Course Announcement: Master Program V5B7

Advanced Topics in Analysis – Functional analytic methods for Global Analysis and Noncommutative Geometry

Informal title: what my students should know about (functional) analysis but the standard course failed to cover.

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Time and Format

Mo 12:15-14, Fr 12:15-14 CET, that is 11:15-13 UTC, 8:15-10 Sao Paulo, 4:15-6 Mountain time, 22:15-24:00 Wollongong. First session: Monday, Nov 23, 2020 @ 12:15. Since the course starts late, it may continue until the end of February.
The course will be held online via Zoom. Interested people should send me an email to receive the Zoom login information.

Target audience

third year undergraduates, graduate students, postdoc researchers

Content

The purpose of this lecture is to cover some topics of functional analysis which are typically relevant in Global Analysis and Noncommutative Geometry. My selection is biased as I will also discuss some of my own work. A detailed breakdown follows below.
The breakdown should be considered tentative. Suggestions by the audience for topics are welcome; also it will depend on the audience which topics to cover in more or less depth. Unexpectedly there is more material than time available, so we will have to make hard decision.

1. (Un)bounded operators in Hilbert space, interpolation, Hilbert complexes We start by fixing some notation and remind the audience of the basics of unbounded operators in Hilbert space and the Spectral Theorem. As a bonus we will discuss the powerful Calderon complex interpolation method and as an application we will look at the paper [BrLe92] on Hilbert complexes. Sturm-Liouville operators form a natural class to understand defect indices.

References [Ped89, 3.1, 3.2, Chap. 5], [BrLe92], [HLV18, Sec. 3], [BrLe01]

2. Compact and trace class operators The trace will be the main theme of the course.
Schatten classes, Calkin's theory of trace ideals, duality theory.

References [Ped89, 3.3, 3.4], [Sim05, Chap. 1 and 2]

3. Determinants of Hilbert space operators Fredholm determinants, Lidskii's theorem, the determinant as an entire function, Hadamard's theory of functions of finite order, regularized Fredholm determinant for operators of the form $I + T$ with T being p–summable after Simon.

References [Sim05, Chap. 3 and 9], [Sim77]

4. Asymptotic Analysis Mellin transform, asymptotic expansions, Hadamard's partie finie, regularized integrals, Laplace transform.
I will advertise that Hadamard's partie finie, though very elementary, is extremely usefull when dealing with asymptotic expansions resp. spectral generating functions.

References [Les97, Sec. 2.1],[Les98, LeTo98, LeVe11, HLV17]

5. Spectral generating functions Heat operator, heat trace, spectral ζ-functions and their relations. We will see that the heat expansion and the meromorphicity of the ζ-function are just two sides of the same medal and that the heat coefficients and the Laurent expansion of ζ at the poles can be computed from each other.

References [BrLe96, BrLe99]

6. The zeta-regularized determinant We will study the zeta-regularized determiant and its relation to Simon's regularized Fredholm determinant. Sturm-Liouville operators give rise to several interesting case studies. In part this is ongoing joint work with Luiz Hartmann.

References [Les98, LeTo98, LeVe11, HLV17]

7. The spectral flow (if time permits) This is independent of the previous topics and could as well be covered immediately after the introductory section.
I will basically follow [Les05] to present the the definition and uniqueness for the spectral flow on spaces of unbounded Fredholm operators. Then I will hint at the recent developments [BCLR].

References [Les05, BBLZ09, BCLR]

[BBLP05] B. Booss-Bavnbek, M. Lesch, and J. Phillips, *Unbounded Fredholm operators and spectral flow*, Canad. J. Math. **57** (2005), no. 2, 225–250.

[BBLZ09] B. Booss Bavnbek, M. Lesch, and C. Zhu, *The Calderón projection: new definition and applications*, J. Geom. Phys. **59** (2009), no. 7, 784–826.

[BCLR] C. Bourne, A. L. Carey, M. Lesch, and A. Rennie, *The KO-valued spectral flow for skew-adjoint Fredholm operators*, arxiv:1907.04981v2 [math.KT].

[BrLe92] J. Brüning and M. Lesch, *Hilbert complexes*, J. Funct. Anal. **108** (1992), no. 1, 88–132.

[BrLe96] J. Brüning and M. Lesch, *On the spectral geometry of algebraic curves*, J. Reine Angew. Math. **474** (1996), 25–66.

[BrLe99] ———, *On the η-invariant of certain nonlocal boundary value problems*, Duke Math. J. **96** (1999), no. 2, 425–468.

[BrLe01] ———, *On boundary value problems for Dirac type operators. I. Regularity and self-adjointness*, J. Funct. Anal. **185** (2001), no. 1, 1–62.

[HLV17] L. Hartmann, M. Lesch, and B. Vertman, *Zeta-determinants of Sturm-Liouville operators with quadratic potentials at infinity*, J. Differential Equations **262** (2017), no. 5, 3431–3465.

[HLV18] ———, *On the domain of Dirac and Laplace type operators on stratified spaces*, J. Spectr. Theory **8** (2018), no. 4, 1295–1348.

[Les97]   M. Lesch, *Operators of Fuchs type, conical singularities, and asymptotic methods*, Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], vol. 136, B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1997.

[Les98]   _____, *Determinants of regular singular Sturm-Liouville operators*, Math. Nachr. **194** (1998), 139–170.

[Les05]   _____, *The uniqueness of the spectral flow on spaces of unbounded self-adjoint Fredholm operators*, Spectral geometry of manifolds with boundary and decomposition of manifolds, Contemp. Math., vol. 366, Amer. Math. Soc., Providence, RI, 2005, pp. 193–224.

[LeTo98] M. Lesch and J. Tolksdorf, *On the determinant of one-dimensional elliptic boundary value problems*, Comm. Math. Phys. **193** (1998), no. 3, 643–660.

[LeVe11] M. Lesch and B. Vertman, *Regular singular Sturm-Liouville operators and their zeta-determinants*, J. Funct. Anal. **261** (2011), no. 2, 408–450.

[Ped89]   G. K. Pedersen, *Analysis now*, Graduate Texts in Mathematics, vol. 118, Springer-Verlag, New York, 1989.

[Sim77]   B. Simon, *Notes on infinite determinants of Hilbert space operators*, Advances in Math. **24** (1977), no. 3, 244–273.

[Sim05]   _____, *Trace ideals and their applications*, second ed., Mathematical Surveys and Monographs, vol. 120, American Mathematical Society, Providence, RI, 2005.