr}_1 : B \times V \to B$ be the first projection and let
\[
\xi \in H^1(B, \mathcal{E}_{f/B}) \otimes \mathcal{O}_V \subset H^1(B, \mathcal{E}_{f/B}) \otimes H^0(V, \mathcal{O}_V) \simeq H^1(B \times V, \mathcal{E}_{\text{pr}_1^*\mathcal{H}/B\times V})
\]
be the element which corresponds to the identity $\text{Id} : V \to H^1(B, \mathcal{E}_{f/B})$. Let
\[
\theta := \text{pr}_1^*\eta(f) + \overline{\exp(\xi)} \in H^1(B \times V, \mathcal{H}_{\text{pr}_1^*\mathcal{H}/B\times V})
\]
where $\text{pr}_1^* : H^1(B, \mathcal{H}_{f/B}) \to H^1(B \times V, \mathcal{H}_{\text{pr}_1^*\mathcal{H}/B\times V})$ is the map induced by $\mathcal{H}_{f/B} \to \text{pr}_1^*\mathcal{H}/B\times V$ and
\[
\overline{\exp} : H^1(B, \mathcal{E}_{\text{pr}_1^*\mathcal{H}/B\times V}) \to H^1(B \times V, \mathcal{H}_{\text{pr}_1^*\mathcal{H}/B\times V})
\]
the map induced by $\exp : \mathcal{E}_{\text{pr}_1^*\mathcal{H}/B\times V} \to \mathcal{H}_{\text{pr}_1^*\mathcal{H}/B\times V}$. Then the smooth torus fibration $\theta : \mathcal{X} \to B \times V$ defining $\Pi$ is defined to be the $(f \times B)$-torsor corresponding to $\theta$. As $\theta$ can be represented by a Čech 1-cocycle $(\theta_{ij})$ with respect to an open cover of $B \times V$ of the form $\{U_i \times V\}$ where $\{U_i\}$ is a good open cover of $B$, it follows that $\Pi$ is locally trivial.

In the case where $f : X \to B$ is a $G$-equivariant smooth torus fibration for some finite group $G$, there is a natural $G$-action on (3.1). The sub-family of (3.2) over $V^G$ is a deformation of $f : X \to B$ preserving the $G$-action [5, Proposition 2.10], called the $G$-equivariant tautological family associated to $f$.

**Theorem 3.1 (Claudon [5]).** Let $G$ be a finite group and $f : X \to B$ a $G$-equivariant smooth torus fibration. Assume that the total space $X$ is a compact Kähler manifold. Then in the $G$-equivariant tautological family $\Pi$
associated to \( f \), the subset of \( V^G := H^1(B, \mathcal{O}_{\mathbb{H}/B})^G \) parameterizing fibrations with a multi-section is dense in \( V^G \). In particular, if the fibers of \( f \) and \( B \) are projective, then \( \Pi \) is an algebraic approximation of \( f \).

**Sketch of proof of Theorem 3.1.** — By Deligne’s theorem, \( H := H^1(B, \mathcal{H}) \) is a pure Hodge structure of degree \( 2g \) (where \( g \) is the relative dimension of \( f \)) satisfying the Hodge symmetry and concentrated in bi-degrees \( (g - 1, g + 1), (g, g), \) and \( (g + 1, g - 1) \) [21, Section 2]. Also, if \( F^*H_C \) denotes the Hodge filtration of \( H \), then \( V := H^1(B, \mathcal{O}_{\mathbb{H}/B}) \) is isomorphic to \( H_C/F^*H_C \) [21, Section 2]. It follows that the composition

\[
\mu : H^1 \to H^1(B, \mathcal{O}_{\mathbb{H}/B}) \to V^G.
\]

is surjective, so \( \mu(H^1_B) \) is dense in \( V \). Since \( G \) is finite, we have

\[
\mu(H^1_B^G) \otimes \mathbb{R} = \mu(H^1_B) \otimes \mathbb{R} = V^G.
\]

Therefore \( \mu(H^1_B^G) \) is dense in \( V^G \).

The Kähler assumption of \( X \) implies that the image of the \( G \)-equivariant class \( \eta_G(f) \in H^1_G(B, \mathcal{J}) \) associated to \( X \) (which is a refinement of \( \eta(f) \), see [5, Section 2.4]) under the connecting morphism

\[
H^1_G(B, \mathcal{J}) \to H^2_G(B, \mathcal{H})
\]

induced by (3.1) is torsion [5, Proposition 2.11]. So there exist \( m \in \mathbb{Z}_{>0} \) and \( t_0 \in V^G \) such that \( mn\eta(f) = \exp(t_0) \).

Therefore \( \eta(f) + \exp \left( t - \frac{t_0}{m} \right) \) is torsion for every \( t \in \mu(H^1_B^G) \), so each of the fibrations \( \mathcal{X}_i \to B \) parameterized by the subset

\[
\mu(H^1_B^G) - \frac{t_0}{m} \subset V^G
\]

in the tautological family (3.2) has an étale multi-section over \( B \) [5, Proposition 2.2]. As \( \mu(H^1_B^G) \) is dense in \( V^G \), so is \( \mu(H^1_B^G) - \frac{t_0}{m} \). Hence \( V^G \) contains a dense subset parameterizing fibrations in the tautological family having a multi-section over \( B \). The last statement follows from the main statement of Theorem 3.1 and Corollary 2.4.

\[\square\]

4 Fibrations in abelian varieties over a curve

In this section, we study deformations of (a priori non-smooth) fibrations in abelian varieties over a curve in a similar spirit of what we did in Section 3. In 4.1 and 4.2, we first focus on the Hodge-theoretic ingredients we need in the proof of Theorem 1.2 based on Zucker’s theory of variations of Hodge structures over a curve [21] and also [8]. Then we introduce and study in 4.3 the notion of bimeromorphic \( f \)-torsors and construct the tautological families associated to them.

4.1 Variation of Hodge structures over a curve: Zucker’s theorem

First we recall Zucker’s result on the variations of Hodge structures (VHS) over a curve [21]. Let \( B^* \) be a smooth quasi-projective curve and \( \mathcal{H} \) an (integral) local system which underlies a VHS of weight \( m \) over \( B^* \). Let \( j : B^* \to B \) be the smooth compactification of \( B^* \).
Theorem 4.1 (Zucker [21, Theorem 7.12]). — Assume that $H$ underlies an $\mathbb{R}$-polarized VHS of weight $m$ and the local monodromies of $H$ around $B \backslash B^*$ are quasi-unipotent. Then $H^i(B, j_!H)$ has a natural Hodge structure of weight $m + i$. Moreover, if $H$ satisfies the Hodge symmetry, then $H^i(B, j_*H)$ satisfies the Hodge symmetry as well.

Here an $\mathbb{R}$-polarized VHS is a VHS admitting a flat bilinear paring defined over $\mathbb{R}$ on the Hodge bundle $H_C = H \otimes \mathbb{C}$ whose restriction to each fiber of $H_C$ is a polarization of the underlying Hodge structure. This is the case when $H = R^m f^*Z$ where $f^* : X^* \to B^*$ is a family of compact Kähler manifolds. The quasi-unipotence of the monodromy action is satisfied for instance, when $H = R^m f^*Z$ where $f^* : X^* \to B^*$ is the smooth part of a locally projective morphism $f : X \to B$ [18, Theorem 3.15].

