

# CONNES-CHERN CHARACTER FOR MANIFOLDS WITH BOUNDARY AND ETA COCHAINS

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ABSTRACT. We represent the Connes-Chern character of the Dirac operator associated to a b-metric on a manifold with boundary in terms of a retracted cocycle in relative cyclic cohomology, whose expression depends on a scaling/cut-off parameter. Blowing-up the metric one recovers the pair of characteristic currents that represent the corresponding de Rham relative homology class, while the blow-down yields a relative cocycle whose expression involves higher eta cochains and their b-analogues. The corresponding pairing formulæ with relative K-theory classes capture information about the boundary, and thus have geometric implications. As a byproduct we obtain a generalization of the Atiyah-Patodi-Singer odd-index theorem, from trivialized flat bundles to any pair of  $K$ -equivalent vector bundles.

## INTRODUCTION

Let  $M$  be a compact smooth  $m$ -dimensional manifold with boundary  $\partial M \neq \emptyset$ . Assuming that  $M$  possesses a  $Spin^c$  structure, the fundamental class in the relative  $K$ -homology group  $K_m(M, \partial M)$  can be realized analytically in terms of the Dirac operator  $D$  (graded if  $m$  is even) associated to a Riemannian metric on  $M$ . More precisely, according to [BDT89, §3], if  $D_e$  is a closed extension of  $D$  satisfying the condition that either  $D_e^*D_e$  or  $D_eD_e^*$  has compact resolvent (*e.g.* both  $D_{\min}$  and  $D_{\max}$  are such), then the bounded operator  $F = D_e(D_e^*D_e + 1)^{-1/2}$  defines a relative Fredholm module over the pair of  $C^*$ -algebras  $(\mathcal{C}(M), \mathcal{C}(\partial M))$ , hence an element  $[D] \in K_m(M, \partial M)$ . Moreover, by [BDT89, §4], the connecting homomorphism maps  $[D]$  to the fundamental class  $[D_\partial] \in K_{m-1}(\partial M)$  corresponding to the Dirac operator  $D_\partial$  associated to the boundary restriction of the metric and of  $Spin^c$  structure.

The map  $\text{Index}_{[D]} : K^m(M, \partial M) \rightarrow \mathbb{Z}$ , defined by the pairing of  $K$ -theory with the  $K$ -homology class of  $[D]$ , can be expressed in cohomological terms by means of Connes'

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Chern character with values in cyclic cohomology [CON85]. Indeed, the relative  $K$ -homology group  $K_m(M, \partial M)$ , viewed as the Kasparov group  $KK_m(\mathcal{C}_0(M \setminus \partial M); \mathbb{C})$ , can be realized as homotopy classes of Fredholm modules over the Fréchet algebra  $\mathcal{J}^\infty(\partial M, M)$  of smooth functions on  $M$  vanishing to any order on  $\partial M$ ;  $\mathcal{J}^\infty(\partial M, M)$  is a local  $C^*$ -algebra,  $H$ -unital and dense in  $\mathcal{C}_0(M \setminus \partial M) = \{f \in \mathcal{C}(M) \mid f|_{\partial M} = 0\}$ . One can therefore define the Connes-Chern character of  $[D]$  by restricting the operator  $F = D_e(D_e^*D_e + 1)^{-1/2}$ , or directly  $D$ , to  $\mathcal{J}^\infty(\partial M, M)$  and regarding it as a finitely summable Fredholm module. The resulting periodic cyclic cocycle corresponds, via the canonical isomorphism between the periodic cyclic cohomology  $HP^{\text{ev/odd}}(\mathcal{J}^\infty(\partial M, M))$  and the de Rham homology  $H_{\text{ev/odd}}^{\text{dR}}(M \setminus \partial M; \mathbb{C})$  (cf. [BRPF08]), to the de Rham class of the current (with arbitrary support) associated to the  $\hat{A}$ -form of the Riemannian metric. In fact, one can even recover the  $\hat{A}$ -form itself out of local cocycle representatives for the Connes-Chern character, as in [CoMo93, Remark 4, p. 119] or [CoMo95, Remark II.1, p. 231]. However, the boundary  $\partial M$  remains conspicuously absent in such representations.

It is the purpose of this paper to provide cocycle representatives for the Connes-Chern character of the fundamental  $K$ -homology class  $[D] \in K_m(M, \partial M)$  that capture and reflect geometric information about the boundary. Our point of departure is Getzler's construction [GET93A] of the Connes-Chern character of  $[D]$ . Cast in the propitious setting of Melrose's b-calculus [MEL93], Getzler's cocycle has however the disadvantage, from the viewpoint of its geometric functionality, of being realized not in the relative cyclic cohomology complex proper but in its entire extension. Entire cyclic cohomology [CON88], devised primarily for handling infinite-dimensional geometries, is less effective as a tool than ordinary cyclic cohomology when dealing with finite-dimensional  $K$ -homology cycles. To remedy this drawback we undertook the task of producing cocycle realizations for the Connes-Chern character directly in the relative cyclic cohomology complex associated to the pair of algebras  $(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M))$ . This is achieved by adapting and implementing in the context of relative cyclic cohomology the retraction procedure of [CoMo93], which converts the entire Connes-Chern character into the periodic one. The resulting cocycles automatically carry information about the boundary, and are thus amenable to geometric applications. A noteworthy byproduct of these considerations is the extension of the statement of the Atiyah-Patodi-Singer odd-index theorem [APS76], from trivialized flat bundles to any pair of equivalent vector bundles in  $K$ -theory.

Here is a quick synopsis of our main results. Throughout the paper, we fix an exact b-metric  $g$  on  $M$ , and denote by  $D$  the corresponding b-Dirac operator. We define for each  $t > 0$  and any  $n \geq m = \dim M$ ,  $n \equiv m \pmod{2}$ , pairs of cochains

$$\left( {}^b\text{ch}_t^n(D), \text{ch}_t^{n+1}(D_\partial) \right) \quad \text{resp.} \quad \left( {}^b\widetilde{\text{ch}}_t^n(D), \text{ch}_t^{n-1}(D_\partial) \right) \quad (0.1)$$

over the pair of algebras  $(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M))$ , given by the following expressions

$$\begin{aligned} {}^b\text{ch}_t^n(\mathbb{D}) &:= \sum_{j \geq 0} {}^b\text{Ch}^{n-2j}(t\mathbb{D}) + B {}^b\text{TCh}_t^{n+1}(\mathbb{D}), \\ \text{ch}_t^{n+1}(\mathbb{D}_\partial) &:= \sum_{j \geq 0} \text{Ch}^{n-2j+1}(t\mathbb{D}_\partial) + B \text{TCh}_t^{n+2}(\mathbb{D}_\partial), \\ {}^b\tilde{\text{ch}}_t^n(\mathbb{D}) &:= {}^b\text{ch}_t^n(\mathbb{D}) + \text{TCh}_t^n(\mathbb{D}_\partial) \circ i^*. \end{aligned} \quad (0.2)$$

In these formulæ,  $\text{Ch}^\bullet(\mathbb{D}_\partial)$  stand for the components of the Jaffe-Lesniewski-Osterwalder cocycle [JLO88] representing the Connes-Chern character in entire cyclic cohomology,  ${}^b\text{Ch}^\bullet(\mathbb{D})$  denote the corresponding b-analogue (cf. (2.4)), while the cochains  $\text{TCh}_t^\bullet(\mathbb{D}_\partial)$ , resp.  ${}^b\text{TCh}_t^\bullet(\mathbb{D})$  (see (6.5)), are manufactured out of the canonical transgression formula as in [CoMo93];  $i : \partial M \rightarrow M$  denotes the inclusion. One checks that

$$\begin{aligned} (b + B)({}^b\text{ch}_t^n(\mathbb{D})) &= \text{ch}_t^{n+1}(\mathbb{D}_\partial) \circ i^* \\ (b + B)({}^b\tilde{\text{ch}}_t^n(\mathbb{D})) &= \text{ch}_t^{n-1}(\mathbb{D}_\partial) \circ i^*, \end{aligned} \quad (0.3)$$

which shows that the cochains (0.1) are cocycles in the relative total  $(b, B)$ -complex of  $(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M))$ . Moreover, the class of this cocycle in periodic cyclic cohomology is independent of  $t > 0$  and of  $n = m + 2k$ ,  $k \in \mathbb{Z}_+$ . Furthermore, its limit as  $t \rightarrow 0$  gives the pair of  $\hat{A}$  currents corresponding to the b-manifold  $M$ , that is

$$\left( \lim_{t \searrow 0} {}^b\text{ch}_t^n(\mathbb{D}), \lim_{t \searrow 0} \text{ch}_t^{n+1}(\mathbb{D}_\partial) \right) = \left( \int_{{}^b\mathbb{M}} \hat{A}({}^b\nabla_g^2) \wedge -, \int_{\partial M} \hat{A}(\nabla_{g_\partial}^2) \wedge - \right). \quad (0.4)$$

The notation in the right hand side requires an explanation. With

$$\hat{A}({}^b\nabla_g^2) = \det \left( \frac{{}^b\nabla_g^2/4\pi i}{\sinh {}^b\nabla_g^2/4\pi i} \right)^{\frac{1}{2}}, \quad \hat{A}(\nabla_{g_\partial}^2) = \det \left( \frac{\nabla_{g_\partial}^2/4\pi i}{\sinh \nabla_{g_\partial}^2/4\pi i} \right)^{\frac{1}{2}}, \quad (0.5)$$

and  $\int_{{}^b\mathbb{M}} : {}^b\Omega^m(M) \rightarrow \mathbb{C}$  denoting the b-integral of b-differential  $m$ -forms on  $M$  associated to the trivialization of the normal bundle to  $\partial M$  underlying the b-structure, both terms in the right hand side of (0.4) are viewed as  $(b, B)$ -cochains associated to currents. More precisely, incorporating the  $2\pi i$  factors which account for the conversion of the Chern character in cyclic homology into the Chern character in de Rham cohomology, for  $M$  even-dimensional this identification takes the form

$$\int_{{}^b\mathbb{M}} \hat{A}({}^b\nabla_g^2) \wedge (f_0, \dots, f_{2q}) = \frac{1}{(2\pi i)^q (2q)!} \int_{{}^b\mathbb{M}} \hat{A}({}^b\nabla_g^2) \wedge f_0 df_1 \wedge \dots \wedge f_{2q}, \quad (0.6)$$

respectively

$$\int_{\partial M} \hat{A}(\nabla_{g_\partial}^2) \wedge (h_0, \dots, h_{2q-1}) = \frac{1}{(2\pi i)^q (2q-1)!} \int_{\partial M} \hat{A}(\nabla_{g_\partial}^2) \wedge h_0 dh_1 \wedge \dots \wedge h_{2q-1}. \quad (0.7)$$

Finally (cf. Theorem 6.1 *infra*), the limit formula (0.4) implies that both  $({}^b\text{ch}_t^n(\mathbb{D}), \text{ch}_t^{n+1}(\mathbb{D}_\partial))$  and  $({}^b\tilde{\text{ch}}_t^n(\mathbb{D}), \text{ch}_t^{n-1}(\mathbb{D}_\partial))$  represent the Chern character of  $[\mathbb{D}] \in K_m(M, \partial M)$ .

Under the assumption that  $\text{Ker } D_\partial = 0$ , we next prove (Theorem 6.2 *infra*) that the pair of retracted cochains  $({}^b\widetilde{\text{ch}}_t^n(D), \text{ch}_t^{n-1}(D_\partial))$  has a limit as  $t \rightarrow \infty$ . For  $n$  even, or equivalently  $M$  even-dimensional, this limit has the expression

$$\begin{aligned} {}^b\widetilde{\text{ch}}_\infty^n(D) &= \sum_{j=0}^{n/2} \kappa^{2j}(D) + B {}^b\text{T}\phi_\infty^{n+1}(D) + \text{T}\phi_\infty^n(D_\partial) \circ i^*, \\ \text{ch}_\infty^{n-1}(D_\partial) &= B \text{T}\phi_\infty^n(D_\partial), \end{aligned} \quad (0.8)$$

with the cochains  $\kappa^\bullet(D)$ , occurring only when  $\text{Ker } D \neq \{0\}$ , given by

$$\kappa^{2j}(D)(a_0, \dots, a_{2j}) = \text{Str}(\varrho_H(a_0) \omega_H(a_1, a_2) \cdots \omega_H(a_{2j-1}, a_{2j})); \quad (0.9)$$

here  $H$  denotes the orthogonal projection onto  $\text{Ker } D$ , and

$$\varrho_H(a) := HaH, \quad \omega_H(a, b) := \varrho_H(ab) - \varrho_H(a)\varrho_H(b), \quad \text{for all } a, b \in \mathcal{C}^\infty(M).$$

When  $M$  is odd-dimensional, the limit cocycle takes the similar but reversed form

$$\begin{aligned} {}^b\widetilde{\text{ch}}_\infty^n(D) &= B {}^b\text{T}\phi_\infty^{n+1}(D) + \text{T}\phi_\infty^n(D_\partial) \circ i^*, \\ \text{ch}_\infty^{n-1}(D_\partial) &= B \text{T}\phi_\infty^n(D_\partial). \end{aligned} \quad (0.10)$$

The absence of cochains of the form  $\kappa^\bullet(D_\partial)$  in the boundary component is due to the fact that  $\text{Ker } D_\partial = 0$ .

The geometric implications become apparent when one inspects the ensuing pairing with  $K$ -theory classes. For  $M$  even-dimensional, a class in  $K^m(M, \partial M)$  can be represented as a triple  $[E, F, h]$ , where  $E, F$  are vector bundles over  $M$ , which we will identify with projections  $p_E, p_F \in \text{Mat}_N(\mathcal{C}^\infty(M))$ , and  $h : [0, 1] \rightarrow \text{Mat}_N(\mathcal{C}^\infty(\partial M))$  is a smooth path of projections connecting their restrictions to the boundary  $E_\partial$  and  $F_\partial$ . For  $M$  odd-dimensional, a representative of a class in  $K^m(M, \partial M)$  is a triple  $(U, V, h)$ , where  $U, V : M \rightarrow U(N)$  are unitaries and  $h$  is a homotopy between their restrictions to the boundary  $U_\partial$  and  $V_\partial$ . In both cases, the Chern character of  $[X, Y, h] \in K^m(M, \partial M)$  is represented by the relative cyclic homology cycle over the algebras  $(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M))$

$$\text{ch}_\bullet([X, Y, h]) = \left( \text{ch}_\bullet(Y) - \text{ch}_\bullet(X), -\text{T}\phi_\bullet(h) \right), \quad (0.11)$$

where  $\text{ch}_\bullet$ , resp.  $\text{T}\phi_\bullet$  denote the components of the standard Chern character in cyclic homology resp. of its canonical transgression (see Subsection 1.3).

The pairing  $\langle [D], [X, Y, h] \rangle$  between the classes  $[D] \in K_m(M, \partial M)$  and  $[X, Y, h] \in K^m(M, \partial M)$  acquires the cohomological expression

$$\begin{aligned} \langle [D], [X, Y, h] \rangle &= \langle ({}^b\widetilde{\text{ch}}_t^n(D), \text{ch}_t^{n-1}(D_\partial)), \text{ch}_\bullet[X, Y, h] \rangle = \\ &= \left\langle \sum_{j \geq 0} {}^b\text{Ch}^{n-2j}(tD) + B {}^b\text{T}\phi_t^{n+1}(D), \text{ch}_\bullet(Y) - \text{ch}_\bullet(X) \right\rangle \\ &\quad + \langle \text{T}\phi_t^n(D_\partial), \text{ch}_\bullet(Y_\partial) - \text{ch}_\bullet(X_\partial) \rangle \\ &\quad - \left\langle \sum_{j \geq 0} \text{Ch}^{n-2j-1}(tD_\partial) + B \text{T}\phi_t^n(D_\partial), \text{T}\phi_\bullet(h) \right\rangle, \end{aligned} \quad (0.12)$$

which holds for any  $t > 0$ . Letting  $t \rightarrow 0$  yields the local form of the pairing formula

$$\begin{aligned} \langle [D], [X, Y, h] \rangle &= \\ &= \int_{\mathfrak{b}M} \hat{A}(\mathfrak{b}\nabla_g^2) \wedge (\text{ch}_\bullet(Y) - \text{ch}_\bullet(X)) - \int_{\partial M} \hat{A}(\nabla_{g_\partial}^2) \wedge \text{Tch}_\bullet(h). \end{aligned} \quad (0.13)$$

It should be pointed out that (0.13) holds in complete generality, without requiring the invertibility of  $D_\partial$ .

When  $M$  is even-dimensional and  $D_\partial$  is invertible, the equality between the above limit and the limit as  $t \rightarrow \infty$  yields, for any  $n = 2\ell \geq m$ , the identity

$$\begin{aligned} &\sum_{0 \leq k \leq \ell} \langle \kappa^{2k}(D), \text{ch}_{2k}(p_F) - \text{ch}_{2k}(p_E) \rangle + \langle B \text{bTch}_\infty^{n+1}(D), \text{ch}_n(p_F) - \text{ch}_n(p_E) \rangle + \\ &\quad + \langle \text{Tch}_\infty^n(D_\partial), \text{ch}_n(p_{F_\partial}) - \text{ch}_n(p_{E_\partial}) \rangle = \\ &= \int_{\mathfrak{b}M} \hat{A}(\mathfrak{b}\nabla_g^2) \wedge (\text{ch}_\bullet(p_F) - \text{ch}_\bullet(p_E)) - \int_{\partial M} \hat{A}(\nabla_{g_\partial}^2) \wedge \text{Tch}_\bullet(h) \\ &\quad + \langle B \text{Tch}_\infty^n(D_\partial), \text{Tch}_{n-1}(h) \rangle, \end{aligned} \quad (0.14)$$

where

$$\text{ch}_{2k}(p) = \begin{cases} \text{tr}_0(p), & \text{for } k = 0, \\ (-1)^k \frac{(2k)!}{k!} \text{tr}_{2k} \left( (p - \frac{1}{2}) \otimes p^{\otimes 2k} \right), & \text{for } k > 0. \end{cases}$$

Like the Atiyah-Patodi-Singer index formula [APS75], the equation (0.14) involves index and eta cochains, only of higher order. Moreover, the same type of identity continues to hold in the odd-dimensional case. Explicitly, it takes the form

$$\begin{aligned} &(-1)^{\frac{n-1}{2}} \left( \frac{n-1}{2} \right)! \left( \langle B \text{bTch}_\infty^{n+1}(D), (V^{-1} \otimes V)^{\otimes \frac{n+1}{2}} - (U^{-1} \otimes U)^{\otimes \frac{n+1}{2}} \rangle + \right. \\ &\quad \left. \langle \text{Tch}_\infty^n(D_\partial), (V_\partial^{-1} \otimes V_\partial)^{\otimes \frac{n+1}{2}} - (U_\partial^{-1} \otimes U_\partial)^{\otimes \frac{n+1}{2}} \rangle \right) = \\ &= \int_{\mathfrak{b}M} \hat{A}(\mathfrak{b}\nabla_g^2) \wedge (\text{ch}_\bullet(V) - \text{ch}_\bullet(U)) - \int_{\partial M} \hat{A}(\nabla_{g_\partial}^2) \wedge \text{Tch}_\bullet(h) \\ &\quad + \langle B \text{Tch}_\infty^n(D_\partial), \text{Tch}_{n-1}(h) \rangle. \end{aligned} \quad (0.15)$$

The relationship between the relative pairing and the Atiyah-Patodi-Singer index theorem can actually be made explicit, and leads to interesting geometric consequences. Indeed, under the necessary assumption that  $M$  is even-dimensional, we show (*cf.* Theorem 7.6) that the above pairing can be expressed as follows:

$$\langle [D], [E, F, h] \rangle = \text{Ind}_{\text{APS}} D^F - \text{Ind}_{\text{APS}} D^E + \text{SF}(h, D_\partial); \quad (0.16)$$

here  $\text{Ind}_{\text{APS}}$  stands for the APS-index, and  $\text{SF}(h, D_\partial)$  denotes the spectral flow along the path of operators  $(D_\partial^+)^{h(s)}$ ;  $D_\partial^+$  is the restriction of  $c(dx)^{-1}D_\partial$  to the positive half spinor bundle and  $c(dx)$  denotes Clifford multiplication by the inward normal vector. On applying the A-P-S index formula [APS75, Eq. (4.3)], the pairing takes the explicit

form

$$\begin{aligned} \langle [\mathbf{D}], [E, F, h] \rangle &= \int_{\mathfrak{b}M} \hat{A}(\mathfrak{b}\nabla_g^2) \wedge (\mathrm{ch}_\bullet(F) - \mathrm{ch}_\bullet(E)) \\ &\quad - \left( \xi(\mathbf{D}_\partial^{+,F\partial}) - \xi(\mathbf{D}_\partial^{+,E\partial}) \right) + \mathrm{SF}(h, \mathbf{D}_\partial), \end{aligned} \quad (0.17)$$

where

$$\xi(\mathbf{D}_\partial^{+,E\partial}) = \frac{1}{2} \left( \eta(\mathbf{D}_\partial^{+,E\partial}) + \dim \mathrm{Ker} \mathbf{D}_\partial^{+,E\partial} \right). \quad (0.18)$$

Comparing this expression with the local form of the pairing (0.13) leads to a generalization of the A-P-S odd-index formula [APS76, Prop. 6.2, Eq. (6.3)], from trivialized flat bundles to pairs of equivalent vector bundles in  $K$ -theory. Precisely (*cf.* Corollary 7.9), if  $E', F'$  are two such bundles on a closed odd-dimensional spin manifold  $N$ , and  $h$  is the homotopy implementing the equivalence of  $E'$  with  $F'$ , then

$$\xi(\mathbf{D}_{g'}^{F'}) - \xi(\mathbf{D}_{g'}^{E'}) = \int_N \hat{A}(\nabla_{g'}^2) \wedge \mathrm{Tch}_\bullet(h) + \mathrm{SF}(h, \mathbf{D}_{g'}), \quad (0.19)$$

where  $\mathbf{D}_{g'}$  denotes the Dirac operator associated to a Riemannian metric  $g'$  on  $N$ ; equivalently,

$$\int_0^1 \frac{1}{2} \frac{d}{dt} (\eta(p_{h(t)} \mathbf{D}_{g'} p_{h(t)})) dt = \int_N \hat{A}(\nabla_{g'}^2) \wedge \mathrm{Tch}_\bullet(h), \quad (0.20)$$

where  $p_{h(t)}$  is the path of projections joining  $E'$  and  $F'$ , and the left hand side is the natural extension of the real-valued index in [APS76, Eq. (6.1)].

Let us briefly comment on the main analytical challenges encountered in the course of proving the results outlined above. In order to compute the limit as  $t \searrow 0$  of the Chern character, one needs to understand the asymptotic behavior of expressions of the form

$$\begin{aligned} &\mathfrak{b}\langle A_0, A_1, \dots, A_k \rangle_{\sqrt{t}\mathbf{D}} \\ &:= \int_{\Delta_k} \mathfrak{b}\mathrm{Tr}(A_0 e^{-\sigma_0 t \mathbf{D}^2} A_1 e^{-\sigma_1 t \mathbf{D}^2} \dots A_k e^{-\sigma_k t \mathbf{D}^2}) d\sigma, \end{aligned} \quad (0.21)$$

where  $\Delta_k$  denotes the standard simplex  $\{\sigma_0 + \dots + \sigma_k = 1, \sigma_j \geq 0\}$  and  $A_0, \dots, A_k$  are  $\mathfrak{b}$ -differential operators. The difficulty here is twofold. Firstly, the  $\mathfrak{b}$ -trace is a *regularized* extension of the trace to  $\mathfrak{b}$ -pseudodifferential operators on the *non-compact* manifold  $M \setminus \partial M$  (recall that the  $\mathfrak{b}$ -metric degenerates at  $\partial M$ ). Secondly, the expression inside the  $\mathfrak{b}$ -trace involves a product of operators. The Schwartz kernel of the product  $A_0 e^{-\sigma_0 t \mathbf{D}^2} A_1 e^{-\sigma_1 t \mathbf{D}^2} \dots A_k e^{-\sigma_k t \mathbf{D}^2}$  does admit a *pointwise* asymptotic expansion (see [Wid79], [CoMo90], [BLFo90]), namely

$$\begin{aligned} &\left( A_0 e^{-\sigma_0 t \mathbf{D}^2} A_1 e^{-\sigma_1 t \mathbf{D}^2} \dots A_k e^{-\sigma_k t \mathbf{D}^2} \right)(p, p) \\ &=: \sum_{j=0}^n a_j(A_0, \dots, A_k, \mathbf{D})(p) t^{\frac{j - \dim M - a}{2}} + O_p(t^{(n+1-a-\dim M)/2}), \end{aligned} \quad (0.22)$$

where  $a = \sum_{j=0}^k \text{ord } A_j$ . However, this asymptotic expansion is only *locally* uniform in  $p$ ; it is not *globally* uniform on the non-compact manifold  $M \setminus \partial M$ . A further complication arises from the fact that the function  $a_j(A_0, \dots, A_k, D)$  is not necessarily integrable. Nevertheless, a *partie finie*-type regularized integral, which we denote by  $\int_{\text{b}M} a_j(A_0, \dots, A_k, D) d \text{vol}$ , does exist and we prove (*cf.* Theorem 5.3) that the corresponding b-trace admits an asymptotic expansion of the form

$$\begin{aligned} & \text{bTr} \left( A_0 e^{-\sigma_0 t D^2} A_1 e^{-\sigma_1 t D^2} \dots A_k e^{-\sigma_k t D^2} \right) \\ &= \sum_{j=0}^n \int_{\text{b}M} a_j(A_0, \dots, A_k, D) d \text{vol} t^{\frac{j - \dim M - a}{2}} + \\ &+ O \left( \left( \prod_{j=1}^k \sigma_j^{-a_j/2} \right) t^{(n+1-a-\dim M)/2} \right). \end{aligned} \quad (0.23)$$

When  $D_\partial$  is invertible and hence  $D$  is a Fredholm operator, we can also prove the following estimate (*cf.* (3.43))

$$\begin{aligned} & | \text{b} \langle A_0(I - H), \dots, A_k(I - H) \rangle_{\sqrt{t}D} | \\ & \leq \tilde{C}_{\delta, \varepsilon} t^{-a/2 - (\dim M)/2 - \varepsilon} e^{-t\delta}, \quad \text{for all } 0 < t < \infty, \end{aligned} \quad (0.24)$$

for any  $\varepsilon > 0$  and any  $0 < \delta < \inf \text{spec}_{\text{ess}} D^2$ . Here,  $H$  is the orthogonal projection onto  $\text{Ker } D$ . This estimate allows us to compute the limit as  $t \nearrow \infty$  and thus derive the formulæ (0.8) and (0.10).

A few words about the organization of the paper are now in order. We start by recalling, in Section 1, some basic material on relative cyclic cohomology [LMP09], b-calculus [MEL93] and Dirac operators. As an illustration of the usefulness of the relative cohomology point of view for manifolds with boundary, in Subsection 1.5 we establish a cohomological analogue of the well-known McKean-Singer formula in the framework of relative cyclic cohomology for the pseudodifferential b-calculus, and employ it to recast in these terms Melrose's approach to the proof of the Atiyah-Patodi-Singer index theorem (*cf.* [MEL93, Introduction]). In Subsection 1.6, refining an observation due to Loya [LOY05], we give an effective formula for the b-trace, which will turn out to be quite a useful technical device.

Gerzler's version of the relative entire Connes-Chern character in the setting of b-calculus [GET93A] is discussed in Section 2. Additional material on b-geometry can be found in Appendix A.

In Section 3 we prove some crucial estimates for the heat kernel of a b-Dirac operator, which are then applied in Section 4 to analyze the short and long time behavior of the components of the b-analogue of the entire Chern character. More standard resolvent and heat kernel estimates are relegated to Appendix B.

Section 5 is devoted to asymptotic expansions for the b-analogues of the Jaffe-Lesniewski-Osterwalder components. The retracted relative cocycle representing the

Connes-Chern character in relative cyclic cohomology is constructed in Section 6, where we also compute the expressions of its small and large scale limits.

Finally, Section 7 derives the ensuing pairing formulæ with the  $K$ -theory, establishes the connection with the Atiyah-Patodi-Singer index theorem, and discusses the geometric consequences. The paper concludes with an explanatory note (Subsection 7.1) elucidating the relationship between the results presented here and the prior work in this direction by Getzler [GET93A] and Wu [WU93].

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1. PRELIMINARIES

1.1. **The general setup.** Associated to a compact smooth manifold  $M$  with boundary  $\partial M$ , there is a commutative diagram of Fréchet algebras with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{J}^\infty(\partial M, M) & \longrightarrow & \mathcal{C}^\infty(M) & \xrightarrow{\ell} & \mathcal{E}^\infty(\partial M, M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \text{Id} & & \downarrow \parallel_{\partial M} & & \\
 0 & \longrightarrow & \mathcal{J}(\partial M, M) & \longrightarrow & \mathcal{C}^\infty(M) & \longrightarrow & \mathcal{C}^\infty(\partial M) & \longrightarrow & 0.
 \end{array} \tag{1.1}$$

$\mathcal{J}(\partial M, M) \subset \mathcal{C}^\infty(M)$  is the closed ideal of smooth functions on  $M$  vanishing on  $\partial M$ ,  $\mathcal{J}^\infty(\partial M, M) \subset \mathcal{J}(\partial M, M)$  denotes the closed ideal of smooth functions on  $M$  vanishing up to infinite order on  $\partial M$ , and  $\mathcal{E}^\infty(\partial M, M)$  is the algebra of Whitney functions over the subset  $\partial M \subset M$ . More generally, for every closed subset  $X \subset M$  the ideal  $\mathcal{J}^\infty(X, M) \subset \mathcal{C}^\infty(M)$  is defined as being

$$:= \{f \in \mathcal{C}^\infty(M) \mid Df|_X = 0 \text{ for every differential operator } D \text{ on } M\}.$$

By Whitney’s extension theorem (cf. [MAL67, TOU72]), the algebra  $\mathcal{E}^\infty(X, M)$  of Whitney functions over  $X \subset M$  is naturally isomorphic to the quotient of  $\mathcal{C}^\infty(M)$  by the closed ideal  $\mathcal{J}^\infty(X, M)$ ; we take this as a definition of  $\mathcal{E}^\infty(X, M)$ . The right vertical arrow in diagram (1.1) is given by the map

$$\mathcal{E}^\infty(X, M) \rightarrow \mathcal{C}^\infty(X), \quad F \mapsto F|_X := F + \mathcal{J}^\infty(X, M),$$

which is a surjection.

Let us check that the Fréchet algebra  $\mathcal{J}^\infty := \mathcal{J}^\infty(\partial M, M)$  is a local  $C^*$ -algebra. First, by Faà di Bruno's formula the unitalization  $\mathcal{J}^{\infty,+}$  of  $\mathcal{J}^\infty$  is seen to be closed under holomorphic calculus in the unitalization  $J^+$  of the algebra  $J := \mathcal{C}_0(M \setminus \partial M)$ . Since  $\mathcal{J}^{\infty,+}$  is also dense in  $J^+$ , it follows that  $\mathcal{J}^\infty := \mathcal{J}^\infty(\partial M, M)$  is indeed a local  $C^*$ -algebra whose  $C^*$ -closure is the  $C^*$ -algebra  $J$ . Using this together with excision in  $K$ -homology (cf. for example [HiRo00]), one can easily check that the space of equivalence classes of Fredholm modules over  $\mathcal{J}^\infty$  coincides naturally with the  $K$ -homology of the pair of  $C^*$ -algebras  $(\mathcal{C}(M), \mathcal{C}(\partial M))$ . Moreover, by [CON94, p. 298] one has the following commutative diagram

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{finitely summable Fredholm} \\ \text{modules over } \mathcal{J}^\infty \end{array} \right\} & \xrightarrow{\text{ch.}} & HP^\bullet(\mathcal{J}^\infty) \\ \downarrow & & \downarrow \\ K^\bullet(J) = KK_\bullet(J, \mathbb{C}) & \longrightarrow & \text{Hom}(K_\bullet(J), \mathbb{C}), \end{array} \quad (1.2)$$

where the right vertical arrow is given by natural pairing between periodic cyclic cohomology and  $K$ -theory, and the lower horizontal arrow by the pairing of  $K$ -theory with  $K$ -homology via the Fredholm index.