Let $H := H \otimes \mathcal{O}_B$, and $F^*H$ be the Hodge filtration on $H$. Let $
abla : \mathfrak{h} \to \mathfrak{h} \otimes \Omega^1_B$ denote the Gauss-Manin connection. Assume that the local monodromies of $H$ around $B \backslash B^*$ are quasi-unipotent, then $H$ has a canonical extension $\bar{H}$ due to Deligne together with a filtration $\mathfrak{F}^*\bar{H}$ and a connection $\bar{\nabla} : \bar{H} \to \bar{H} \otimes \Omega^1_B(\log \Sigma)$ extending $\mathfrak{F}^*H$ and $\nabla$ [7, Remarques II.5.5 (i)]. The connection $\bar{\nabla}$ satisfies Griffiths’ transversality $\bar{\nabla}(F^p\bar{H}) \subset F^{p-1}\bar{H} \otimes \Omega^1_B(\log \Sigma)$ so that $\bar{\nabla}$ induces a map

$$\nabla_p : (\mathfrak{h} \mid \mathfrak{F}^p\mathfrak{h}) \to (\mathfrak{h} \mid \mathfrak{F}^{p-1}\mathfrak{h}) \otimes \Omega^1_B(\log \Sigma).$$

for each $p$. We define

$$(\mathfrak{h} \mid \mathfrak{F}^p\mathfrak{h})_h := \ker(\nabla_p)$$

to be the horizontal part of $\mathfrak{h} \mid \mathfrak{F}^p\mathfrak{h}$.

The inclusion $H \hookrightarrow \mathfrak{h}$ can be extended to $j_!H \hookrightarrow \mathfrak{h}$ over $B$. Since the local sections of $H$ are flat with respect to the Gauss-Manin connection and $\mathfrak{h} \mid \mathfrak{F}^p\mathfrak{h}$ is locally free [8, p. 130], the image of the composition $j_!H \hookrightarrow \mathfrak{h} \to \mathfrak{h} \mid \mathfrak{F}^p\mathfrak{h}$ lies in $(\mathfrak{h} \mid \mathfrak{F}^p\mathfrak{h})_h$.

We will only be interested in the case where $i = 1$ in Theorem 4.1. The following result can be found in the proof of [21, Theorem 9.2] as a simple corollary of [21, Proposition 9.1].

Proposition 4.2 (Zucker). — Let $H$ be as in Theorem 4.1 and let $F^*H$ denote the Hodge filtration on $H := H^1(B, j_*H) \otimes \mathbb{C}$. Then for every integer $p$, the map $j_!H \to (\mathfrak{h} \mid \mathfrak{F}^p\mathfrak{h})_h$ induces an isomorphism $H/F^pH \cong H^1(B, (\mathfrak{h} \mid \mathfrak{F}^p\mathfrak{h})_h)$.

What will be useful for our purpose is the following corollary.
Corollary 4.3. — Let $H$ be as in Theorem 4.1. Assume that $H$ underlies a VHS of weight $2g - 1$ concentrated in bi-degrees $(g, g - 1)$ and $(g - 1, g)$ and satisfying the Hodge symmetry. Then the image of

$$H^1(B, j_! H \otimes \mathbb{Q}) \to H^1(B, \mathcal{H} / \mathcal{F} \mathcal{H})$$

induced by $j_! H \to \mathcal{H} / \mathcal{F} \mathcal{H}$ is dense in $H^1(B, \mathcal{H} / \mathcal{F} \mathcal{H})$.

Proof. — Since $H$ is concentrated in bi-degrees $(g, g - 1)$ and $(g - 1, g)$, we have $\mathcal{F} \mathcal{H}^{-1} \mathcal{H} = \mathcal{H}$, so $\nabla_\mathcal{H} = 0$ and

$$(\mathcal{H} / \mathcal{F} \mathcal{H})_b = \mathcal{H} / \mathcal{F} \mathcal{H}.$$ 

Thus by Proposition 4.2, we have $H / \mathcal{F}H \cong H^1(B, \mathcal{H} / \mathcal{F} \mathcal{H})$. Since the VHS underlain by $H$ satisfies the Hodge symmetry and is concentrated in bi-degrees $(g, g - 1)$ and $(g - 1, g)$, the Hodge structure $H = H^1(B, j_! H)$ satisfies also Hodge symmetry and is concentrated in bi-degrees $(g + 1, g - 1), (g, g)$, and $(g - 1, g + 1)$ by [21, Theorem 7.12]. Thus the natural map $H^1(B, j_! H) \otimes \mathbb{R} \to H / \mathcal{F}H \cong H^1(B, \mathcal{H} / \mathcal{F} \mathcal{H})$ is surjective. Corollary 4.3 now follows from the density of $H^1(B, j_! H \otimes \mathbb{Q}) = H^1(B, j_! H) \otimes \mathbb{Q}$ in $H^1(B, j_! H) \otimes \mathbb{R}$. 

\[ \square \]

4.2 The canonical extension of $\mathcal{J}$

The variations of Hodge structures that we will encounter in the proof of Theorem 1.2 will satisfy the assumption of Corollary 4.3. In order to simplify the notation, we set $\mathcal{E} := \mathcal{E}_{W/B} := \mathcal{H} / \mathcal{F} \mathcal{H}$, and similarly $\mathcal{E} := \mathcal{E}_{W/B} := \mathcal{H} / \mathcal{F} \mathcal{H}$. For these VHSs, recall that the sheaf of sections of the intermediate Jacobian fibration $\mathcal{J}$ associated to $H$ is defined by $\mathcal{J} := \mathcal{J}_{W/B} := \mathcal{E} / H$, and the canonical extension of $\mathcal{J}$ is defined by the exact sequence [20, (2.5)]

\begin{equation}
0 \longrightarrow j_! H \longrightarrow \mathcal{E} \xrightarrow{\exp} \mathcal{J} := \mathcal{J}_{W/B} \longrightarrow 0. \tag{4.1}
\end{equation}

For the VHSs coming from a family of varieties, there is another way to define $\mathcal{J}$ due to Deligne, El Zein, and Zucker and we recall the construction following [8]. Let $f : X \to B$ be a morphism of compact Kähler manifolds with $\dim B = 1$. Assume that the fibers of $f$ are normal crossing divisors. Let $j : B^* \hookrightarrow B$ be a Zariski open subset parameterizing smooth fibers of $f$ and set $i : X^* := f^{-1}(B^*) \hookrightarrow X$. Let $f^* : X^* \to B^*$ denote the restriction of $f$ to $X^*$. We assume that the local monodromies of $R^{2g - 1}f^* \mathcal{L}$ around $\Sigma = B \setminus B^*$ are quasi-unipotent.