A Dirac, resp. a b-Dirac operator on  $M^m$  determines a Fredholm module over  $\mathcal{J}^\infty$  and therefore a class in the  $K$ -homology of the pair  $(\mathcal{C}(M), \mathcal{C}(\partial M))$ . In this article, we are concerned with geometric representations of the Connes-Chern character of such a class and of the ensuing pairing with the  $K$ -theory of the pair  $(\mathcal{C}(M), \mathcal{C}(\partial M))$ .

**1.2. Relative cyclic cohomology.** As in [LMP09], we associate to a short exact sequence of Fréchet algebras

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{A} \xrightarrow{\sigma} \mathcal{B} \longrightarrow 0, \quad (1.3)$$

with  $\mathcal{A}$  and  $\mathcal{B}$  unital, a short exact sequence of mixed complexes

$$0 \longrightarrow (C^\bullet(\mathcal{B}), b, B) \longrightarrow (C^\bullet(\mathcal{A}), b, B) \longrightarrow (Q^\bullet, b, B) \longrightarrow 0, \quad (1.4)$$

where  $C^\bullet(\mathcal{A})$  denotes the Hochschild cochain complex of a Fréchet algebra  $\mathcal{A}$ ,  $b$  the Hochschild coboundary, and  $B$  is the Connes coboundary (cf. [CON85, CON94]). Recall that the *Hochschild cohomology* of  $\mathcal{A}$  is computed by the complex  $(C^\bullet(\mathcal{A}), b)$ , the *cyclic cohomology* of  $\mathcal{A}$  is the cohomology of the total complex

$(\text{Tot}_{\oplus}^\bullet \mathcal{BC}^{\bullet,\bullet}(\mathcal{A}), b + B)$ , where

$$\mathcal{BC}^{p,q}(\mathcal{A}) = \begin{cases} C^{q-p}(\mathcal{A}) := (\mathcal{A}^{\hat{\otimes}^{q-p+1}})^*, & \text{for } q \geq p \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

while the *periodic cyclic cohomology* of  $\mathcal{A}$  is the cohomology of the total complex  $(\text{Tot}_{\oplus}^\bullet \mathcal{BC}_{\text{per}}^{\bullet,\bullet}(\mathcal{A}), b + B)$ , where

$$\mathcal{BC}_{\text{per}}^{p,q}(\mathcal{A}) = \begin{cases} C^{q-p}(\mathcal{A}) := (\mathcal{A}^{\hat{\otimes}^{q-p+1}})^*, & \text{for } q \geq p, \\ 0, & \text{else.} \end{cases}$$

In [LMP09] we noted that the relative cohomology theories, or in other words the cohomologies of the quotient mixed complex  $(Q^\bullet, b, B)$ , can be calculated from a particular mixed complex quasi-isomorphic to  $Q^\bullet$ , namely from the direct sum mixed complex

$$(C^\bullet(\mathcal{A}) \oplus C^{\bullet+1}(\mathcal{B}), \tilde{b}, \tilde{B}),$$

where

$$\tilde{b} = \begin{pmatrix} b & -\sigma^* \\ 0 & -b \end{pmatrix}, \quad \text{and} \quad \tilde{B} = \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}. \quad (1.5)$$

In particular, the relative Hochschild cohomology  $HH^\bullet(\mathcal{A}, \mathcal{B})$  is computed by the complex  $(C^\bullet(\mathcal{A}) \oplus C^{\bullet+1}(\mathcal{B}), \tilde{b})$ , the relative cyclic cohomology  $HC^\bullet(\mathcal{A}, \mathcal{B})$  by  $(\text{Tot}_\oplus^\bullet \mathcal{BC}^{\bullet,\bullet}(\mathcal{A}) \oplus \text{Tot}_\oplus^{\bullet+1} \mathcal{BC}^{\bullet,\bullet}(\mathcal{B}), \tilde{b} + \tilde{B})$ , and the relative periodic cyclic cohomology  $HP^\bullet(\mathcal{A}, \mathcal{B})$  by  $(\text{Tot}_\oplus^\bullet \mathcal{BC}_{\text{per}}^{\bullet,\bullet}(\mathcal{A}) \oplus \text{Tot}_\oplus^{\bullet+1} \mathcal{BC}_{\text{per}}^{\bullet,\bullet}(\mathcal{B}), \tilde{b} + \tilde{B})$ .

Note that of course

$$\begin{aligned} & (\text{Tot}_\oplus^\bullet \mathcal{BC}^{\bullet,\bullet}(\mathcal{A}) \oplus \text{Tot}_\oplus^{\bullet+1} \mathcal{BC}^{\bullet,\bullet}(\mathcal{B}), \tilde{b} + \tilde{B}) \\ & \simeq (\text{Tot}_\oplus^\bullet \mathcal{BC}^{\bullet,\bullet}(\mathcal{A}, \mathcal{B}), \tilde{b} + \tilde{B}), \end{aligned} \quad (1.6)$$

where  $\mathcal{BC}^{p,q}(\mathcal{A}, \mathcal{B}) := \mathcal{BC}^{p,q}(\mathcal{A}) \oplus \mathcal{BC}^{p,q+1}(\mathcal{B})$ .

Dually to relative cyclic cohomology, one can define relative cyclic homology theories. We will use these throughout this article as well, and in particular their pairing with relative cyclic cohomology. For the convenience of the reader we recall their definition, referring to [LMP09] for more details. The short exact sequence (1.3) gives rise to the following short exact sequence of homology mixed complexes

$$0 \rightarrow (K_\bullet, b, B) \rightarrow (C_\bullet(\mathcal{A}), b, B) \rightarrow (C_\bullet(\mathcal{B}), b, B) \rightarrow 0, \quad (1.7)$$

where here  $b$  denotes the Hochschild boundary, and  $B$  the Connes boundary. The kernel mixed complex  $K_\bullet$  is quasi-isomorphic to the direct sum mixed complex

$$(C_\bullet(\mathcal{A}) \oplus C_{\bullet+1}(\mathcal{B}), \tilde{b}, \tilde{B}),$$

where

$$\tilde{b} = \begin{pmatrix} b & 0 \\ -\sigma_* & -b \end{pmatrix}, \quad \text{and} \quad \tilde{B} = \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}. \quad (1.8)$$

This implies that the relative cyclic homology  $HC_\bullet(\mathcal{A}, \mathcal{B})$  is the homology of  $(\text{Tot}_\oplus^\bullet \mathcal{BC}_{\bullet,\bullet}(\mathcal{A}, \mathcal{B}), \tilde{b} + \tilde{B})$ , where  $\mathcal{BC}_{p,q}(\mathcal{A}, \mathcal{B}) = \mathcal{BC}_{p,q}(\mathcal{A}) \oplus \mathcal{BC}_{p,q+1}(\mathcal{B})$ . Likewise, the relative periodic cyclic homology  $HP_\bullet(\mathcal{A}, \mathcal{B})$  is the homology of  $(\text{Tot}_\oplus^\bullet \mathcal{BC}_{\bullet,\bullet}^{\text{per}}(\mathcal{A}, \mathcal{B}), \tilde{b} + \tilde{B})$ , where  $\mathcal{BC}_{p,q}^{\text{per}}(\mathcal{A}, \mathcal{B}) = \mathcal{BC}_{p,q}^{\text{per}}(\mathcal{A}) \oplus \mathcal{BC}_{p,q+1}^{\text{per}}(\mathcal{B})$ .

By [LMP09, Prop. 1.1], the relative cyclic (co)homology groups inherit a natural pairing

$$\langle -, - \rangle_\bullet : HC^\bullet(\mathcal{A}, \mathcal{B}) \times HC_\bullet(\mathcal{A}, \mathcal{B}) \rightarrow \mathbb{C}, \quad (1.9)$$

which will be called the *relative cyclic pairing*, and which on chains and cochains is defined by

$$\begin{aligned} \langle -, - \rangle : & \left( \mathcal{BC}^{p,q}(\mathcal{A}) \oplus \mathcal{BC}^{p,q+1}(\mathcal{B}) \right) \times \left( \mathcal{BC}_{p,q}(\mathcal{A}) \oplus \mathcal{BC}_{p,q+1}(\mathcal{B}) \right) \rightarrow \mathbb{C}, \\ & ((\varphi, \psi), (\alpha, \beta)) \mapsto \langle \varphi, \alpha \rangle + \langle \psi, \beta \rangle. \end{aligned} \quad (1.10)$$

This formula also describes the pairing between the relative periodic cyclic (co)homology groups.

Returning to diagram (1.1), we can now express the (periodic) cyclic cohomology of the pair  $(\mathcal{C}^\infty(M), \mathcal{E}^\infty(\partial M, M))$  resp. of the pair  $(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M))$  in terms of the cyclic cohomology complexes of  $\mathcal{C}^\infty(M)$  and  $\mathcal{E}^\infty(\partial M, M)$  resp.  $\mathcal{C}^\infty(M)$ . We note that the ideal  $\mathcal{J}^\infty(\partial M, M)$  is H-unital, since  $(\mathcal{J}^\infty(\partial M, M))^2 = \mathcal{J}^\infty(\partial M, M)$  (cf. [BRPF08]). Hence excision holds true for the ideal  $\mathcal{J}^\infty(\partial M, M)$ , and any of the above cohomology theories of  $\mathcal{J}^\infty(\partial M, M)$  coincides with the corresponding relative cohomology of the pair  $(\mathcal{C}^\infty(M), \mathcal{E}^\infty(\partial M, M))$ . In particular, we have the following chain of quasi-isomorphisms

$$\begin{aligned} \mathrm{Tot}_\oplus^\bullet \mathcal{BC}^{\bullet,\bullet}(\mathcal{J}^\infty(\partial M, M)) &\sim_{\mathrm{qism}} \mathrm{Tot}_\oplus^\bullet \mathcal{BC}^{\bullet,\bullet}(\mathcal{C}^\infty(M), \mathcal{E}^\infty(\partial M), M) \sim_{\mathrm{qism}} \\ &\sim_{\mathrm{qism}} \mathrm{Tot}_\oplus^\bullet \mathcal{BC}^{\bullet,\bullet}(\mathcal{C}^\infty(M)) \oplus \mathrm{Tot}_\oplus^{\bullet+1} \mathcal{BC}^{\bullet,\bullet}(\mathcal{E}^\infty(\partial M, M)). \end{aligned} \quad (1.11)$$

Next recall from [BRPF08] that the map

$$\begin{aligned} \mathrm{Tot}_\oplus^k \mathcal{BC}_{\mathrm{per}}^{\bullet,\bullet}(\mathcal{C}^\infty(M)) &\rightarrow \mathrm{Tot}_\oplus^k \mathcal{BC}_{\mathrm{per}}^{\bullet,\bullet}(\mathcal{E}^\infty(\partial M, M)), \\ \psi &\mapsto \left( (\mathcal{E}^\infty(\partial M, M))^{\hat{\otimes} k+1} \ni F_0 \otimes \dots \otimes F_k \mapsto \psi(F_{0\parallel X} \otimes \dots \otimes F_{k\parallel X}) \right) \end{aligned}$$

between the periodic cyclic cochain complexes is a quasi-isomorphism. As a consequence of the Five Lemma one obtains quasi-isomorphisms

$$\begin{aligned} \mathrm{Tot}_\oplus^\bullet \mathcal{BC}_{\mathrm{per}}^{\bullet,\bullet}(\mathcal{J}^\infty(\partial M, M)) &\sim_{\mathrm{qism}} \mathrm{Tot}_\oplus^\bullet \mathcal{BC}_{\mathrm{per}}^{\bullet,\bullet}(\mathcal{C}^\infty(M), \mathcal{E}^\infty(\partial M)) \sim_{\mathrm{qism}} \\ &\sim_{\mathrm{qism}} \mathrm{Tot}_\oplus^\bullet \mathcal{BC}_{\mathrm{per}}^{\bullet,\bullet}(\mathcal{C}^\infty(M)) \oplus \mathrm{Tot}_\oplus^{\bullet+1} \mathcal{BC}_{\mathrm{per}}^{\bullet,\bullet}(\mathcal{C}^\infty(\partial M)). \end{aligned} \quad (1.12)$$

In this paper we will mainly work with the relative complexes over the pair of algebras  $(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M))$ , because its cycles carry geometric information about the boundary, which is lost when considering only cycles over the ideal  $\mathcal{J}^\infty(\partial M, M)$ . In this respect we note that periodic cyclic cohomology satisfies excision by [CuQU93, CuQU94], hence in the notation of (1.3),  $HP^\bullet(\mathcal{J})$  is canonically isomorphic to  $HP^\bullet(\mathcal{A}, \mathcal{B})$ .

**1.3. The Chern-character.** For future reference, we recall the Chern character and its transgression in cyclic homology, both in the even and in the odd case.

**1.3.1. Even case.** The Chern character of an idempotent  $e \in \mathrm{Mat}_\infty(\mathcal{A}) := \varinjlim_{N \rightarrow \infty} \mathrm{Mat}_N(\mathcal{A})$  is the class in  $HP_0(\mathcal{A})$  of the cycle given by the formula

$$\mathrm{ch}_\bullet(e) := \mathrm{tr}_0(e) + \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{k!} \mathrm{tr}_{2k} \left( \left( e - \frac{1}{2} \right) \otimes e^{\otimes (2k)} \right), \quad (1.13)$$

where  $e^{\otimes (2k)}$  is an abbreviation for the  $2k$ -fold tensor product  $e \otimes \dots \otimes e$ , and  $\mathrm{tr}_{2k}$  denotes the generalized trace map  $\mathrm{Mat}_N(\mathcal{A})^{\otimes j} \rightarrow \mathcal{A}^{\otimes j}$ .

If  $(e_s)_{0 \leq s \leq 1}$  is a smooth path of idempotents, then the transgression formula reads

$$\frac{d}{ds} \mathrm{ch}_\bullet(e_s) = (b + B) \phi_{\mathbf{h}_\bullet}(e_s, (2e_s - 1)\dot{e}_s); \quad (1.14)$$

here the secondary Chern character  $\phi_{\mathbf{h}_\bullet}$  is given by

$$\phi_{\mathbf{h}_\bullet}(e, h) := \iota(h) \mathrm{ch}_\bullet(e), \quad (1.15)$$

where the map  $\iota(h)$  is defined by

$$\begin{aligned} \iota(h)(a_0 \otimes a_1 \otimes \dots \otimes a_l) \\ = \sum_{i=0}^l (-1)^i (a_0 \otimes \dots \otimes a_i \otimes h \otimes a_{i+1} \otimes \dots \otimes a_l). \end{aligned} \quad (1.16)$$

A relative  $K$ -theory class in  $K_0(\mathcal{A}, \mathcal{B})$  can be represented by a triple  $(p, q, h)$  with projections  $p, q \in \text{Mat}_N(\mathcal{A})$  and  $h : [0, 1] \rightarrow \text{Mat}_N(\mathcal{B})$  a smooth path of projections with  $h(0) = \sigma(p)$ ,  $h(1) = \sigma(q)$  (cf. [HiRo00, Def. 4.3.3], see also [LMP09, Sec. 1.6]). The Chern character of  $(p, q, h)$  is represented by the relative cyclic cycle

$$\text{ch}_\bullet(p, q, h) = \left( \text{ch}_\bullet(q) - \text{ch}_\bullet(p), -\text{T}\phi_\bullet(h) \right), \quad (1.17)$$

where

$$\text{T}\phi_\bullet(h) = \int_0^1 \phi_\bullet(h(s), (2h(s) - 1)\dot{h}(s)) ds. \quad (1.18)$$

That the r.h.s. of Eq. (1.17) is a relative cyclic cycle follows from the transgression formula Eq. (1.14). From a secondary transgression formula [LMP09, (1.43)] one deduces that (1.17) indeed corresponds to the standard Chern character on  $K_0(\mathcal{J})$  under excision.

1.3.2. *Odd case.* The odd case parallels the even case in many aspects. Given an element  $g \in \text{GL}_\infty(\mathcal{A}) := \lim_{N \rightarrow \infty} \text{GL}_N(\mathcal{A})$ , the odd Chern character is the following normalized periodic cyclic cycle:

$$\text{ch}_\bullet(g) = \sum_{k=0}^{\infty} (-1)^k k! \text{tr}_{2k+1} \left( (g^{-1} \otimes g)^{\otimes(k+1)} \right). \quad (1.19)$$

If  $(g_s)_{0 \leq s \leq 1}$  is a smooth path in  $\text{GL}_\infty(\mathcal{A})$ , the transgression formula (cf. [GET93B, Prop. 3.3]) reads

$$\frac{d}{ds} \text{ch}_\bullet(g_s) = (b + B) \phi_\bullet(g_s, \dot{g}_s), \quad (1.20)$$

where the secondary Chern character  $\phi_\bullet$  is defined by

$$\begin{aligned} \phi_\bullet(g, h) = \text{tr}_0(g^{-1}h) + \\ + \sum_{k=0}^{\infty} (-1)^{k+1} k! \sum_{j=0}^k \text{tr}_{2k+2} \left( (g^{-1} \otimes g)^{\otimes(j+1)} \otimes g^{-1}h \otimes (g^{-1} \otimes g)^{\otimes(k-j)} \right). \end{aligned} \quad (1.21)$$

A relative  $K$ -theory class in  $K_1(\mathcal{A}, \mathcal{B})$  can be represented by a triple  $(U, V, h)$ , where  $U, V \in \text{Mat}_N(\mathcal{A})$  are unitaries and  $h : [0, 1] \rightarrow \text{Mat}_N(\mathcal{B})$  is a path of unitaries joining  $\sigma(U)$  and  $\sigma(V)$ . Putting

$$\text{T}\phi_\bullet(h) = \int_0^1 \phi_\bullet(h_s, \dot{h}_s) ds, \quad (1.22)$$

the Chern character of  $(U, V, h)$  is represented by the relative cyclic cycle

$$\text{ch}_\bullet(U, V, h) = \left( \text{ch}_\bullet(V) - \text{ch}_\bullet(U), -\text{T}\phi_\bullet(h) \right). \quad (1.23)$$

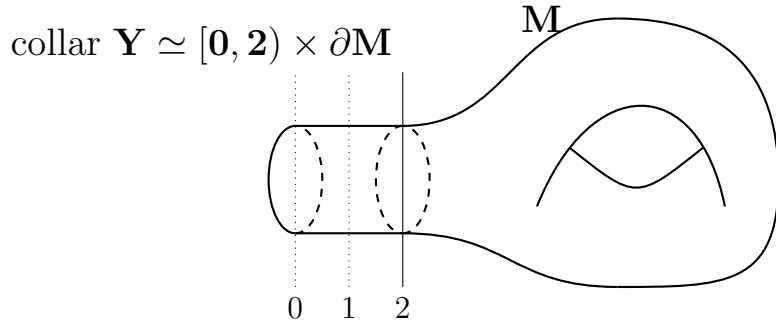


FIGURE 1. The compact manifold  $M$  with boundary. The picture of the collar does not capture the  $b$ -metric.

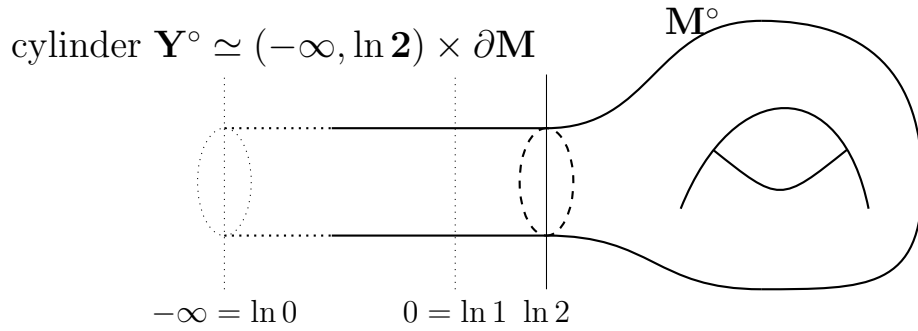


FIGURE 2. Interior of  $M$  with cylindrical coordinates on the collar  $(0, 2) \times \partial M \simeq (-\infty, \ln 2) \times \partial M$  via  $(r, \eta) \mapsto (\ln r, \eta)$ . This picture correctly captures the metric on the collar.

Again the cycle property follows from the transgression formula Eq. (1.20) and with the aid of a secondary transgression formula [LMP09, (1.15)] one shows that (1.23) corresponds to the standard Chern character on  $K_1(\mathcal{J})$  under excision [LMP09, Thm. 1.7].

**1.4. The  $b$ -trace.** From now on we assume that  $M$  is a compact manifold with boundary,  $r : Y \rightarrow [0, 2)$  a boundary defining function, and that  $g_b$  is an exact  $b$ -metric on  $M$  (see Figure 1).

These data are the main ingredients of the so-called  $b$ -calculus, which we will use in what follows. The primary source for the  $b$ -calculus is the monograph [MEL93] by Melrose, and for the reader's convenience we included some basic definitions in Appendix A.

We do recall here however the definition of the  $b$ -trace (cf. [MEL93]), since this notion plays an essential role in our work. It will often be convenient to choose cylindrical coordinates (see Figure 2)  $(x, \eta) : Y^\circ \rightarrow (-\infty, \ln 2) \times \partial M$  over a collar  $Y \subset M$  with a boundary defining function  $r : Y \rightarrow [0, 2)$  (see Appendix A.1 for details and for occasionally unexplained notation used below).

When using these coordinates, we view the interior  $M^\circ$  as a manifold with cylindrical ends, and have in this picture  $M^\circ \cong (-\infty, 0] \times \partial M \cup_{\partial M} \overline{M}^\circ$ . All explicit calculations

will be done in cylindrical coordinates. We also refer to the survey paper by P. Loya [LOY05] where the b-calculus is developed from this point of view.

Let  $E$  be a smooth Hermitian vector bundle over  $M$ . Whenever convenient, we will tacitly identify elements of  $\Gamma^\infty((-\infty, 0] \times \partial M; E)$ , the sections of  $E$  over  $(-\infty, 0] \times \partial M$ , with  $\Gamma^\infty(\partial M; E|_{\partial M})$ -valued smooth functions on  $(-\infty, 0]$ , *i.e.* elements of  $\mathcal{C}^\infty((-\infty, 0], \Gamma^\infty(\partial M; E|_{\partial M}))$ , in the obvious way; *cf.* Subsection 3.1 and Appendix A.3. Accordingly, we define for  $u \in \Gamma_{\text{cpt}}^\infty((-\infty, 0] \times \partial M; E)$  the Fourier transform in the cylinder variable,  $\hat{u}(\lambda) \in \Gamma^\infty(\partial M; E|_{\partial E})$ , by

$$\hat{u}(\lambda, p) := \int_{-\infty}^{\infty} e^{-ix\lambda} u(x, p) dx. \quad (1.24)$$

Now assume that  $A \in {}^b\Psi^{-\infty}(M; E)$  is a smoothing b-pseudodifferential operator. In general,  $A$  is not trace class in the usual sense. However, since  $A$  has a smooth Schwartz kernel it is locally trace class, in the sense that  $\psi A \varphi$  is trace class for any pair of smooth functions  $\psi, \varphi : M \rightarrow \mathbb{R}$  having compact support in  $M^\circ$ . Using the notation from Appendix A.3, let  $\tilde{A}$  be the operator induced on the cylinder  $(-\infty, 0] \times \partial M$ .

We define an operator valued symbol  $A_\partial(x, \lambda) : \Gamma^\infty(E|_{\partial M}) \rightarrow \Gamma^\infty(E|_{\partial M})$  as follows: for  $v \in \Gamma^\infty(E|_{\partial M})$  put

$$(A_\partial(x, \lambda)v)(p) := \left( e^{-ix\lambda} \tilde{A}(e^{i\cdot\lambda} \otimes v) \right)(x, p). \quad (1.25)$$

For  $v \in \Gamma^\infty(E|_{\partial M})$  we have  $(e^{i\cdot\lambda} \otimes v)^\wedge = 2\pi\delta_\lambda \otimes v$ , hence for  $u \in \Gamma_{\text{cpt}}^\infty((-\infty, 0] \times \partial M; E)$  one then obtains

$$\begin{aligned} (\tilde{A}u)(x, p) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\lambda} (A_\partial(x, \lambda)\hat{u}(\lambda, -))(p) d\lambda = \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x-\tilde{x})\lambda} (A_\partial(x, \lambda)u(\tilde{x}, -))(p) d\tilde{x} d\lambda. \end{aligned} \quad (1.26)$$

We note in passing that  $A_\partial(x, \lambda)$  can also be constructed from the global symbol  $\tilde{a}$  as  $A_\partial(x, \lambda) = \text{Op}(\tilde{a}(x, \lambda, -))$  (*cf.* Appendices A.2 and A.3). For the properties of  $\tilde{a}$  see [LOY05, Sec. 2.2] and Appendix A.3. In the small b-calculus  $\tilde{a}(x, \lambda, -)$  and hence  $A_\partial(x, \lambda)$  are entire in  $\lambda$  while in the full b-calculus they are meromorphic [MEL93]. In any case one has

$$A_\partial(x, \lambda) = \mathcal{I}(A)(\lambda) + O(e^x), \quad x \rightarrow -\infty, \quad (1.27)$$

with a family  $\mathcal{I}(A)(\lambda), \lambda \in \mathbb{R}$ , of classical pseudodifferential operators on  $\partial M$  in the parameter dependent calculus; *cf. e.g.* [LMP09, Sec. 2.1] for a brief summary of the parameter dependent calculus.

The operator valued function  $\lambda \in \mathbb{R} \mapsto \mathcal{I}(A)(\lambda)$  is called the *indicial family* of  $A$ . In terms of the global symbol  $\tilde{a}$ , one has  $\mathcal{I}(A)(\lambda) = \text{Op}(\tilde{a}_0(\lambda, -))$ , where  $\tilde{a}_0(\lambda, -)$  is the first term in the asymptotic expansion of  $\tilde{a}$  with respect to  $x \rightarrow -\infty$ ; *cf.* Appendix A.4.

Denote by  $k(x, \tilde{x})$ ,  $x, \tilde{x} \geq 0$ , the  $\mathcal{L}(L^2(\partial M; E|_{\partial M}))$ -valued kernel of  $\tilde{A}$ . In terms of  $A_{\partial}(x, \lambda)$  the kernel  $k(x, \tilde{x})$  is given by

$$k(x, \tilde{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(x-\tilde{x})\lambda} A_{\partial}(x, \lambda) d\lambda. \quad (1.28)$$

Hence, as  $R \rightarrow \infty$  one has in view of (1.27)

$$\begin{aligned} \mathrm{Tr}(A|_{\{x \geq -R\}}) &= \\ &= \mathrm{Tr}(A|_{M^1}) + \int_{-R}^0 \mathrm{Tr}_{\partial M}(k(x, x)) dx \\ &= \mathrm{Tr}(A|_{M^1}) + \frac{1}{2\pi} \int_{-\infty}^0 \int_{-\infty}^{\infty} \mathrm{Tr}_{\partial M}(A_{\partial}(x, \lambda) - \mathcal{I}(A)(\lambda)) d\lambda dx \\ &\quad + R \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Tr}_{\partial M}(\mathcal{I}(A)(\lambda)) d\lambda + O(e^{-R}), \quad R \rightarrow \infty. \end{aligned} \quad (1.29)$$

The *finite part* of this expansion is called the *b-trace* of  $A$ :

$${}^b\mathrm{Tr}(A) := \mathrm{Tr}(A|_{M^1}) + \frac{1}{2\pi} \int_{-\infty}^0 \int_{-\infty}^{\infty} \mathrm{Tr}_{\partial M}(A_{\partial}(x, \lambda) - \mathcal{I}(A)(\lambda)) d\lambda dx. \quad (1.30)$$

Hence

$$\begin{aligned} \mathrm{Tr}(A|_{\{x \geq -R\}}) &= \\ &= {}^b\mathrm{Tr}(A) + R \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Tr}_{\partial M}(\mathcal{I}(A)(\lambda)) d\lambda + O(e^{-R}), \quad R \rightarrow \infty. \end{aligned} \quad (1.31)$$

Its name notwithstanding, the b-trace is not a trace. One has, however, the following crucial formula.

**Proposition 1.1** ([MEL93, Prop. 5.9], [LOY05, Thm. 2.5]). *Assume that  $A \in {}^b\Psi_{\mathrm{cl}}^m(M; E)$  and  $K \in {}^b\Psi_{\mathrm{cl}}^{-\infty}(M; E)$ . Then*

$${}^b\mathrm{Tr}(AK - KA) = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \mathrm{Tr}_{\partial M} \left( \frac{d\mathcal{I}(A)(\lambda)}{d\lambda} \mathcal{I}(K)(\lambda) \right) d\lambda. \quad (1.32)$$

*Proof in a special case.* It is instructive to prove this in the special case that  $A$  is a Dirac operator  $\mathbf{D}$ . We will see in Subsection 1.7 below that on the cylinder  $\mathbf{D}$  takes the form  $\mathbf{D} = \Gamma \frac{d}{dx} + \mathbf{D}_{\partial}$  and that  $\mathcal{I}(\mathbf{D})(\lambda) = i\Gamma\lambda + \mathbf{D}_{\partial}$ .

After choosing cut-off functions w.l.o.g. we may assume that  $K$  is supported in the interior of the cylinder and given by

$$\begin{aligned} (Ku)(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\lambda} k(x, \lambda) \hat{u}(\lambda) d\lambda \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x-y)\lambda} k(x, \lambda) u(y) dy d\lambda \end{aligned} \quad (1.33)$$

with an operator valued symbol  $k(x, \lambda)$ . Then

$$\begin{aligned} (\mathbf{D}K u)(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\lambda} (i\lambda\Gamma + \mathbf{D}_{\partial}) k(x, \lambda) \hat{u}(\lambda) d\lambda \\ &\quad + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\lambda} \Gamma \partial_x k(x, \lambda) \hat{u}(\lambda) d\lambda. \end{aligned} \quad (1.34)$$

Furthermore, since  $(\mathbf{D}u)^\wedge = (i\Gamma\lambda + \mathbf{D}_{\partial})\hat{u}$  we have

$$(\mathbf{K}\mathbf{D})u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\lambda} k(x, \lambda) (i\lambda\Gamma + \mathbf{D}_{\partial}) \hat{u}(\lambda) d\lambda. \quad (1.35)$$

Consequently

$$\mathrm{Tr}_{\partial M}((\mathbf{D}K - \mathbf{K}\mathbf{D})(x, x)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Tr}_{\partial M}(\Gamma \partial_x k(x, \lambda)) d\lambda, \quad (1.36)$$

and hence, since by assumption  $K$  is supported in  $(-\infty, 0) \times \partial M$  and taking (1.27) into account, we find

$$\begin{aligned} \int_{-R}^0 \mathrm{Tr}_{\partial M}((\mathbf{D}K - \mathbf{K}\mathbf{D})(x, x)) dx &= \frac{-1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Tr}_{\partial M}(\Gamma k(-R, \lambda)) d\lambda \\ &\xrightarrow{R \rightarrow \infty} \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \mathrm{Tr}_{\partial M} \left( \frac{d\mathcal{I}(\mathbf{D})(\lambda)}{d\lambda} \mathcal{I}(K, \lambda) \right) d\lambda. \end{aligned} \quad (1.37) \quad \square$$

Eq. (1.31) immediately entails the following result.