Let $D := f^{-1}(\Sigma)$. For $H := R^{2g - 1}f^* \mathcal{L}$, we have

$$\mathcal{H} \cong R^{2g - 1}f_! \Omega_{X/B}^*(\log D)$$

and

$$\mathcal{F} \mathcal{H} \cong R^{2g - 1}f_! F^* \Omega_{X/B}^*(\log D),$$

where $F^* \Omega_{X/B}^*(\log D)$ is the naïve filtration on $\Omega_{X/B}^*(\log D)$ (cf. [8, p. 130]). Let

$$\mathcal{O}_{X/B}^*(\log D) := \Omega_{X/B}^*(\log D) / F^* \Omega_{X/B}^*(\log D).$$
In the bounded derived category of sheaves of abelian groups over \( X \), we define the morphism
\[
\Phi^p : R_i, Z \rightarrow R_i, C \simeq \Omega^*_X(\log D) \rightarrow \Omega^*_{X/B}(\log D) \rightarrow \sigma_p \Omega^*_{X/B}(\log D)
\]
and define the relative Deligne complex to be
\[
D_{X/B}(\varphi) := \text{Cone}(\Phi^p)[-1].
\]
Applying \( Rf_\ast \) to \( D_{X/B}(\varphi) \) yields the long exact sequence
\[
\cdots \rightarrow R^{q-1} f_\ast \sigma_p \Omega^*_{X/B}(\log D) \rightarrow R^q f_\ast D_{X/B}(\varphi) \rightarrow R^q (f \circ i)_\ast, Z \rightarrow \cdots
\]
The canonical extension of \( \mathcal{J} \) is defined to be
\[
\mathcal{J} := \text{coker} \left( R^{2g-1} (f \circ i)_\ast, Z \rightarrow R^{2g-1} f_\ast \sigma_p \Omega^*_{X/B}(\log D) \right).
\]
Note that since the spectral sequence
\[
E^{pq}_{1} = R^{q} f_\ast \Omega^*_{X/B}(\log D) \Rightarrow R^{p+q} f_\ast \Omega^*_{X/B}(\log D)
\]
degenerates at \( E_1 \) (see [8, p. 130]), we have \( R^{2g-1} f_\ast \sigma_p \Omega^*_{X/B}(\log D) \simeq \mathcal{E} \). So for \( p = g \), breaking up the long exact sequence (4.2) at \( R^{2g-1} f_\ast \sigma_p \Omega^*_{X/B}(\log D) \) yields
\[
0 \rightarrow \text{Im} \left( R^{2g-1} (f \circ i)_\ast, Z \rightarrow \mathcal{J} \right) \rightarrow \mathcal{E} \rightarrow \mathcal{J} \rightarrow 0.
\]
In order to compare the above construction of \( \mathcal{J} \) with the one defined by (4.1), it suffices to prove the following isomorphism.

**Lemma 4.4.** — We have the isomorphism
\[
j_\ast, H \simeq \text{Im} \left( \psi : R^{2g-1} (f \circ i)_\ast, Z \rightarrow \mathcal{E} \right).
\]

Since a proof of Lemma 4.4 is hard to find in the literature, we provide one below.

**Proof.** — As the restriction of \( \psi \) to \( B^\ast \) is the inclusion \( H \hookrightarrow \mathcal{E} \), it suffices to prove (4.4) around a neighborhood \( p \in B \setminus B^\ast \). Let \( \Delta \subset B \) a sufficiently small disc centered at \( p \). Let \( \Delta^\ast := \Delta - \{ p \} \) and \( X_p^\ast := f^{-1}(\Delta^\ast) \). Since the natural morphism \( \phi : R^{2g-1} (f \circ i)_\ast, Z \rightarrow j_\ast, H \) is surjective (cf. the proof of [8, Proposition 7]), it suffices to show that
\[
\ker(\phi) = \ker(\psi)
\]
over \( \Delta \). Given \( \alpha \in H^{2g-1}(X_p^\ast, Z) \). On the one hand,
\[
\alpha \in \ker \left( \phi(\Delta) : H^{2g-1}(X_p^\ast, Z) \rightarrow H^0(\Delta^\ast, R^{2g-1} f_\ast, Z) \right)
\]
if and only if its restriction to a general fiber \( X_s \) of \( X_p^\ast \rightarrow \Delta^\ast \), which is an element in
\[
H^{2g-1}(X_s, Z) \subset H^{2g-1}(X_s, C)/p^n H^{2g-1}(X_s, C),
\]
is zero. On the other hand, since \( \mathcal{D} \) is locally free [8, p. 130],

\[
\alpha \in \ker \left( \psi(\Lambda) : H^{2g-1}(X_{\mathcal{D}}^*, Z) \to H^0(\Lambda, \mathcal{D}) \right)
\]

if and only if the restriction of \( \psi(\Lambda)(\alpha) \) to a general fiber \( \mathcal{D} = H^{2g-1}(X_{\mathcal{D}}^*, \mathcal{C})/F^g H^{2g-1}(X_{\mathcal{D}}^*, \mathcal{C}) \) of \( \mathcal{D} \) is zero. Since the two restrictions coincide, we have (4.5). \hfill \( \square \)

We can also break up (4.2) at \( R^{2g} f_* D_{\mathcal{X}/\mathcal{B}}(g) \), which yields the short exact sequence

\[
0 \longrightarrow \mathcal{J} \longrightarrow R^{2g} f_* D_{\mathcal{X}/\mathcal{B}}(g) \longrightarrow H^{g,\mathfrak{d}}(X/B) \longrightarrow 0 \tag{4.6}
\]

where

\[
H^{g,\mathfrak{d}}(X/B) := \ker \left( R^{2g}(f \circ \iota)_* Z \to R^{2g} f_* \sigma_\mathcal{D}^* \Omega_{X/B}^* (\log D) \right).
\]

**Lemma 4.5.** — The diagram

\[
\begin{array}{ccc}
H^0(B, R^{2g}(f \circ \iota)_* Z) & \xrightarrow{d_2} & H^2(B, R^{2g-1}(f \circ \iota)_* Z) \\
\uparrow & & \downarrow \\
H^0(B, H^{g,\mathfrak{d}}(X/B)) & \xrightarrow{\delta} & H^1(B, \mathcal{J}) \xrightarrow{\psi} H^2(B, j_* \mathcal{H})
\end{array}
\tag{4.7}
\]

is commutative, where the first row is the \( d_2 \) map in the second page of the Leray spectral sequence

\[
E_2^{g,\mathfrak{d}} = H^p(B, R^q(f \circ \iota)_* Z) \Rightarrow H^{p+q}(X^*, Z)
\]

and the morphisms in the second row are the connecting morphisms induced by (4.6) and (4.1) respectively.

**Proof.** — It suffices to apply Lemma 6.1 to the short exact sequence of bounded complexes representing the distinguished triangle

\[
R f_* D_{\mathcal{X}/\mathcal{B}}(g) \longrightarrow R f_* R_! Z \longrightarrow R f_* \sigma_\mathcal{D}^* \Omega_{X/B}^* (\log D) \longrightarrow R f_* D_{\mathcal{X}/\mathcal{B}}(g)[1] \tag{4.8}
\]

and note that \( j_* \mathcal{H} \cong \text{Im} \left( R^{2g-1}(f \circ \iota)_* Z \to \mathcal{D} \right) \) by Lemma 4.4. \hfill \( \square \)

When \( g \) is the relative dimension of the fibration \( f : X \to B \), we have the following result.

**Lemma 4.6.** — Let \( g = \dim X - \dim B \). We have

\[
H^{g,\mathfrak{d}}(X/B) = R^{2g}(f \circ \iota)_* Z.
\]

In particular, the morphism \( H^0(B, H^{g,\mathfrak{d}}(X/B)) \to H^0(B, R^{2g}(f \circ \iota)_* Z) \) in (4.7) is the identity.

**Proof.** — By definition of \( H^{g,\mathfrak{d}}(X/B) \), it suffices to show that the sheaf \( R^{2g} f_* \sigma_\mathcal{D}^* \Omega_{X/B}^* (\log D) \) is zero. Let \( \mathcal{V} := R^{2g} f^* Z \). Since \( f : X \to B \) is of relative dimension \( g \), \( \mathcal{V} \) is a pure VHS concentrated in bi-degree \( (g, g) \). The sheaf \( \mathcal{V} := R^{2g} f^* \Omega_{X/B}^* (\log D) \) is the canonical extension of \( \mathcal{V} := \mathcal{V} \otimes \mathcal{O}_{\mathcal{B}}^* \), so \( R^{2g} f_* \sigma_\mathcal{D}^* \Omega_{X/B}^* (\log D) = \mathcal{V} \otimes F^g \mathcal{V} \) is locally free [8, p.130]. It follow from \( \mathcal{V} / F^g \mathcal{V} = 0 \) that \( \mathcal{V} / F^g \mathcal{V} = 0 \). \hfill \( \square \)

### 4.3 Bimeromorphic \( J \)-torsors and their tautological families

Bimeromorphic \( J \)-torsors are defined as follows.
**Definition 4.7.** — Let \( \phi : J \to B^* \) be a Jacobian fibration over a smooth quasi-projective curve \( B^* \) and \( B \) the smooth compactification of \( B^* \). Assume that \( \phi \) is projective. A bimeromorphic \( J \)-torsor is a morphism \( f : X \to B \) satisfying the following properties:

i) The restriction \( f^* : X^* \to B^* \) of \( f \) to \( X^* := f^{-1}(B^*) \) is a \( J \)-torsor.

ii) The total space \( X \) is locally bimeromorphically Kähler over \( B \). Namely, for every \( b \in B \), there exists a neighborhood \( U \subset B \) of \( b \) such that \( f^{-1}(U) \) is bimeromorphic to a Kähler manifold.

iii) The fibration \( f \) has local sections at every point of \( B \). Namely, for every \( b \in B \), there exists a neighborhood \( U \subset B \) of \( b \) such that \( f \) has a local section \( \sigma_U : U \to f^{-1}(U) \) over \( U \).

The following lemma allows us to apply results obtained in 4.1 and 4.2 to study bimeromorphic \( J \)-torsors. An (integral) local system \( H \) over a Zariski open \( B^* \) of a smooth curve \( B \) is called locally geometric if for every \( b \in B \), there exists a neighborhood \( U \subset B \) of \( b \) and a projective morphism \( f_U : X_U \to U \) such that \( \mathcal{H}|_{U \times B^*} = \left( R^f_{\xi} Z \right)|_{U \times B^*} \), for some \( i \in \mathbb{Z} \).

**Lemma 4.8.** — If \( f : X \to B \) is a bimeromorphic \( J \)-torsor, then the VHS underlain by \( H := R^{\dim X - \dim B} f_* Z \) is locally geometric where \( g = \dim X - \dim B \). In particular, the local monodromies of \( H \) around \( B \setminus B^* \) are quasi-unipotent.

**Proof.** — On the one hand, since \( f \) is locally bimeromorphically Kähler and fibers of \( f \) are algebraic, by [1, Corollaire du Théorème 2] \( f \) is locally Moishezon. On the other hand, there exists by assumption some modification \( \tilde{f} : \tilde{X} \to B \) along singular fibers such that for every \( b \in B \), there exists a neighborhood \( U \subset B \) of \( b \) such that \( \tilde{f}^{-1}(U) \) is Kähler. So \( \tilde{f} \) is locally projective by [2, Theorem 10.1] and since \( R^{\dim X - \dim B} f_* Z_{B^*} = R^{\dim X - \dim B} f_* Z_{B^*} \), it follows that \( H \) is locally geometric.

Let \( f : X \to B \) be a bimeromorphic \( J \)-torsor. When \( B \setminus B^* \neq \emptyset \), we let \( \{p_1, \ldots, p_m\} := B \setminus B^* \). By assumption, there exists a good open cover \( \{U_i\}_{i=1, \ldots, n} \) of \( B \) such that \( p_j \in U_i \) if and only if \( i = j \) and that \( f_j : X_i := f^{-1}(U_i) \to U_i \) has a section \( \sigma_i : U_i \to X_i \) for all \( i \). For each \( i > m \) (resp. \( i \leq m \)), let \( U^*_i := U_i \) (resp. \( U^*_i = U_i \setminus \{p_i\} \)). By Definition 4.7.i), for each \( i \) there exists a biholomorphic map

\[
\eta_i : U^*_i = f^{-1}(U^*_i) \to \phi^{-1}(U^*_i) = \tilde{f}_i,
\]

over \( U^*_i \) sending \( \sigma_i(U^*_i) \) to the zero-section of \( \tilde{f}_i \to U^*_i \). Let \( U_{ij} = U_i \cap U_j \) and \( X_{ij} = f^{-1}(U_{ij}) \). Whenever \( i \neq j \), we have \( U_i \cap U_j = U_{ij} \cap U_{ij} \), so we can glue the fibrations \( X_i \to U_i \) using the transition maps

\[
(\eta_{ij})^{-1} \circ \eta_{ij} : X_{ij} \to X_{ij}.
\]

We call the thus obtained fibration \( p : \tilde{J} \to B \) the Jacobian fibration associated to \( f : X \to B \), which is a compactification of \( J \to B^* \). By construction, the zero-section of \( \tilde{J} \to B^* \) extends to a section of \( \tilde{f} : \tilde{J} \to B \).

**Lemma 4.9.** — The bimeromorphic class of the Jacobian fibration \( p : \tilde{J} \to B \) constructed above is independent of the bimeromorphic \( J \)-torsor \( f \). Also, the sheaf \( \tilde{J}_* \) is contained in the sheaf of germs of holomorphic sections of \( p : \tilde{J} \to B \).

**Remark 4.10.** — The sheaf of germs of holomorphic sections of \( \tilde{J} \to B \) is a sheaf of abelian groups. Indeed, since \( f \) is a fibration over a curve, every local meromorphic section is holomorphic. For every open subset
$U \subset B$, meromorphic sections of $p : J \to B$ over $U$ form an abelian group coming from the group variety structure of $J \to B^*$.

Proof. — By Lemma 4.8, the VHS underlain by $H$ is locally geometric and is therefore admissible [15, Theorem 14.51]. Accordingly, the Jacobian fibration $J \to B^*$ admits a Saito compactification $\overline{J}_S \to B$ [16, Theorem 0.8] and there is a bimeromorphic map $\nu : \overline{J} \to \overline{J}_S$ over $B$ [16, Example 2.10]. This proves the first statement of Lemma 4.9.

As $\overline{J}_S \to B$ is a compactification of Zucker’s extension $\overline{J}_Z \to B$ [16, p.237–238] constructed in [20, Section 2] and since $\mathcal{J}$ is contained in the sheaf of germs of holomorphic sections of $\overline{J}_Z \to B$ [20, Section 2], we conclude that $\mathcal{J}$ is identified (via $\nu$) with a subsheaf of sheaf of germs of holomorphic sections of $p : \overline{J} \to B$.