**Corollary 1.2.** *Let  $A \in {}^b\Psi_{\mathrm{cl}}^m(M; E)$  be a classical b-pseudodifferential operator of order  $m < \dim M$ . If the indicial family  $\mathcal{I}(A)$  vanishes, then  $A$  is trace class, and*

$$\mathrm{Tr}(A) = {}^b\mathrm{Tr}(A).$$

**1.5. The relative McKean–Singer formula and the APS Index Theorem.** In the introduction to [MEL93], the author explained in detail his elegant approach to the A-P-S index theorem, based on the b-calculus. The relative cohomology point of view allows to make this approach even more appealing. Indeed, we will show that the APS index can be obtained as the pairing between a natural relative cyclic 0-cocycle, one of whose components is the b-trace, and a relative cyclic 0-cycle constructed out of the heat kernel. This pairing leads in fact to a relative version of the McKean–Singer formula.

We start, a bit more abstractly, by considering an exact sequence of algebras

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{A} \xrightarrow{\sigma} \mathcal{B} \longrightarrow 0; \quad (1.38)$$

$\mathcal{A}, \mathcal{B}$  are assumed to be unital,  $\sigma$  is assumed to be a unital homomorphism. Let  $\tau$  be a hypertrace on  $\mathcal{J}$ , i.e.  $\tau$  satisfies

$$\tau(xa) = \tau(ax) \text{ for } a \in \mathcal{A}, x \in \mathcal{J}. \quad (1.39)$$

Let  $\bar{\tau} : \mathcal{A} \longrightarrow \mathbb{C}$  be a linear extension (regularization) of  $\tau$  to  $\mathcal{A}$ , which is not assumed to be tracial. Nevertheless,  $\bar{\tau}$  induces a cyclic 1-cocycle on  $\mathcal{B}$  as follows:

$$\mu(\sigma(a_0), \sigma(a_1)) := \bar{\tau}([a_0, a_1]). \quad (1.40)$$

Because of Eq. (1.39),  $\mu$  does indeed depend only on  $\sigma(a_0), \sigma(a_1)$ . Moreover, the pair  $(\bar{\tau}, \mu)$  is a relative cyclic cocycle. Namely, in the notation of (1.5), Eq. (1.40) translates into  $\tilde{b}(\bar{\tau}, \mu) = 0$ .

The relevant example for this paper is the exact sequence

$$0 \longrightarrow {}^b\Psi_{\text{tr}}^{-\infty}(M; E) \longrightarrow {}^b\Psi^{-\infty}(M; E)^+ \xrightarrow{\mathcal{I}} \mathcal{S}(\mathbb{R}, \Psi^{-\infty}(\partial M; E))^+ \longrightarrow 0, \quad (1.41)$$

where  $M$  is a compact manifold with boundary equipped with an exact b-metric. The operator trace  $\text{Tr}$  on  ${}^b\Psi_{\text{tr}}^{-\infty}(M; E) = \text{Ker } \mathcal{I}$  satisfies (1.39), and the b-trace  ${}^b\text{Tr}$  provides its linear extension to  ${}^b\Psi^{-\infty}(M; E)^+$ .

Assume now that the algebra  $\mathcal{A}$  is represented as bounded operators on some Hilbert space  $\mathcal{H}$  and that  $\mathcal{J} \subset \mathcal{L}^1(\mathcal{H})$  consists of trace class operators. Let  $D$  be a self-adjoint unbounded operator affiliated with  $\mathcal{A}$ , *i.e.* bounded continuous functions of  $D$  belong to  $\mathcal{A}$ . Furthermore, we assume that we are in a graded (even) situation and denote the grading operator by  $\alpha$ . Finally we assume that  $D$  is a Fredholm operator and that the orthogonal projection  $P_{\text{Ker } D} \in \mathcal{J}$ .

We note that in the case of a Dirac operator  $D$  on the b-manifold  $M$  it is well-known that  $D$  is Fredholm if and only if the tangential operator  $D_{\partial}$  (see Subsection 1.7 and Eq. (3.1) below) is invertible.

Define

$$A_0(t) := \alpha D e^{-\frac{1}{2}tD^2}, \quad (1.42)$$

$$A_1(t) := \int_t^{\infty} D e^{(\frac{t}{2}-s)D^2} ds. \quad (1.43)$$

Since  $D$  is affiliated with  $\mathcal{A}$ ,  $A_j(t) \in \mathcal{A}$  for  $t > 0$ .

1.5.1. *Case 1:*  $e^{-tD^2} \in \mathcal{J}$ , for  $t > 0$ . Under this assumption  $A_j(t) \in \mathcal{J}$ , for  $t > 0$ . Moreover,  $e^{-tD^2} \in C_0^\lambda(\mathcal{J}) = C_0(\mathcal{J})/((1-\lambda)C_0(\mathcal{J}))$  defines naturally a class in  $H_0^\lambda(\mathcal{J})$ .

**Lemma 1.3.** *The class of  $\alpha e^{-tD^2}$  in  $H_0^\lambda(\mathcal{J})$  equals that of  $\alpha P_{\text{Ker } D}$ . In particular, it is independent of  $t$ .*

*Proof.* We calculate

$$\begin{aligned} b(A_0 \otimes A_1) &= 2\alpha \int_t^{\infty} D^2 e^{-sD^2} ds \\ &= 2\alpha(e^{-tD^2} - P_{\text{Ker } D}), \end{aligned} \quad (1.44)$$

proving that  $\alpha e^{-tD^2}$  and  $\alpha P_{\text{Ker } D}$  are cohomologous.  $\square$

As an immediate corollary one recovers the classical McKean–Singer formula. Indeed, since the trace  $\tau$  defines a class in  $H_\lambda^0(\mathcal{J})$  one finds

$$\text{Ind } D = \text{Tr}(\alpha P_{\text{Ker } D}) = \langle \tau, \alpha P_{\text{Ker } D} \rangle = \langle \tau, \alpha e^{-tD^2} \rangle = \text{Tr}(\alpha e^{-tD^2}). \quad (1.45)$$

1.5.2. *Case 2: The general case.* The heat operator  $e^{-tD^2}$  gives a class in  $H_0^\lambda(\mathcal{A})$ . The pairing of  $e^{-tD^2}$  with  $\bar{\tau}$  cannot be expected to be independent of  $t$  since  $\bar{\tau}$  is not a trace. It is however a component of the relative cyclic 0-cocycle  $(\bar{\tau}, \mu)$ . Therefore, we are led to construct a relative cyclic homology class from  $e^{-tD^2}$ . By Eq. (1.8) the relative cyclic complex is given by

$$C_n^\lambda(\mathcal{A}, \mathcal{B}) := C_n^\lambda(\mathcal{A}) \oplus C_{n+1}^\lambda(\mathcal{B}), \quad \tilde{b} := \begin{pmatrix} b & 0 \\ -\sigma_* & -b \end{pmatrix}. \quad (1.46)$$

From (1.44) we infer

$$\sigma(\alpha e^{-tD^2}) = \sigma(\alpha e^{-tD^2} - \alpha P_{\text{Ker } D}) = \frac{1}{2}b(\sigma(A_0) \otimes \sigma(A_1)), \quad (1.47)$$

hence

$$\tilde{b} \begin{pmatrix} \alpha e^{-tD^2} \\ -\frac{1}{2}\sigma(A_0) \otimes \sigma(A_1) \end{pmatrix} = 0, \quad (1.48)$$

*i.e.* the pair  $\text{EXP}_t(D) := (\alpha e^{-tD^2}, -\frac{1}{2}\sigma(A_0) \otimes \sigma(A_1))$  is a relative cyclic homology class. Furthermore, since

$$\tilde{b} \begin{pmatrix} \frac{1}{2}A_0 \otimes A_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha(e^{-tD^2} - P_{\text{Ker } D}) \\ -\frac{1}{2}\sigma(A_0) \otimes \sigma(A_1) \end{pmatrix}, \quad (1.49)$$

the class of  $\text{EXP}_t(D)$  in  $H_0^\lambda(\mathcal{A}, \mathcal{B})$  equals that of the pair  $(\alpha P_{\text{Ker } D}, 0)$  which, via excision, corresponds to the class of  $\alpha P_{\text{Ker } D} \in HP_0(\mathcal{J})$ . We have thus proved the following result.

**Lemma 1.4.** *The class of the pair  $(\alpha e^{-tD^2}, -\frac{1}{2}\sigma(A_0) \otimes \sigma(A_1))$  in  $HC_0^\lambda(\mathcal{A}, \mathcal{B})$  equals that of  $(\alpha P_{\text{Ker } D}, 0)$ . In particular, it is independent of  $t$ .*

Pairing with the relative 0-cocycle  $(\bar{\tau}, \mu)$  we now obtain the following relative version of the McKean–Singer formula:

$$\begin{aligned} \text{Ind } D &= \text{Tr}(\alpha P_{\text{Ker } D}) = \langle (\bar{\tau}, \mu), (\alpha e^{-tD^2}, -\frac{1}{2}\sigma(A_0) \otimes \sigma(A_1)) \rangle \\ &= \bar{\tau}(\alpha e^{-tD^2}) - \frac{1}{2}\mu(\sigma(A_0), \sigma(A_1)) \\ &= \bar{\tau}(\alpha e^{-tD^2}) - \frac{1}{2}\bar{\tau}([A_0, A_1]). \end{aligned} \quad (1.50)$$

Once known, this identity can also be derived quite directly. Indeed, since  $[A_0, A_1] = 2\alpha \int_t^\infty D^2 e^{-sD^2} ds$ ,

$$\begin{aligned} \bar{\tau}(\alpha e^{-tD^2}) - \frac{1}{2}\bar{\tau}([A_0, A_1]) &= \bar{\tau}(\alpha e^{-tD^2}) - \bar{\tau}\left(\alpha \int_t^\infty D^2 e^{-sD^2} ds\right) \\ &= \bar{\tau}(\alpha e^{-tD^2}) + \int_t^\infty \frac{d}{ds} \bar{\tau}(\alpha e^{-sD^2}) ds \\ &= \lim_{s \rightarrow \infty} \bar{\tau}(\alpha e^{-sD^2}). \end{aligned} \quad (1.51)$$

Let us show that in the case of the Dirac operator on a b-manifold the second summand is nothing but the  $\eta$ -invariant of the tangential operator. Indeed, in a collar of

the boundary  $D$  takes the form

$$D = \begin{pmatrix} 0 & -\frac{d}{dx} + A \\ \frac{d}{dx} + A & 0 \end{pmatrix} =: \Gamma \frac{d}{dx} + D_\partial, \quad D_\partial = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}.$$

Hence, one calculates using Proposition 1.1

$$\begin{aligned} & \frac{1}{2} \mu(\mathcal{I}(A_0, \lambda), \mathcal{I}(A_1, \lambda)) \\ &= \frac{-1}{4\pi i} \int_{-\infty}^{\infty} \text{Tr}_{\partial M} \left( \frac{d\mathcal{I}(A_0, \lambda)}{d\lambda} \mathcal{I}(A_1, \lambda) \right) d\lambda \\ &= \frac{-1}{4\pi} \int_{-\infty}^{\infty} \text{Tr}_{\partial M} \left( \alpha \Gamma \int_t^{\infty} (i\lambda \Gamma + D_\partial) e^{-s(D_\partial^2 + \lambda^2)} ds \right) d\lambda \\ &= \frac{-1}{4\sqrt{\pi}} \int_t^{\infty} \frac{1}{\sqrt{s}} \text{Tr}_{\partial M} \left( \begin{pmatrix} -A & 0 \\ 0 & -A \end{pmatrix} e^{-sA^2} \right) ds \\ &= \frac{1}{2\sqrt{\pi}} \int_t^{\infty} \frac{1}{\sqrt{s}} \text{Tr}_{\partial M} (A e^{-sA^2}) ds =: \frac{1}{2} \eta_t(A). \end{aligned} \tag{1.52}$$

Thus if  $D$  is Fredholm we have for each  $t > 0$

$$\text{Ind } D = {}^b\text{Tr}(\alpha e^{-tD^2}) - \frac{1}{2} \eta_t(A), \tag{1.53}$$

and taking the limit as  $t \searrow 0$  gives the APS index theorem in the  $b$ -setting.

**1.6. A formula for the  $b$ -trace.** In this subsection we give an explicit formula for the  $b$ -trace, based on an observation of Loya [LOY05], which provides a convenient tool for subsequent computations.

We first briefly review the Hadamard partie finie integral in the special case of  $b$ -functions. Let  $f \in {}^b\mathcal{C}^\infty((-\infty, 0])$ . From the asymptotic expansion (see Eq. (A.4))

$$f(x) \sim_{x \rightarrow -\infty} f_0^- + f_1^- e^x + f_2^- e^{2x} + \dots \tag{1.54}$$

we infer

$$\int_{-R}^0 f(x) dx = f_0^- R + c + O(e^{-R}), \quad R \rightarrow -\infty. \tag{1.55}$$

The *partie finie integral* of  $f$  is then defined to be the constant term in the asymptotic expansion (1.55), i.e.

$$\int_{-R}^0 f(x) dx =: f_0^- R + \text{Pf} \int_{-\infty}^0 f(x) dx + O(e^{-R}), \quad R \rightarrow -\infty. \tag{1.56}$$

The definition of the partie finie integral has an obvious extension to  $b$ -functions on manifolds with cylindrical ends. Because of its importance, we single it out as a definition-proposition.

**Definition and Proposition 1.5.** *Let  $M^\circ$  be a Riemannian manifold with cylindrical ends and  $M$  the (up to diffeomorphism) unique compact manifold with boundary having  $M^\circ$  as its interior. For a function  $f \in {}^b\mathcal{C}^\infty(M^\circ)$  one has*

$$\int_{x \geq -R} f d \text{vol} =: c \log R + \int_{{}^b M} f d \text{vol} + O(e^{-R}) \quad \text{as } R \rightarrow \infty.$$

This means that  $\int_{\text{b}M} f d \text{vol}$  is the finite part in the asymptotic expansion of  $\int_{x \geq -R} f d \text{vol}$  as  $R \rightarrow \infty$ . More generally, if  $\omega \in {}^b\Omega^m(M)$  is a (top degree) b-differential form, i.e. a form whose coefficients are in  ${}^b\mathcal{C}^\infty(M^\circ)$ , then  $\int_{\text{b}M} \omega$  is defined accordingly as the finite part of  $\int_{x \geq -R} \omega$  as  $R \rightarrow \infty$ .

In local coordinates  $y_1, \dots, y_n$  on  $\partial M$ , b-differential forms are sums of terms of the form

$$\omega = f(x, y) dx \wedge dy_{j_1} \wedge \dots \wedge dy_{j_{p-1}} + g(x, y) dy_{l_1} \wedge \dots \wedge dy_{j_p} \quad (1.57)$$

with b-smooth functions  $f, g$ . Putting  $\iota^* \omega := g_0(y) dy_{l_1} \wedge \dots \wedge dy_{j_p}$  extends to a pullback  $\iota^* : {}^b\Omega^p(M) \rightarrow \Omega^p(\partial M)$ . It is easy to see that Stokes' Theorem holds for  $\int_{\text{b}M}$  and  $\iota$ :

$$\int_{\text{b}M} d\omega = \int_{\partial M} \iota^* \omega. \quad (1.58)$$

For a b-pseudodifferential operator  $A \in {}^b\Psi_{\text{cl}}^\bullet(M; E)$  of order  $< -\dim M$  the b-trace is nothing but the partie finie integral of its kernel over the diagonal:

$${}^b\text{Tr}(A) = \int_{\text{b}M} \text{tr}_p(K_A(p, p)) d \text{vol}(p), \quad (1.59)$$

where now  $K_A(\cdot, \cdot)$  denotes the Schwartz kernel of  $A$  and  $\text{tr}_p$  denotes the fiber trace on  $E_p$ .