Let $\mathcal{E}(B, \Delta, H)$ be the set of bimeromorphic classes of bimeromorphic $J$-torsors. Assume that $\mathcal{E}(B, \Delta, H) \neq \emptyset$, then we can construct a map

$$\Phi : H^1(B, \mathcal{J}) \to \mathcal{E}(B, \Delta, H)$$

as follows. First we fix a Jacobian fibration $p : \overline{J} \to B$ associated to some element in $\mathcal{E}(B, \Delta, H)$ and a good open cover $\{U_i\}$ of $B$ such that $U_{ij} := U_i \cap U_j \subset B^*$ for every $i$ and $j$. Let $\eta \in H^1(B, \mathcal{J})$, represented by a 1-cocycle $[\eta_{ij}]$ defined over $U_{ij}$. Since $U_{ij} \subset B^*$, the sections $\eta_{ij}$ define a 1-cocycle of translations

$$\text{tr}(\eta_{ij}) : p^{-1}(U_{ij}) \to p^{-1}(U_{ij})$$

over $U_{ij}$ and can be used to glue the $p^{-1}(U_i) \to U_i$ and form a bimeromorphic $J$-torsor $f : X \to B$. We call $f$ the bimeromorphic $J$-torsor twisted by $[\eta_{ij}]$. Suppose that $[\eta_{ij}']$ is another 1-cocycle representing $\eta$ and let $f' : X' \to B$ be the bimeromorphic $J$-torsor twisted by $[\eta_{ij}']$. Let $\eta_i \in \mathcal{J}(U_i)$ be local sections such that $\eta_{ij} \eta_i - \eta_i \eta_{ij} = \eta_{ij} - \eta_{ij}'$. As $\eta_i$ is a holomorphic section of $p : \overline{J} \to B$ (Lemma 4.9), the translation map $\text{tr}(\eta_i)$, which is biholomorphic over $U_i \cap B^*$, extends to a bimeromorphic map $\text{tr}(\eta_i) : p^{-1}(U_i) \to p^{-1}(U_i)$ over $U_i$ [14, Proposition 1.6]. These bimeromorphic maps define a bimeromorphic map $X \to X'$ over $B$, which shows that $f$ and $f'$ have the same image in $\mathcal{E}(B, \Delta, H)$. Thus $\Phi$ is well-defined. Finally, it is easy to see that the construction of $\Phi$ does not depend on the choice of the Jacobian fibrations $\overline{J} \to B$, because these fibrations have the same bimeromorphic class (Lemma 4.9).

The following lemma shows that $\Phi$ is surjective, which is a generalization of [5, Remark 2.3].

**Lemma 4.11.** — Let $f : X \to B$ be a bimeromorphic $J$-torsor and assume that the singular fibers of $f$ are normal crossing divisors. Let $1 \in H^0(B, R^{2\varphi}(f \circ 0), \mathcal{Z})$ be a pre-image of $1 \in \mathcal{Z} = H^0(B^*, R^{2\varphi} f^* \mathcal{Z})$. If $\eta \in H^1(B, \mathcal{J})$ denotes the image of $\overline{1}$ under the composition

$$H^0(B, R^{2\varphi}(f \circ 0), \mathcal{Z}) \cong H^0(B, H^{2\varphi}(X/B)) \longrightarrow H^1(B, \mathcal{J})$$

(4.10)

induced by Lemma 4.6 and the short exact sequence (4.6), then $\Phi(\eta) = [f] \in \mathcal{E}(B, \Delta, H)$.

Proof. — As before, let $\{U_i\}$ be a good open cover of $B$ such that $U_{ij} := U_i \cap U_j \subset B^*$. By definition, a 1-cocycle $[\eta_{ij}]$ with respect to the open cover $\{U_i\}$ which represents $\eta$ can be constructed as follows. Let
be the restriction of \( \delta \) to \( U_i \), and let \( \sigma_i \in R^{2g} f_* D_{X/B}(g)(U_i) \) be a lift of \( \delta \), by virtue of (4.6) and Lemma 4.6. We define \( [\eta_{ij}] \) be the 1-cocycle obtained by taking the Čech differential of \( [\sigma_i] \). Over \( U_i^* := U_i \cap B^* \), the arrow on the right side of (4.6) gives \( R^{2g} f_* D_{X/B}(g)(U_i^*) \rightarrow \mathbb{Z} \) and the subset of \( R^{2g} f_* D_{X/B}(g)(U_i^*) \) which maps to 1 in \( \mathbb{Z} \) is identified with the set of sections of \( f^{-1}(U_i^*) \rightarrow U_i^* \). In particular, each \( \sigma_{i|U_i^*} \) is a local section \( U_i^* \rightarrow f^{-1}(U_i^*) \) of \( f : X \rightarrow B \) over \( U_i^* \). By construction,

\[
\eta_{ij} = \eta_i(\sigma_i(U_{ij})) - \eta_j(\sigma_j(U_{ij})) \in \mathcal{J} \cap \mathcal{U}_{ij}
\]

where \( \eta_i \) and \( \eta_j \) are the maps (4.9), and therefore \( \Phi(\eta) = [f] \).

We have a more precise statement about the surjectivity of \( \Phi \).

**Lemma 4.12.** — Let \( c : H^1(B, \mathcal{J}) \rightarrow H^2(B, j, \mathcal{H}) \) be the morphism induces by (4.1). The restriction of \( \Phi \) to \( \ker(c) \) is surjective.

**Proof.** — Let \( f : X \rightarrow B \) be a bimeromorphic \( j \)-torsor. Up to replacing \( f \) by another bimeromorphic model, we can assume that the singular fibers of \( f \) are normal crossing divisors. Let \( f^* : X^* \rightarrow B^* \) be the restriction of \( f \) to \( X^* := f^{-1}(B^*) \) and let \( i : X^* \hookrightarrow X \) be the inclusion. We have the following commutative diagram

\[
\begin{array}{ccc}
H^2(X^*, \mathbb{Z}) & \longrightarrow & H^0(B^*, R^{2g} f^* \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^2(X^*, \mathbb{Z}) & \longrightarrow & H^0(B, R^{2g} (f \circ i)_*, \mathbb{Z}) \\
\end{array}
\]

(4.11)

where the horizontal arrows on the left side come from the Leray spectral sequence, and the one at the bottom right is (4.10). Note that since \( B^* \) is an affine curve, we have \( E^{2g}_{2,2} = H^0(B^*, R^{2g} f^* \mathbb{Z}) = 0 \) whenever \( p \geq 2 \). So the Leray spectral sequence associated to \( f^* \) degenerates at \( E_2 \), in particular the upper horizontal arrow \( H^2(X^*, \mathbb{Z}) \rightarrow H^0(B^*, R^{2g} f^* \mathbb{Z}) \) in (4.11) is surjective. Let \( \alpha \in H^2(X^*, \mathbb{Z}) \) be a pre-image of 1 in \( H^0(B^*, R^{2g} f^* \mathbb{Z}) \) and let \( \eta \in H^1(B, \mathcal{J}) \) be the image of \( \alpha \) following the arrows in (4.11). By Lemma 4.11, we have \( \Phi(\eta) = [f] \).

To show that \( c(\eta) = 0 \), note that by Lemma 4.5 and 4.6 there is a commutative diagram

\[
\begin{array}{ccc}
H^2(X^*, \mathbb{Z}) & \longrightarrow & H^0(B, R^{2g} (f \circ i)_*, \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^1(B, \mathcal{J}) & \longrightarrow & H^2(B, j, \mathcal{H})
\end{array}
\]

(4.12)

Since \( \eta \) is the image of some element in \( H^2(X^*, \mathbb{Z}) \) and the first row of (4.12) is exact, it follows that \( c(\eta) = 0 \).

Let \( m \in \mathbb{Z}_{>0} \) and let \( f_m : X_m \rightarrow B \) be a bimeromorphic \( j \)-torsor such that \( \Phi(m\eta) = [f_m] \). There exists a finite étale morphism \( X^* \rightarrow X_m^* \), called multiplication-by-\( m \) [5, p.5] defined on the smooth part of \( f \) and by [14, Proposition 1.6] this map has a meromorphic extension \( m : X \rightarrow X_m \) over \( B \). As a consequence, we obtain the following result.

**Lemma 4.13.** — If \( \eta \in H^1(B, \mathcal{J}) \) is a torsion element, then the bimeromorphic \( j \)-torsors represented by \( \Phi(\eta) \) have multi-sections.
There exists a family of bimeromorphic $J$-torsors

$$\eta_1$$

where $m\eta = 0$ for some $m \in \mathbb{Z}_{>0}$, then we have a multiplication-by-$m$ map $m : X \rightarrow J$ over $B$. As $m$ is generically finite, the pre-image of a global section of $J \rightarrow B$ under $m$ is a multi-section of $X \rightarrow B$. \qed

We will also need the following lemma in the proof of Theorem 1.2.

**Lemma 4.14.** — Let $f : X \rightarrow B$ be a bimeromorphic $J$-torsor and assume that $f$ is $G$-equivariant for some finite group $G$. Let $\eta \in H^1(B, \mathcal{F})$ be any lift of $f$ by $\Phi$. Then

$$\Phi([G] \cdot \eta) = \Phi \left( \sum_{g \in G} g^* \eta \right).$$

**Proof.** — As before, let $\{U_i\}$ be a good open cover of $B$ such that $U_{ij} := U_i \cap U_j \subset B^*$ and that $f_{iX} : X_i := f^{-1}(U_i) \rightarrow U_i$ has a section $\sigma_i : U_i \rightarrow X_i$ for every $i$. We assume that the open cover $\{U_i\}_{i=1}^n$ is $G$-invariant and let $G$ act on $I$ by $G_{ij} \cdot \eta_{ij} = \eta^{-1}(\sigma_i(U_{ij})).$ Let $p : \tilde{J} \rightarrow B$ be the Jacobian fibration associated to $f$ and fix isomorphisms $\eta_i : X_i \rightarrow \tilde{J}$ over $U_i$ sending $\sigma_i$ to the 0-section. By construction, $\eta$ is represented by the 1-cocycle $\eta_{ij} = \eta_i(\sigma_i(U_{ij})) - \eta_j(\sigma_j(U_{ij})).$

The $G$-action on $f$ induces a natural $G$-action on $\tilde{J} \rightarrow B^*$ and on $\mathcal{F}$. Since the $G$-action on $\tilde{J} \rightarrow B^*$ preserves the 0-section, it extends to a meromorphic $G$-action on $\tilde{J} \rightarrow B$ by [14, Proposition 1.6]. For every $g \in G$, let $\psi_g : X \rightarrow X$ and $\phi_g : \tilde{J} \rightarrow \tilde{J}$ denote the action of $g$ on $X$ and on $\tilde{J}$ respectively. Then

$$\eta_i \circ \psi_g \circ \eta^{-1} \circ \phi_g^{-1} : \tilde{J} \rightarrow \tilde{J} := p^{-1}(U_i)$$

is the translation by some section $\eta_i^g$ of $\tilde{J} \rightarrow U_i$ and the 1-cocycle $\{\eta_{ij}\}$ satisfies

$$\eta_{ij} - (g^* \eta)_ij = \eta^g_{ij} - \eta^g_{ij},$$

(4.13)

where $(g^* \eta)_ij := g \cdot \eta(\sigma g(\sigma_i)) \in \mathcal{F}(U_{ij})$, which represents $g^* \eta$. Summing up (4.13) over $g \in G$, we have

$$|G| \cdot \eta_{ij} - \sum_{g \in G} (g^* \eta)_ij = \eta^G_{ij} - \eta^G_{ij},$$

where $\eta^G_{ij} = \sum_{g \in G} \eta^g_{ij}$, which is a section of $\tilde{J} \rightarrow U_i$. So if $X_1 \rightarrow B$ and $X_2 \rightarrow B$ are the bimeromorphic $J$-torsors twisted by $|G| \cdot \eta_{ij}$ and $\sum_{g \in G} (g^* \eta)_ij$ respectively, then the translations $\text{tr}(\eta^G_{ij}) : \tilde{J} \rightarrow \tilde{J}$ glue to a bimeromorphic map $X_1 \rightarrow X_2$ over $B$, which proves Lemma 4.14. \qed

Finally, we construct the tautological family associated to a $G$-equivariant bimeromorphic $J$-torsor.

**Proposition-Definition 4.15.** — Let $G$ be a finite group and $f : X \rightarrow B$ a $G$-equivariant bimeromorphic $J$-torsor. There exists a family of bimeromorphic $J$-torsors

$$\Pi : \mathcal{X} \xrightarrow{\Phi} B \times V \rightarrow V := H^1(B, \mathcal{E})^G$$

parameterized by $V$ which satisfies the following properties.
i) The family contains \( f : X \to B \) as the central fiber and for all \( t \in V \), the bimeromorphic \( J \)-torsor \( \mathcal{X}_t \to B \) parameterized by \( t \) corresponds to

\[
\Phi(\eta(f) + \exp(t)) \in \mathcal{E}(B, \Delta, H)
\]

where \( \eta(f) \in H^1(B, \mathcal{F}) \) is any lift of \( [f] \in \mathcal{E}(B, \Delta, H) \) by \( \Phi \) and \( \exp : H^1(B, \mathcal{F}) \to H^1(B, \mathcal{F}) \) is the map induced by \( \exp : \mathcal{E} \to \mathcal{F} \) in (4.1).

ii) The family \( \Pi \) preserves the \( G \)-action on \( f \) and is \( G \)-equivariantly locally trivial.

The family \( \Pi \) is called the \( G \)-equivariant tautological family associated to \( f \).

Proof. — The construction of the tautological family is similar to the one constructed for smooth torus fibrations. By abus of notation, let \( \text{pr}_1 \) denote both the first projections \( B \times V \to B \) and \( B^* \times V \to B^* \).

Let \( \{U_i\}_{i \in I} \) be a \( G \)-invariant good open cover of \( B \) such that \( U_{ij} \subset B^* \) for every \( i \neq j \) and let \( G \) act on \( I \) such that \( g^{-1}(U_i) = U_{gi} \). Since \( V \) is simply connected, the product \( \{U_i \times V\} \) is a good open cover of \( B \times V \).

Let \( \{\eta_{ij}\} \) be a 1-cocycle representing \( \eta(f) \) and let \( \eta_{ij} \in \mathcal{F}_{\text{pr}_1^{-1}H/\times V}(U_{ij} \times V) \) be the pullback of \( \eta_{ij} \) under \( U_{ij} \times V \to U_{ij} \). Let \( \{\xi_{ij} : V \to \mathcal{E}(U_{ij})\} \) be a \( G \)-invariant 1-cocycle representing \( \xi \), which we may assume that \( \xi_{ij}(0) = 0 \). Finally, let

\[
\lambda_{ij} := \eta_{ij} + \exp(U_{ij} \times V)(\xi_{ij}) \in \mathcal{F}_{\text{pr}_1^{-1}H/\times V}(U_{ij} \times V)
\]

where \( \exp : \mathcal{E}_{\text{pr}_1^{-1}H/\times V} \to \mathcal{F}_{\text{pr}_1^{-1}H/\times V} \) is the exponential map.