Next we mention a useful formula for the partie finie integral in terms of a convergent integral. By Eq. (A.5), the asymptotic expansion (1.54) may be differentiated, hence  $\partial_x f = O(e^x)$ ,  $x \rightarrow -\infty$ , is integrable and thus integration by parts yields

$$\begin{aligned} \int_{-R}^0 f(x) dx &= Rf(-R) - \int_{-R}^0 x \partial_x f(x) dx \\ &= Rf_0^- - \int_{-\infty}^0 x \partial_x f(x) dx + O(Re^{-R}), \quad R \rightarrow -\infty. \end{aligned} \quad (1.60)$$

Hence

$$\text{Pf} \int_{-\infty}^0 f(x) dx = \int_{-\infty}^0 x \partial_x f(x) dx, \quad (1.61)$$

where the integrand on the right hand side is summable in the Lebesgue sense. Using the tools from the previous paragraphs, we can now prove the following theorem about the representation of the b-trace as a trace of certain trace class operators.

**Proposition 1.6.** *Let  $M$  be a compact manifold with boundary and an exact b-metric  $g_b$ . Fix a collar  $(r, \eta) : Y \rightarrow [0, 2) \times \partial M$  of the boundary  $\partial M$  as described in Appendix A.1, and let  $(x, \eta) : \overline{Y^1} \rightarrow (-\infty, 0] \times \partial M$  denote the corresponding diffeomorphism onto the cylinder  $(-\infty, 0] \times \partial M$ . Assume that  $A \in {}^b\Psi_{\text{cl}}^\infty(M; E)$  is a classical*

*b*-pseudodifferential operator of order  $< -\dim M$ , and that its kernel is supported within the cylinder  $(-\infty, 0) \times \partial M$ . Then  $x[\frac{d}{dx}, A]$  is trace class and one has

$$\begin{aligned} {}^b\mathrm{Tr}(A) &= -\mathrm{Tr}\left(x\left[\frac{d}{dx}, A\right]\right) = \\ &= -\int_{(-\infty, 0) \times \partial M} x \frac{d}{dx} \mathrm{tr}_{x,q}(K_A(x, q; x, q)) d\mathrm{vol}(x, q), \end{aligned} \tag{1.62}$$

where  $K_A$  denotes the Schwartz kernel of  $A$ .

*Proof.* The condition on the support of  $A$  is necessary since the operators  $x$  and  $\frac{d}{dx}$  are only defined on the cylinder. However, Proposition 1.6 can be extended to arbitrary  $A \in {}^b\Psi_{\mathrm{cl}}^{<-\dim M}(M; E)$  in a straightforward way: choose a pair of cut-off functions  $\varphi, \psi \in \mathcal{C}^\infty(M)$  with support within the cylinder  $(-\infty, 0) \times \partial M$  and such that  $\varphi(x) = 1$  for  $x \leq -2$ ,  $\varphi(x) = 0$ , for  $x \geq -3/2$ ,  $\psi(x) = 1$  for  $x \leq -1$  and  $\psi(x) = 0$  for  $x \geq -1/2$ . Finally, choose a cut-off function  $\chi \in \mathcal{C}_0^\infty(M \setminus \partial M)$  with compact support and  $\chi(1 - \varphi) = 1 - \varphi$ . The definition of the *b*-trace then immediately shows that

$${}^b\mathrm{Tr}(\varphi A) = {}^b\mathrm{Tr}(\psi \varphi A) = {}^b\mathrm{Tr}(\varphi A \psi)$$

and hence

$$\begin{aligned} {}^b\mathrm{Tr}(A) &= {}^b\mathrm{Tr}(\varphi A \psi) + \mathrm{Tr}((1 - \varphi)A\chi) = \\ &= -\mathrm{Tr}\left(x\left[\frac{d}{dx}, \varphi A \psi\right]\right) + \mathrm{Tr}((1 - \varphi)A\chi). \end{aligned}$$

The fact that  $x[\frac{d}{dx}, A]$  is trace class follows by Prop. 1.2, since the indicial family of the commutator  $[\frac{d}{dx}, A]$  vanishes. We provide two variants of proof for (1.62).

*1st Variant.* From equations (1.59) and (1.61) we infer

$$\begin{aligned} {}^b\mathrm{Tr}(A) &= \int_{{}^b(-\infty, 0) \times \partial M} \mathrm{tr}_{x,q}(K_A(x, q; x, q)) d\mathrm{vol}(x, q) \\ &= -\int_{(-\infty, 0) \times \partial M} x \frac{d}{dx} \mathrm{tr}_{x,q}(K_A(x, q; x, q)) d\mathrm{vol}(x, q). \end{aligned} \tag{1.63}$$

This proves the second line of (1.62). The first line follows, since the kernel of  $[\frac{d}{dx}, A]$  is given by  $[\frac{d}{dx}, K_A](x, p; y, q) = \partial_x K_A(x, p; y, q) + \partial_y K_A(x, p; y, q)$  which for  $x = y$  equals  $\frac{d}{dx} K_A(x, p; x, p)$ , cf. Eq. (1.61).

*2nd Variant.* For  $\Re z > 0$  the operator  $e^{zx}A$  is trace class and the function

$$z \mapsto \mathrm{Tr}(e^{zx}A) \tag{1.64}$$

is holomorphic for  $\Re z > 0$  and it extends meromorphically to  $\Re z > -1$ , 0 is a simple pole and the residue at 0 equals  ${}^b\text{Tr}(A)$  (cf. [LOY05]). Hence

$$\begin{aligned} {}^b\text{Tr}(A) &= \frac{d}{dz} z \text{Tr}(e^{zx} A) \Big|_{z=0} \\ &= \frac{d}{dz} \text{Tr} \left( \left[ \frac{d}{dx}, e^{zx} \right] A \right) \Big|_{z=0} \\ &= \frac{d}{dz} \text{Tr} \left( \left[ \frac{d}{dx}, e^{zx} A \right] - e^{zx} \left[ \frac{d}{dx}, A \right] \right) \Big|_{z=0} \\ &= - \text{Tr} \left( x \left[ \frac{d}{dx}, A \right] \right), \end{aligned} \tag{1.65}$$

since for  $\Re z > 0$  the trace of the commutator  $\text{Tr}(\left[\frac{d}{dx}, e^{zx} A\right])$ , thanks to the decay of  $e^{zx}$ , does vanish. The last claim follows as above.  $\square$

**1.7. b-Clifford modules and b-Dirac operators.** To treat both the even and the odd cases simultaneously we make use of the Clifford supertrace (cf. e.g. [GET93A, Appendix]). Denote by  $\text{Cl}_q$  the complex Clifford algebra on  $q$  generators, that is  $\text{Cl}_q$  is the universal  $C^*$ -algebra on unitary generators  $e_1, \dots, e_q$  subject to the relations

$$e_j e_k + e_k e_j = -2\delta_{jk}. \tag{1.66}$$

Let  $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$  be a  $\mathbb{Z}_2$ -graded Hilbert space with grading operator  $\alpha$ . We assume additionally that  $\mathcal{H}$  is a  $\mathbb{Z}_2$ -graded right  $\text{Cl}_q$ -module. Denote by  $\text{c}^r : \mathcal{H} \otimes \text{Cl}_q \rightarrow \mathcal{H}$  the right  $\text{Cl}_q$ -action and define operators  $E_j : \mathcal{H} \rightarrow \mathcal{H}$  for  $j = 1, \dots, q$  by  $E_j := \text{c}^r(-\otimes e_j)$ . Then the  $E_j$  are unitary operators on  $\mathcal{H}$  which anti-commute with  $\alpha$ .

The  $C^*$ -algebra  $\mathcal{L}(\mathcal{H})$  of bounded linear operators on  $\mathcal{H}$  is naturally  $\mathbb{Z}_2$ -graded, too. For operators  $A, B \in \mathcal{L}(\mathcal{H})$  of pure degree  $|A|, |B|$  the *supercommutator* is defined by

$$[A, B]_{\mathbb{Z}_2} := AB - (-1)^{|A||B|} BA. \tag{1.67}$$

Furthermore denote by  $\mathcal{L}_{\text{Cl}_q}(\mathcal{H})$  the *supercommutant* of  $\text{Cl}_q$  in  $\mathcal{H}$ , that is  $\mathcal{L}_{\text{Cl}_q}(\mathcal{H})$  consists of those  $A \in \mathcal{L}(\mathcal{H})$  for which  $[A, E_j]_{\mathbb{Z}_2} = 0$ ,  $j = 1, \dots, q$ . For  $K \in \mathcal{L}_{\text{Cl}_q}^1(\mathcal{H}) = \{A \in \mathcal{L}_{\text{Cl}_q}(\mathcal{H}) \mid A \text{ trace class}\}$  one puts

$$\text{Str}_q(K) := (4\pi)^{-q/2} \text{Tr}(\alpha E_1 \cdot \dots \cdot E_q K). \tag{1.68}$$

The following properties of  $\text{Str}_q$  are straightforward to verify.

**Lemma 1.7.** *For  $K, K_1, K_2 \in \mathcal{L}_{\text{Cl}_q}^1(\mathcal{H})$ , one has*

- (1)  $\text{Str}_q K = 0$ , if  $|K| + q$  is odd.
- (2)  $\text{Str}_q$  vanishes on super-commutators:  $\text{Str}_q([K_1, K_2]_{\mathbb{Z}_2}) = 0$ .

Again, let  $M$  be a compact manifold with boundary,  $r : Y \rightarrow [0, 2)$  a boundary defining function, and  $g_b$  an exact b-metric on  $M$ . Associated to these data is the bundle  ${}^b\text{Cl}(M) := \text{Cl}({}^bT^*M)$  of Clifford algebras. Its fiber over  $p \in M$  is given by the Clifford algebra generated by elements of  ${}^bT_p^*M$  subject to the relations

$$\xi \cdot \zeta + \zeta \cdot \xi = -2g_b(\xi, \zeta) \quad \text{for all } \xi, \zeta \in {}^bT_p^*M. \tag{1.69}$$

**Definition 1.8** (cf. [GET93A, Sec. 5]). Let  $q$  be a natural number. By a *degree  $q$  b-Clifford module* over  $M$  one then understands a  $\mathbb{Z}_2$ -graded complex vector bundle  $W \rightarrow M$  together with a Hermitian metric  $\langle -, - \rangle$ , a Clifford action  $\mathfrak{c} = \mathfrak{c}^1 : {}^bT^*M \otimes W \rightarrow W$ , and an action  $\mathfrak{c}^r : W \otimes \text{Cl}_q \rightarrow W$  such that both actions are graded and unitary and supercommute with each other. A *b-Clifford superconnection* on a degree  $q$  b-Clifford module  $W$  over  $M$  is a b-superconnection

$${}^b\mathbb{A} : {}^b\Omega^\bullet(M, W) := \Gamma^\infty(M, \Lambda^\bullet({}^bT^*M) \otimes W) \rightarrow {}^b\Omega^\bullet(M, W)$$

which supercommutes with the action of  $\text{Cl}_q$ , satisfies

$$[{}^b\mathbb{A}, \mathfrak{c}(\omega)]_{\mathbb{Z}_2} = \mathfrak{c}({}^b\nabla\omega) \quad \text{for all } \omega \in {}^b\Omega^1(M),$$

and is metric in the sense that

$$\langle {}^b\mathbb{A}\xi, \zeta \rangle + \langle \xi, {}^b\mathbb{A}\zeta \rangle = d \langle \xi, \zeta \rangle \quad \text{for all } \xi, \zeta \in {}^b\Omega^\bullet(M, W).$$

Here, and in what follows,  ${}^b\nabla$  denotes the Levi-Civita b-connection belonging to  $g_b$ . In the remainder of this article, we assume that a b-Clifford superconnection is always of product form near the boundary. This means that over  $Y^s$  for some  $s$  with  $0 < s < 2$  the superconnection has the form

$${}^b\mathbb{A}|_{Y^s} = \eta^* \nabla^\partial + \eta^* \omega^\partial \wedge -,$$

where  $\eta : Y \rightarrow \partial M$  is the boundary projection from Appendix A,  $\nabla^\partial$  is a metric connection on the restricted bundle  $W|_{\partial M}$  and  $\omega^\partial \in \Omega^\bullet(\partial M; \text{End}(W|_{\partial M}))$ . Recall that the pull-back covariant derivative  $(\eta^* \nabla^\partial)$  on  $W|_Y$  is uniquely defined by requiring for  $\xi \in \Gamma^\infty(Y, W)$  that

$$(\eta^* \nabla^\partial)_V \xi = \begin{cases} r \frac{\partial \xi}{\partial r}, & \text{if } V = r \frac{\partial}{\partial r}, \\ \nabla_{\tilde{V}} \xi, & \text{if } V = \tilde{V} \circ \eta \text{ for some } \tilde{V} \in \Gamma^\infty(\partial M, T\partial M). \end{cases}$$

The *Dirac operator* associated to a degree  $q$  b-Clifford module  $W$  and a b-Clifford superconnection  ${}^b\mathbb{A}$  is defined as the b-differential operator

$$D := \mathfrak{c}^1 \circ {}^b\mathbb{A} : \Gamma^\infty(M, W) \rightarrow \Gamma^\infty(M, \Lambda^\bullet({}^bT^*M) \otimes W) \rightarrow \Gamma^\infty(M, W).$$

In this paper the term “Dirac operator” will always refer to the Dirac operator associated to a Clifford (super)connection in the above sense. Such Dirac operators are automatically formally self-adjoint. By a *Dirac-type operator* we understand a first order differential operator such that the principal symbol of its square is scalar (cf. [TAY96]).

Note that the b-metric on  $M$  and the metric structure on  $W$  give rise to the Hilbert space  $\mathcal{H} = L^2(M; W)$  of square integrable sections of the b-Clifford module. By assumption,  $\text{Cl}_q$  acts on  $L^2(M; W)$ , hence by Eq. (1.68) one obtains a supertrace  $\text{Str}_q : \mathcal{L}_{\text{Cl}_q}^1(L^2(M; W)) \rightarrow \mathbb{C}$ . Similarly the b-trace gives rise to a b-supertrace

$$\begin{aligned} {}^b\text{Str}_q &: {}^b\Psi_{\text{cl}, \text{Cl}_q}^{< -\dim M} (M; W) \longrightarrow \mathbb{C}, \\ {}^b\text{Str}_q(K) &:= (4\pi)^{-q/2} \text{Tr}(\alpha E_1 \cdot \dots \cdot E_q K). \end{aligned} \tag{1.70}$$

Here  ${}^b\Psi_{\text{cl}, \text{Cl}_q}^\bullet(M; W)$  denote the classical b-pseudodifferential operators which lie in the supercommutant of  $\text{Cl}_q$  in  $\mathcal{H}$  (cf. (1.67) *supra*).

Next consider the natural embedding  $T^*\partial M \hookrightarrow {}^bT_{|\partial M}^*M$ . By the universal property of Clifford algebras one obtains an embedding of Clifford bundles  $\text{Cl}(\partial M) \hookrightarrow {}^b\text{Cl}({}^bT_{|\partial M}^*M)$ . Moreover, the decomposition  ${}^bT_{|\partial M}^*M = T^*M \oplus \mathbb{R} \cdot r \frac{\partial}{\partial r}$  induced by  $g_b$  even gives rise to a splitting  ${}^b\text{Cl}({}^bT_{|\partial M}^*M) \rightarrow \text{Cl}(\partial M)$ . Let now  $W \rightarrow M$  be a degree  $q$  b-Clifford module over  $M$ . Then  $\text{Cl}(\partial M)$  acts on  $W_{|\partial M}$  via the embedding  $\text{Cl}(\partial M) \hookrightarrow {}^b\text{Cl}({}^bT_{|\partial M}^*M)$ . We denote the resulting left action of the boundary Clifford bundle on  $W_{|\partial M}$  again by  $\mathbf{c}$ . Moreover, the action  $W_{|\partial M} \otimes \text{Cl}_q \rightarrow W_{|\partial M}$  extends to a right action  $\mathbf{c}_\partial^r : W_{|\partial M} \otimes \text{Cl}_{q+1} \rightarrow W_{|\partial M}$  by putting

$$E_j = \mathbf{c}_\partial^r(w, e_j) := \begin{cases} \mathbf{c}^r(w, e_j), & \text{for } w \in W_p, p \in \partial M, j = 1, \dots, q, \\ -\mathbf{c}^l\left(\frac{dr}{r}, w\right), & \text{for } w \in W_p, p \in \partial M, j = q + 1, \end{cases} \quad (1.71)$$

cf. the beginning of this Subsection 1.7. It is now easy to check that  $W_{|\partial M}$  together with  $\mathbf{c}$  and  $\mathbf{c}_\partial^r$  as Clifford actions becomes a degree  $q + 1$  Clifford-module over  $\partial M$ .

Now we have the ingredients for the b-supertrace of a supercommutator:

**Proposition 1.9** ([GET93A, Cor. 5.5]). *Let  $D$  be a Dirac operator on a  $q$ -graded b-Clifford bundle  $W$ , and  $K \in {}^b\Psi_{\text{cl}, \text{Cl}_q}^{-\infty}(M; W)$ . On  $\partial M$  put  $E_{q+1} := -\mathbf{c}^l\left(\frac{dr}{r}\right) = -\mathbf{c}^l(dx)$ . Then*

$${}^b\text{Str}_q([D, K]_{\mathbb{Z}_2}) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \text{Str}_{q+1, \partial M}(\mathcal{I}(K, \lambda)) d\lambda. \quad (1.72)$$

We do not claim here that  $\mathcal{I}(K, \lambda)$  commutes with  $\Gamma$  in the graded sense. This is not necessary for the definition of  $\text{Str}_{q+1, \partial M}$ .

**Remark 1.10.** Another consequence of the previous considerations which we single out for future reference is the structure of a Dirac operator on a cylinder  $\mathbb{R} \times \partial M$  (cf. (A.3)). Since all structures are product,  $D$  takes the form

$$D = \mathbf{c}(dx) \frac{d}{dx} + D_\partial =: \Gamma \left( \frac{d}{dx} + A \right). \quad (1.73)$$

Here  $\Gamma = \mathbf{c}(dx)$  is Clifford multiplication by the normal vector  $\frac{d}{dx}$  and  $D_\partial := \Gamma A$  is the *tangential operator*.  $D_\partial$  is a Dirac operator on the boundary. Moreover, one has the relations

$$\Gamma^* = -\Gamma, \quad \Gamma^2 = -I, \quad A^t = A, \quad \Gamma A + A\Gamma = \Gamma D_\partial + D_\partial \Gamma = 0. \quad (1.74)$$

Now let  $u$  be a section of the Clifford bundle  $W$  over the cylinder  $\mathbb{R} \times M$ . Then

$$\begin{aligned} (Du)(x, p) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \mathbf{c}(dx) \frac{d}{dx} + D_\partial \right) e^{ix\lambda} \hat{u}(\lambda, p) d\lambda = \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\lambda} (i\mathbf{c}(dx)\lambda + D_\partial) \hat{u}(\lambda, p) d\lambda. \end{aligned} \quad (1.75)$$

By Eqs. (1.26) and (1.27) this proves the following:

**Proposition 1.11.** *Let  $M$  be a compact manifold with boundary and let  $g_b$  be an exact b-metric on  $M$ . Furthermore, let  $D$  be a Dirac operator on  $M$ . Then the indicial family of  $D$  is given by  $\mathcal{I}(D)(\lambda) = i\lambda c(dx) + D_\partial$ .*

**1.8. The relative Connes-Chern character of a Dirac operator over a manifold with boundary.** In this section,  $M$  is a compact manifold with boundary,  $g_0$  is a Riemannian metric which is smooth up to the boundary, and  $W \rightarrow M$  is a degree  $q$  Clifford module. We choose a Hermitian metric  $h$  on  $W$  together with a Clifford connection which is unitary with respect to  $h$ . Let  $D = D(\nabla, g_0)$  be the associated Dirac operator; we suppress the dependence on  $h$  from the notation. Then  $D$  is a densely defined operator on the Hilbert space  $\mathcal{H}$  of square-integrable sections of  $W$ .

According to [BDT89, Prop. 3.1], as outlined in the introduction,  $D$  defines a relative Fredholm module over the pair of  $C^*$ -algebras  $(\mathcal{C}(M), \mathcal{C}(\partial M))$ . Recall that the relative Fredholm module is given by  $F = D_e(D_e^*D_e + 1)^{-1/2}$ , where  $D_e$  is a closed extension of  $D$  such that either  $D_e^*D_e$  or  $D_eD_e^*$  has compact resolvent (*e.g.* both  $D_{\min}$  and  $D_{\max}$  satisfy this condition), and that the  $K$ -homology class  $[F]$  does not depend on the particular choice of  $D_e$  (see [BDT89, Prop. 3.1]). Furthermore, [BDT89, §2] shows that over the  $C^*$ -algebra  $\mathcal{C}_0(M \setminus \partial M)$ , whose  $K$ -homology is by excision isomorphic to  $K^\bullet(\mathcal{C}_0(M \setminus \partial M))$ , one has even more freedom to choose closed extensions of  $D$ , and in particular the self-adjoint extension  $D_{\text{APS}}$  obtained by imposing APS boundary conditions yields the same  $K$ -homology class as  $[F]$  in  $K^\bullet(\mathcal{C}_0(M \setminus \partial M))$ .

It is well-known that  $D_{\text{APS}}$  has an  $m^+$ -summable resolvent (*cf.* *e.g.* [GRSE95]). Moreover, multiplication by  $f \in \mathcal{J}^\infty(\partial M, M)$  preserves the domain of  $D_{\text{APS}}$  and  $[D_{\text{APS}}, f] = c(df)$  is bounded. Thus  $D_{\text{APS}}$  defines naturally an  $m^+$ -summable Fredholm module over the local  $C^*$ -algebra  $\mathcal{J}^\infty(\partial M, M) \subset \mathcal{C}_0(M \setminus \partial M)$ . Since by excision in  $K$ -homology  $K^\bullet(\mathcal{C}_0(M \setminus \partial M))$  is naturally isomorphic to  $K^\bullet(\mathcal{C}(M), \mathcal{C}(\partial M))$ , one concludes that the class  $[F]$  of the relative Fredholm module coincides under this isomorphism with the class  $[D]$  of the  $m^+$ -summable Fredholm module over  $\mathcal{J}^\infty(\partial M, M)$ .

Let us now consider the Connes–Chern character of  $[D]$ . According to [CoMo93], it can be represented by the truncated JLO-cocycle of the operator  $D$  (with  $n \geq m$  of the same parity as  $m$ ):

$$\text{ch}_t^n(D) = \sum_{k \geq 0} \text{Ch}^{n-2k}(tD) + BT \phi_t^{n+1}(D). \quad (1.76)$$

Recall from [JLO88] that the JLO-cocycle is given by

$$\begin{aligned} \text{Ch}^k(D)(a_0, \dots, a_k) &= \int_{\Delta_k} \text{Str}_q(a_0 e^{-\sigma_0 D^2} [D, a_1] \dots [D, a_k] e^{-\sigma_k D^2}) d\sigma, \\ &\text{for } a_0, \dots, a_k \in \mathcal{J}^\infty(\partial M, M). \end{aligned} \quad (1.77)$$

Note that the cyclic cohomology class of  $\text{ch}_t^n(D)$  is independent of  $t$ , and that  $\text{ch}_t^\bullet$  is the Chern–Connes character as given in Diagram (1.2). To obtain the precise form of the Chern–Connes character  $\text{ch}_t^\bullet(D) \in HP^\bullet(\mathcal{J}^\infty(\partial M, M))$  one notes first that  $HP^\bullet(\mathcal{J}^\infty(\partial M, M)) \cong H_\bullet^{\text{dR}}(M, \partial M; \mathbb{C})$  by [BRPF08] and then has to calculate the limit  $\lim_{t \searrow 0} \text{ch}_t^n(D)(a_0, \dots, a_k)$ . Since  $\text{Ch}^k$  is continuous with respect to the Fréchet topology on  $\mathcal{J}^\infty(\partial M, M)$ , and  $\mathcal{C}_0^\infty(M \setminus \partial M)$  is dense in  $\mathcal{J}^\infty(\partial M, M)$ , it suffices to consider the

case where all  $a_j$  have compact support in  $M \setminus \partial M$ . But in that case one can use standard local heat kernel analysis or Getzler's asymptotic calculus as in [CoMo90] or [BLFo90] to show that for  $n \geq m$  and same parity as  $m$

$$\lim_{t \searrow 0} [\text{ch}_t^n]_k(\mathbb{D})(a_0, \dots, a_k) = \int_M \omega_{\mathbb{D}} \wedge a_0 da_1 \wedge \dots \wedge da_k. \quad (1.78)$$

Here,  $[\text{ch}_t^n]_k$  denotes the component of  $\text{ch}_t^n$  of degree  $k$  and  $\omega_{\mathbb{D}}$  is the local index form of  $\mathbb{D}$ . By Poincaré duality the class of the current (1.78) in  $H_{\bullet}^{\text{dR}}(M, \partial M)$  depends only on the absolute de Rham cohomology class of  $\omega_{\mathbb{D}}$  in  $H_{\text{dR}}^{\bullet}(M)$ . By the transgression formulae this cohomology class is independent of  $\nabla$  and  $g$ . Summing up and using Diagram (1.2), we obtain the following statement.

**Proposition 1.12.** *Let  $M$  be a compact manifold with boundary and Riemannian metric  $g_0$ , and let  $W$  be a degree  $q$  Clifford module over  $M$ . For any choice of a Hermitian metric  $h$  and unitary Clifford connection  $\nabla$  on  $W$  the Dirac operator  $\mathbb{D} = \mathbb{D}(\nabla, g_0)$  defines naturally a class  $[\mathbb{D}] \in K_m(M \setminus \partial M)$ . The Connes-Chern character of  $[\mathbb{D}]$  is independent of the choice of  $\nabla$  and  $g_0$ . In particular the index map  $\text{Index}_{[\mathbb{D}]} : K^{\bullet}(M, \partial M) \rightarrow \mathbb{Z}$ , defined by the pairing with  $K$ -theory, is independent of  $\nabla$  and  $g_0$ .*

## 2. THE b-ANALOGUE OF THE ENTIRE CHERN CHARACTER

The degree  $q$  Clifford module approach outlined in Subsection 1.7 has advantages when dealing with manifolds with boundary, because the formulae for the JLO cocycle and its transgression (*cf.* (2.6), (2.7) below) become simpler. To make the connection to the standard even and odd Chern character without Clifford action, from now on we will also consider *ungraded* Clifford modules without auxiliary Clifford right action. Therefore we assume that either

- we are in the graded case with  $q$  Clifford matrices  $E_1, \dots, E_q$  where  $\mathbb{D}$  is odd, and  $\text{Str}_q$  denotes the Clifford trace defined in Subsection 1.7,

or

- we are in the ungraded case, when there are no Clifford matrices and no grading operator; this case can be conveniently dealt with by putting  $q = -1$  (which is odd!),  $\alpha = 1$  and  $\text{Str}_q := \text{Tr} = \text{Tr}(\alpha \cdot)$ .

From now on, we assume that  $\mathbb{D}_t$ ,  $t \in (0, \infty)$ , is a family of self-adjoint differential operators of the form  $\mathbb{D}_t = f(t)\mathbb{D}$  with  $\mathbb{D}$  the Dirac operator of a  $q$ -graded b-Clifford module ( $q \geq -1$  according to the previous explanation) with b-Clifford superconnection  $(W, {}^b\mathbb{A})$  and  $f : (0, \infty) \rightarrow \mathbb{R}$  a smooth function.  $\mathbb{D}_t$  are Dirac type operators in the sense of [TAY96].

Following Getzler [GET93A, Sec. 6], we define for  $A_0, \dots, A_k \in {}^b\Psi_{\text{cl}}^{\infty}(M, W)$

$$\begin{aligned} {}^b\langle A_0, \dots, A_k \rangle_{\mathbb{D}_t} &:= \int_{\Delta_k} {}^b\text{Str}_q (A_0 e^{-\sigma_0 \mathbb{D}_t^2} \dots A_k e^{-\sigma_k \mathbb{D}_t^2}) d\sigma \\ &= {}^b\text{Str}_q ((A_0, \dots, A_k)_{\mathbb{D}_t}), \end{aligned} \quad (2.1)$$

where

$$\Delta_k := \{ \sigma = (\sigma_0, \dots, \sigma_k) \in \mathbb{R}^{k+1} \mid \sigma_j \geq 0, \sigma_0 + \dots + \sigma_k = 1 \} \quad (2.2)$$

denotes the standard  $k$ -simplex and

$$(A_0, \dots, A_k)_D := \int_{\Delta_k} A_0 e^{-\sigma_0 D^2} \dots A_k e^{-\sigma_k D^2} d\sigma. \quad (2.3)$$

Put

$${}^b\text{Ch}^k(D)(a_0, \dots, a_k) := {}^b\langle a_0, [D, a_1], \dots, [D, a_k] \rangle, \quad (2.4)$$

$$\begin{aligned} {}^b\phi^k(D, V)(a_0, \dots, a_k) := \\ \sum_{0 \leq j \leq k} (-1)^{j \deg V} {}^b\langle a_0, [D, a_1], \dots, [D, a_j], V, [D, a_{j+1}], \dots, [D, a_k] \rangle. \end{aligned} \quad (2.5)$$

The operation  ${}^b\phi$  will mostly be used with  $V = \dot{D}_t$  as a second argument. Here  $D_t$  is considered of odd degree regardless of the value of  $q$ .

**Remark 2.1.** For  $q = 0$ ,  $k$  even resp.  $q = -1$ ,  $k$  odd  ${}^b\text{Ch}^k(D)$ ,  ${}^b\phi^k(D, \dot{D})$  are the  $b$ -analogues of the even and odd JLO Chern character and its transgression.

The following result is crucial for this paper. It is essentially due to Getzler [GET93A, Thm. 6.2], although the following version is not stated explicitly in his paper.

**Theorem 2.2.** *For  $q \geq 0$  we have the following two equations for  ${}^b\text{Ch}^\bullet(D)$  and  ${}^b\phi(D, \dot{D})$ :*

$$b {}^b\text{Ch}^{k-1}(D_t) + B {}^b\text{Ch}^{k+1}(D_t) = \text{Ch}^k(D_{\partial,t}) \circ i^*, \quad (2.6)$$

$$\frac{d}{dt} {}^b\text{Ch}^k(D_t) + b {}^b\phi^{k-1}(D_t, \dot{D}_t) + B {}^b\phi^{k+1}(D_t, \dot{D}_t) = -\phi^k(D_{\partial,t}, \dot{D}_{\partial,t}) \circ i^*. \quad (2.7)$$

These formulæ will be repeatedly used in Section 6 and thereafter. For notational convenience we will omit the symbol  $\circ i^*$  whenever the context makes clear that this composition is required.

The theorem can be derived from [GET93A, Thm. 6.2] by introducing the form valued expression

$${}^b\langle\langle A_0, \dots, A_k \rangle\rangle := \int_{\Delta_k} {}^b\text{Str}_q (A_0 e^{-\sigma_0(i dD_t + D_t^2)} \dots A_k e^{-\sigma_k(i dD_t + D_t^2)}) d\sigma, \quad (2.8)$$

and the combined Chern character  $\text{Ch}_G^\bullet$ , defined as

$$\begin{aligned} \text{Ch}_G^k(D_t)(a_0, \dots, a_k) \\ := {}^b\langle\langle a_0, [D_t, a_1], \dots, [D_t, a_k] \rangle\rangle, \quad a_0, \dots, a_k \in C^\infty(M). \end{aligned} \quad (2.9)$$

For this, Getzler proves

$$(-i d + b + B) \text{Ch}_G^\bullet(D_t) = \text{Ch}_G^\bullet(D_{\partial,t}) \circ i^*. \quad (2.10)$$

**Remark 2.3.** Note that in this paper we use self-adjoint Dirac operators while Getzler uses skew-adjoint ones in [GET93A]. Accordingly, our Dirac operators differ by a factor  $-i$  from the Dirac operators in [GET93A]. This explains the appearance of such  $i$ -factors in our formulæ, which are not present in [GET93A].