Let \( p : J \to B \) be the Jacobian fibration associated to \( f \) and we fix biholomorphic maps \( \eta_i : X_i := f^{-1}(U_i) \to J_i \), such that \( \eta_i \circ \eta_{ij}^{-1} = \text{tr}(\eta_{ij}) \).

Let \( I_{ij} = p^{-1}(U_i) \) and \( J_{ij} = p^{-1}(U_{ij}) \). We define \( g : \mathcal{X} \to B \) to be the fibration obtained by gluing the \( J_{ij} \times V \to U_{ij} \times V \) along \( J_{ij} \times V \to U_{ij} \times V \) via the translations \( \text{tr}(\lambda_{ij}) : I_{ij} \times V \to J_{ij} \times V \).

By construction, the central fiber of \( \Pi \) is \( f : X \to B \) and for all \( t \in V \), the bimeromorphic \( J \)-torsor parameterized by \( t \) represents \( \eta(f) + \exp(t) \in H^1(B, \mathcal{F}) \), which proves i).

To construct the \( G \)-action on \( \mathcal{X} \) and prove ii), we fix biholomorphic maps \( \lambda_i : X_i \times V \to J_i \times V \) such that \( \lambda_i \circ \lambda_{ij}^{-1} = \text{tr}(\lambda_{ij}) \). For every \( g \in G \), let \( \psi^g_i : X_{gi} \to X_i \) be the restriction to \( X_{gi} \) of the action of \( g \) on \( X \). Define

\[
\Psi^g_i := \lambda_i^{-1} \circ (\eta_i \circ \psi^g_i \circ \eta_{gi}^{-1}) \times \text{Id}_V \circ \lambda_{gi} : X_{gi} \times V \to X_i \times V
\]

Then we can glue the \( \Psi^g_i \) together and obtain a biholomorphic map \( \Psi^g : \mathcal{X} \to \mathcal{X}^* \) such that \( g \mapsto \Psi^g \) is a \( G \)-action on \( \mathcal{X} \) extending the \( G \)-action on \( X \). With this \( G \)-action on \( \mathcal{X} \), \( q \) is \( G \)-equivariant. It follows easily from the construction that \( \Pi \) is \( G \)-equivariantly locally trivial, which proves ii).

5 The tautological family is an algebraic approximation

We continue to use the notations introduced in Section 4. Now we prove that the tautological family constructed at the end of Section 4 is an algebraic approximation.
Proof of Theorem 1.2. — Let $B^* \subset B$ be the Zariski open subset parameterizing smooth fibers of $f$ and let $f : B^* \to B^*$ be the Jacobian fibration associated to $f_{|X^*} : X^* := f^{-1}(B^*) \to B^*$. Then obviously $f$ is a $G$-equivariant bimeromorphic $f$-torsor. By Lemma 4.12, $[f] \in \mathfrak{e}(B, \Delta, H)$ can be lifted to an element $\eta(f) \in H^1(B, \mathfrak{e})$ by $\Phi$ such that $c(\eta(f)) = 0$. By Proposition-Definition 4.15, the $G$-equivariant tautological family

$$
\Pi : \mathcal{X} \to B \times V \to V := H^1(B, \mathfrak{e})^G
$$

associated to $f : X \to B$ is a $G$-equivariantly locally trivial deformation of $f$ and each $t \in V$ parameterizes a bimeromorphic $f$-torsor $f_i : \mathcal{X}_i \to B$ such that $\Phi(\eta(f)) + \exp(t) = [f_i]$. Since the fibers of $f_i$ are projective, by Corollary 2.4 it suffices to show that there exists a dense subset $V_{\text{tors}} \subset V$ parameterizing bimeromorphic $f$-torsors with multi-sections in the tautological family.

For every $\xi \in H^1(B, \mathfrak{f})$, let $f^\xi : X^\xi \to B$ be a bimeromorphic $f$-torsor such that $\Phi(\xi) = [f^\xi]$. Let $\eta := \sum_{g \in G} g^* \eta(f)$. For each $t \in V$, since there exists a generically finite map

$$
\mathcal{X}_i \to \chi^{\mathfrak{e}G}(\eta(f)) + \exp(t) \to \chi^{\mathfrak{e}G}(\eta(G))
$$

over $B$ where the first map is the multiplication-by-$|G|$ and the second one is the bimeromorphic map given by Lemma 4.14, according to Lemma 4.13 it suffices to show that the subset

$$
V_{\text{tors}} := \{ t \in V \mid \eta + \exp(t) \in H^1(B, \mathfrak{f})^G \}
$$

is dense in $V$.

As $c(\eta(f))$ and the map $c$ is $G$-equivariant, by the exact sequence

$$
H^1(B, j_! H) \xrightarrow{\phi_1} H^1(B, \mathfrak{e}) \xrightarrow{\exp} H^1(B, \mathfrak{f}) \xrightarrow{c} H^2(B, j_! H),
$$

induced by (4.1), there exist $\beta \in H^1(B, \mathfrak{e})^G = V$ such that $\exp(\beta) = \eta$. Let $\phi : H^1(B, j_! H)_Q \to H^1(B, \mathfrak{e})$ be the $\mathbb{Q}$-tensorization of $\phi_1$ and let $W := H^1(B, j_! H)^G$. Since $G$ is finite, we have $\text{Im}(\phi)^G = \phi(W)$. By Corollary 4.3, we have $\text{Im}(\phi) \otimes \mathbb{R} = H^1(B, \mathfrak{e})^G$, so

$$
\phi(W) \otimes \mathbb{R} = \text{Im}(\phi)^G \otimes \mathbb{R} = H^1(B, \mathfrak{e})^G = V.
$$

In particular since $\phi(W)$ is a $\mathbb{Q}$-vector space, $\phi(W)$ is dense in $V$, so $\phi(W) - \beta$ is dense in $V$.

By (5.1), we have $\exp(\phi(W)) \subset H^1(B, \mathfrak{f})_{\text{tors}}$, so $\phi(W) - \beta \subset V_{\text{tors}}$. Therefore $V_{\text{tors}}$ is dense in $V$.

Proof of Corollary 1.3. — Let $\{\Delta_i\}$ be a good open cover of $B$. Since fibers of $f : X \to B$ are algebraic, by [1, Corollaire du Théorème 2] the restriction of $f$ to $f^{-1}(\Delta_i)$ is Moishezon. Thus for each $i$, there exists a connected multi-section $Z_i \subset X_i := f^{-1}(\Delta_i)$ of $f_{|X_i}$. Up to refining the open cover, we can assume that each $\Delta_i$ is a disc and the multi-section $Z_i$ is étale over $\Delta_i - \{o_i\}$ where $o_i$ is the center of $\Delta_i$.

Let $\Sigma := \{o_i \mid i \in I\} \subset B$ be the set of centers of $\Delta_i$. Let $d_i$ denote the degree of $Z_i \to \Delta_i$ and $d := \text{lcm}(d_i)$. Up to adding more points to $\Sigma$, the fundamental group $\pi_1(B, \Sigma)$ is free, so there exists a cyclic cover $\tilde{B} \to B$ of degree $d$ branched along $\Sigma$ by Riemann’s extension theorem; let $G := \text{Gal}(\tilde{B}/B)$ denote the corresponding Galois group. Let $\tilde{f} : \tilde{X} \to \tilde{B}$ denote the base change of $f : X \to B$ by $\tilde{B} \to B$. By construction, the fibration $\tilde{f}$
is $G$-equivariant and a general fiber of $\tilde{f}$ is still an abelian variety. Also, the pre-image of the multi-section $Z_i \subset X_i \to \Delta_i$ in $\tilde{X}$ contains a local section of $\tilde{f} : \tilde{X} \to \tilde{B}$ for each $i$. Up to replacing $\tilde{X}$ with a minimal $G$-equivariant Kähler desingularization of it, we can assume that $\tilde{X}$ is smooth. Therefore $\tilde{f} : \tilde{X} \to \tilde{B}$ is a $G$-equivariant fibration satisfying the hypotheses of Theorem 1.2.

By Theorem 1.2, there exists an algebraic approximation
$$\Pi : \tilde{X} \to \tilde{B} \times V \to V$$
of $\tilde{f}$ which is $G$-equivariantly locally trivial. So the quotient of $\Pi$ by $G$ is an algebraic approximation of $f : X \to B$, which is locally trivial by Lemma 2.2. □

6 Appendix: A lemma about the Grothendieck spectral sequences

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of small abelian categories. We assume that $\mathcal{A}$ has enough injectives. The following result is presumably well-known, yet a proof is difficult to find in the literature.

**Lemma 6.1.** — Let
$$0 \to L^* \to M^* \to N^* \to 0 \quad (6.1)$$
be a short exact sequence of bounded complexes in $\mathcal{A}$, which induces the long exact sequence
$$\cdots \to H^{q-1}(L^*) \xrightarrow{\phi} H^{q-1}(M^*) \to H^{q-1}(N^*) \xrightarrow{\rho} H^q(L^*) \to H^q(M^*) \to H^q(N^*) \to \cdots \quad (6.2)$$
Then
$$R^p F(H^q(M^*)) \xrightarrow{d_2} R^{p+2} F(H^{q-1}(M^*)) \quad (6.3)$$
is commutative where the morphism in the top row is the $d_2$ map of the spectral sequence
$$E_2^{pq} = R^p F(H^q(M^*)) \Rightarrow R^{p+q} F(M^*),$$
and the bottom row are the connecting morphisms induced by the exact sequences
$$0 \to \ker(\psi) \to H^{q-1}(N^*) \to \Im(\rho) \to 0 \quad (6.4)$$
$$0 \to \Im(\rho) \to H^q(L^*) \to \ker(\psi) \to 0. \quad (6.5)$$

**Proof.** — Let
$$0 \to (L^{**}, \delta_1, \delta_2) \to (M^{**}, \delta_1, \delta_2) \to (N^{**}, \delta_1, \delta_2) \to 0 \quad (6.6)$$
be an exact sequence of Cartan-Eilenberg resolutions extending $(6.1)$. We define $L^{**} := F(L^{**})$, $M^{**} := F(M^{**})$ and $N^{**} := F(N^{**})$. Since there will be no ambiguity, let $\delta_i$ also denote $F(\delta_i)$. We set
$$\ker(\psi)^* := \ker(H^q(M^{**}, \delta_1) \to H^q(N^{**}, \delta_1)),$$
$$\Im(\delta)^* := \Im(H^{q-1}(N^{**}, \delta_1) \to H^q(L^{**}, \delta_1)),$$
which are injective resolutions of $\ker(\psi)$, $\Im(\delta)$ and $\ker(\phi)$ respectively.

The map $g \circ f$ is constructed as follows. Let $\alpha \in R^pF(\ker(\psi))$ which is represented by some element $a_0 \in f(\ker(\psi))$. By the right-exactness of (6.5), $a_0$ is further represented by some element $\alpha$ in the image of the monomorphism $f_{\theta}^p \to f_{\theta}^p$. By definition, $f(\alpha)$ is represented by some element in $f(\Im(\rho)^{p+1})$ which is further represented by $\delta_{2}^{p}(\tilde{a})$ as an element in $f_{\theta}^{p+1}$. Now by the right-exactness of (6.4), there exists $\beta_0 \in K^{q-p+1}$ which represents a pre-image of the class of $\delta_{2}^{q}(\tilde{a})$ in $H^q(I^{p+1}, \delta_1)$ under the morphism $H^q(I^{p+1}, \delta_1) \to H^q(I^{p+1}, \delta_1)$. Since $J_{\theta}^{q-p+1} \to K^{q-p+1}$ is surjective, we can find $\beta \in J_{\theta}^{q-p+1}$ mapping to $\beta_0 \in K^{q-p+1}$. Again by definition, the class of $\delta_{2}^{q-1,p+1}(\beta)$ in $F(\ker(\phi)^{q+2})$ is an element representing $(g \circ f)(\alpha)$.

Next we recall the construction of the map $d_2$. Let

$$a \in R^pF(H^q(M^*)) = H^q(I^{\theta}, \delta_1, \delta_2)$$

which is represented by $\alpha \in f_{\theta}^p$. Since $\delta_{2}^{p}(\tilde{a}) = 0$ in $H^q(I^{p+1}, \delta_1)$ there exists $\beta \in J_{\theta}^{q-1,p+1}$ such that

$$\delta_{1}^{q-1,p+1}(\beta) = \delta_{2}^{q}(\tilde{a}) \in J_{\theta}^{p+1}.$$

Let $\beta' := \delta_{2}^{q-1,p+1}(\beta)$. Note that

$$\delta_{1}^{q-1,p+2}(\beta') = \delta_{2}^{q,p+1} \delta_{1}^{q-1,p+1}(\beta) = \delta_{2}^{q,p+1} \delta_{2}^{q}(\tilde{a}) = 0,$$

we have $\beta' \in \ker \delta_{1}^{q-1,p+2}$. Let $\tilde{\beta}$ be the class of $\beta'$ in $H^q(I^{\theta}, \delta_1)$. Since

$$\delta_{2}^{q-1,p+2}(\beta') = \delta_{2}^{q,p+2} \delta_{2}^{q-1,p+1}(\beta) = 0,$$

we have

$$\tilde{\beta} \in \ker \delta_{2} : H^q(I^{\theta}, \delta_1) \to H^q(I^{\theta}, \delta_1),$$

and $d_2(\alpha)$ is defined to be the class of $\tilde{\beta}$ in $R^{p-1}F(H^{q-1,2}(M^*)) = H^{q-1,2}(I^{\theta}, \delta_1, \delta_2)$.

The construction of $d_2(\alpha)$ is independent of the choices of $\alpha$ and $\beta$. If $a = h(\alpha')$ for some $\alpha \in R^pF(\ker(\psi))$, then we can choose the same $\tilde{a}$ and $\beta$ as in the construction of $(g \circ f)(\alpha')$. Therefore $(g \circ f)(\alpha')$ and $(d_2 \circ h)(\alpha')$ are both represented by $\delta_{2}^{q-1,p+1}(\beta)$, which proves that (6.3) is commutative.

\[\Box\]

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