By carefully tracing all the signs and  $i$ -factors involved in the graded form valued Clifford calculus, as well as due to the various conventions, it turns out that separating (2.10) into its scalar and 1-form parts, using [GET93A, Lem. 2.5]

$$\begin{aligned} {}^b\langle\langle A_0, \dots, A_k \rangle\rangle &= \\ &= {}^b\langle A_0, \dots, A_k \rangle - i \sum_{j=0}^k {}^b\langle A_0, \dots, A_j, dt \wedge \dot{D}_t, A_{j+1}, \dots, A_k \rangle, \end{aligned} \quad (2.11)$$

one obtains Eqs. (2.6) and (2.7).

However, for completeness, we will give a more direct argument in Subsection 2.2, without using operator valued forms. The proof below follows the lines of the standard proof for the JLO cocycle representing the Chern character of a  $\theta$ -summable Fredholm module (cf. [JLO88], [GESz89]).

**2.1. Cocycle and transgression formulæ for the even/odd b-Chern character (without Clifford covariance).** Recall from Remark 2.1 that for  $q = 0$  and  $k$  even resp.  $q = -1$  and  $k$  odd  ${}^b\text{Ch}^\bullet(D)$  is the b-analogue of the even, resp. odd, JLO Chern character. We shall relate the ungraded ( $q = -1$ ) case to the graded case with  $q = 1$ .

Starting with an ungraded Dirac operator  $D_t$  acting on the Hilbert space  $\mathcal{H}$ , put

$$\tilde{\mathcal{H}} := \mathcal{H} \oplus \mathcal{H}, \quad \alpha := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{D}_t := \begin{pmatrix} 0 & D_t \\ D_t & 0 \end{pmatrix}. \quad (2.12)$$

Then  $\tilde{D}_t$  is odd with respect to the grading operator  $\alpha$  and it anti-commutes with

$$E_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2.13)$$

Note that

$$\tilde{D}_t = \alpha E_1 (D_t \otimes I_2) \quad (2.14)$$

with  $I_2$  being the  $2 \times 2$  identity matrix.

**Proposition 2.4.** *Let  $D_t$  be ungraded ( $q = -1$ ) and let  $\tilde{D}_t = \alpha E_1 (D_t \otimes I_2)$  be the associated 1-graded ( $q = 1$ ) operator. Then for  $k$  odd*

$$\begin{aligned} {}^b\text{Ch}^k(\tilde{D}_t) &= \frac{1}{\sqrt{\pi}} {}^b\text{Ch}^k(D_t), \\ {}^b\phi^h{}^{k-1}(\tilde{D}, \dot{\tilde{D}}_t) &= \frac{1}{\sqrt{\pi}} {}^b\phi^h{}^{k-1}(D_t, \dot{D}_t). \end{aligned} \quad (2.15)$$

Needless to say that these formulæ are valid as well for  $\text{Ch}^\bullet$  and  $\phi^h{}^\bullet$ .

*Proof.* Using Proposition 1.9 we find for  $k$  odd:

$$\begin{aligned}
& {}^b\langle a_0, [\tilde{D}_t, a_1], \dots, [\tilde{D}_t, a_k] \rangle_{\tilde{D}_t} \\
&= {}^b\langle (\alpha E_1)^k a_0, [D_t \otimes I_2, a_1], \dots, [D_t \otimes I_2, a_k] \rangle_{D_t \otimes I_2} \\
&= \int_{\Delta_k} \frac{1}{\sqrt{4\pi}} {}^b\text{Tr} \left( \underbrace{(\alpha E_1)^{k+1}}_{=1} a_0 e^{-\sigma_0 D_t^2} [D_t, a_1] \dots [D_t, a_k] e^{-\sigma_k D_t^2} \otimes I_2 \right) \\
&= \frac{1}{\sqrt{\pi}} {}^b\langle a_0, [D_t, a_1], \dots, [D_t, a_k] \rangle_{D_t}.
\end{aligned} \tag{2.16}$$

The calculation for  ${}^b\text{Ch}^{k-1}(\tilde{D}_t, \dot{\tilde{D}}_t)$  is completely analogous.  $\square$

Now we are ready to translate (2.6) and (2.7) into formulæ for the standard even and odd Chern character without Clifford action.

2.1.1.  $q = 0$ . A priori we are in the standard even situation without Clifford right action. However,  $D_\partial$  is viewed as  $\text{Cl}_1$  covariant with respect to the Clifford action given by  $E_1 = -\Gamma$ . On the boundary,  $\Gamma$  gives a natural identification of the even and odd half spinor bundle and with respect to the splitting into half spinor bundles  $D$  takes the form:

$$D = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\Gamma} \frac{d}{dx} + \underbrace{\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}}_{D_\partial}; \tag{2.17}$$

$A$  is an ungraded Dirac type operator acting on the positive half spinor bundle (it is the operator whose positive spectral projection gives the APS boundary condition). In the notation of Eq. (2.12), we have  $D_\partial = \tilde{A}$ ,  $E_1 = -\Gamma$ . Thus, Proposition 2.4 and Theorem 2.2 give the following result.

**Proposition 2.5.** *Let  $M$  be an even dimensional compact manifold with boundary with an exact b-metric and let  $D_t = f(t)D$  be as before. Writing  $D$  in a collar of the boundary (in cylindrical coordinates) in the form*

$$D =: \begin{pmatrix} 0 & -\frac{d}{dx} + A \\ \frac{d}{dx} + A & 0 \end{pmatrix}, \tag{2.18}$$

*$A$  is an ungraded Dirac type operator acting on the positive half spinor bundle restricted to the boundary. Furthermore we have with  $A_t = f(t)A$*

$${}^b\text{Ch}^{k-1}(D_t) + B {}^b\text{Ch}^{k+1}(D_t) = \frac{1}{\sqrt{\pi}} \text{Ch}^k(A_t) \circ i^*, \tag{2.19}$$

$$\begin{aligned}
& \frac{d}{dt} {}^b\text{Ch}^k(D_t) + {}^b\text{Ch}^{k-1}(D_t, \dot{D}_t) + B {}^b\text{Ch}^{k+1}(D_t, \dot{D}_t) \\
&= -\frac{1}{\sqrt{\pi}} \text{Ch}^k(A_t, \dot{A}_t) \circ i^*.
\end{aligned} \tag{2.20}$$

2.1.2.  $q = -1$ . Now let  $D$  be ungraded and put  $\widetilde{D}, \alpha, E_1$  as in Eqs. (2.12), (2.13), (2.14). Then by Proposition 2.4 we have

$${}^b\text{Ch}^k(D_t)(a_0, \dots, a_k) = \sqrt{\pi} {}^b\text{Ch}^k(\widetilde{D}_t)(a_0, \dots, a_k), \quad (2.21)$$

$${}^b\text{Ch}^k(D_t, \dot{D}_t)(a_0, \dots, a_k) = \sqrt{\pi} {}^b\text{Ch}^k(\widetilde{D}_t, \dot{\widetilde{D}}_t)(a_0, \dots, a_k). \quad (2.22)$$

In the collar of the boundary, we write as usual  $D = \Gamma \frac{d}{dx} + D_\partial$ , and thus

$$\widetilde{D} = \underbrace{\begin{pmatrix} 0 & \Gamma \\ \Gamma & 0 \end{pmatrix}}_{=: \widetilde{\Gamma}} \frac{d}{dx} + \underbrace{\begin{pmatrix} 0 & D_\partial \\ D_\partial & 0 \end{pmatrix}}_{=: \widetilde{D}_\partial}. \quad (2.23)$$

$\widetilde{D}_\partial$  is 2-graded with respect to

$$E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_2 = -\widetilde{\Gamma} = \begin{pmatrix} 0 & -\Gamma \\ -\Gamma & 0 \end{pmatrix}. \quad (2.24)$$

Note that

$$\alpha E_1 E_2 = -\Gamma \otimes I_2, \quad \widetilde{D}_\partial = \alpha E_1 (D_\partial \otimes I_2). \quad (2.25)$$

For even  $k$  we have

$$\begin{aligned} & \text{Str}_2(a_0 e^{-\sigma_0 \widetilde{D}_{\partial,t}^2} [\widetilde{D}_{\partial,t}, a_1] \cdots [\widetilde{D}_{\partial,t}, a_k] e^{-\sigma_k \widetilde{D}_{\partial,t}^2}) \\ &= \frac{1}{4\pi} \text{Tr}(\alpha E_1 E_2 (\alpha E_1)^k (a_0 e^{-\sigma_0 D_{\partial,t}^2} [D_{\partial,t}, a_1] \cdots [D_{\partial,t}, a_k] e^{-\sigma_k D_{\partial,t}^2}) \otimes I_2) \\ &= -\frac{1}{2\pi} \text{Tr}(\Gamma a_0 e^{-\sigma_0 D_{\partial,t}^2} [D_{\partial,t}, a_1] \cdots [D_{\partial,t}, a_k] e^{-\sigma_k D_{\partial,t}^2}). \end{aligned} \quad (2.26)$$

With respect to the grading given by  $-i\Gamma$  we can now write

$$\frac{-1}{2\pi} \text{Tr}(\Gamma \cdot) = \frac{1}{2\pi i} \text{Str}_0. \quad (2.27)$$

Together with (2.21) and (2.22) we have thus proved:

**Proposition 2.6.** *Let  $M$  be an odd dimensional compact manifold with boundary with an exact b-metric and let  $D$  be an ungraded Dirac operator. Writing  $D$  in a collar of the boundary (in cylindrical coordinates) in the form*

$$D =: \Gamma \frac{d}{dx} + D_\partial, \quad (2.28)$$

$D_\partial$  is a graded Dirac type operator with respect to the grading operator  $-i\Gamma$ . Furthermore, we have

$${}^b\text{Ch}^{k-1}(D_t) + B {}^b\text{Ch}^{k+1}(D_t) = \frac{1}{2\sqrt{\pi}i} \text{Ch}^k(D_\partial) \circ i^*, \quad (2.29)$$

$$\begin{aligned} & \frac{d}{dt} {}^b\text{Ch}^k(D_t) + {}^b\text{Ch}^{k-1}(D_t, \dot{D}_t) + B {}^b\text{Ch}^{k+1}(D_t, \dot{D}_t) \\ &= -\frac{1}{2\sqrt{\pi}i} \text{Ch}^k(D_t^\partial, \dot{D}_t^\partial) \circ i^*. \end{aligned} \quad (2.30)$$

**2.2. Sketch of Proof of Theorem 2.2.** Recall that Theorem 2.2 is stated for  $q \geq 0$ , hence in this subsection all Dirac operators will be  $q$ -graded with  $q \geq 0$ .

**Proposition 2.7.** *Let  $A_0, \dots, A_k \in {}^b\Psi_{\text{cl}, \text{Cl}_q}^\bullet(M; W)$ . Assume that for all but one index  $j_0$  the indicial family is independent of  $\lambda$  and commutes with the actions of  $E_1, \dots, E_q$  and  $E_{q+1} = -\Gamma$ . For the possible exception  $j_0$  we assume that  $A_{j_0}$  is proportional to  $\dot{D}_t$ . Then*

$${}^b\langle A_0, \dots, A_k \rangle = (-1)^\varepsilon {}^b\langle A_k, A_0, \dots, A_{k-1} \rangle, \quad (2.31)$$

where  $\varepsilon = |A_k|(|A_0| + \dots + |A_{k-1}|)$ .

$$\begin{aligned} {}^b\langle A_0, \dots, A_k \rangle &= \sum_{j=0}^k {}^b\langle A_0, \dots, A_j, 1, A_{j+1}, \dots, A_k \rangle \\ &= \sum_{j=0}^k (-1)^{\varepsilon_j} {}^b\langle 1, A_j, \dots, A_k, A_0, \dots, A_{j-1} \rangle, \end{aligned} \quad (2.32)$$

where  $\varepsilon_j = (|A_0| + \dots + |A_{j-1}|)(|A_j| + \dots + |A_k|)$ .

For  $j < k$

$$\begin{aligned} &{}^b\langle A_0, \dots, A_{j-1}, [\mathbf{D}^2, A_j], A_{j+1}, \dots, A_k \rangle \\ &= {}^b\langle A_0, \dots, A_{j-2}, A_{j-1}A_j, A_{j+1}, \dots, A_k \rangle \\ &\quad - {}^b\langle A_0, \dots, A_{j-1}, A_jA_{j+1}, A_{j+2}, \dots, A_k \rangle. \end{aligned} \quad (2.33)$$

Similarly, for  $j = k$

$$\begin{aligned} &{}^b\langle A_0, \dots, A_{k-1}, [\mathbf{D}^2, A_k] \rangle \\ &= {}^b\langle A_0, \dots, A_{k-2}, A_{k-1}A_k \rangle \\ &\quad - (-1)^{|A_k|(|A_0| + \dots + |A_{k-1}|)} {}^b\langle A_k A_0, \dots, A_{k-1} \rangle. \end{aligned} \quad (2.34)$$

Note that these formulæ are the same as in Getzler-Szenes [GESz89, Lemma 2.2]. In particular there is no boundary term. The proof proceeds exactly as the proofs of [GET93A, Lemma 6.3] (1),(2), (4), and we omit the details. We only note that one has to make heavy use of the following lemma in order to show the vanishing of certain terms:

**Lemma 2.8** (Berezin Lemma). *Let  $K \in \mathcal{L}_{\text{Cl}_q}^1(\mathcal{H})$ . Then for  $j < q$*

$$\text{Tr}(\alpha E_1 \cdots E_j K) = 0.$$

*Proof.* If  $j + q$  is odd then moving  $\alpha$  past  $E_1 \cdots E_j K$  and using the trace property gives

$$\begin{aligned} \text{Tr}(\alpha E_1 \cdots E_j K) &= -\text{Tr}(E_1 \cdots E_j K \alpha) \\ &= -\text{Tr}(\alpha E_1 \cdots E_j K) = 0. \end{aligned} \quad (2.35)$$

If  $j + q$  is even then, since  $j < q$ ,  $E_q$  anti commutes with  $\alpha E_1 \cdots E_j K$  and hence similarly

$$\begin{aligned} \mathrm{Tr}(\alpha E_1 \cdots E_j K) &= -\mathrm{Tr}(E_q^2 \alpha E_1 \cdots E_j K) = \mathrm{Tr}(E_q \alpha E_1 \cdots E_j K E_q) \\ &= \mathrm{Tr}(E_q^2 \alpha E_1 \cdots E_j K) = -\mathrm{Tr}(\alpha E_1 \cdots E_j K) = 0. \quad \square \end{aligned}$$

We will make repeated use of the equations (2.31)–(2.34). Now we can proceed as for a  $\theta$ -summable Fredholm module. Following [GBVF01, p. 451] we start with the supercommutator

$$\int_{\Delta_k} {}^b\mathrm{Str}_q([\mathbf{D}_t, a_0 e^{-\sigma_0 \mathbf{D}_t^2} [\mathbf{D}_t, a_1] \cdots [\mathbf{D}_t, a_k] e^{-\sigma_k \mathbf{D}_t^2}]) d\sigma. \quad (2.36)$$

As in [GET93A, bottom of p. 37] one shows, using Proposition 1.9 and the fact that  $\int e^{-\lambda^2} d\lambda = \sqrt{\pi}$ , that this supercommutator equals

$$\langle a_{0,\partial}, [\mathbf{D}_t^\partial, a_{1,\partial}], \dots, [\mathbf{D}_t^\partial, a_{k,\partial}] \rangle_{\mathbf{D}_t^\partial}. \quad (2.37)$$

It is important to note that here we are in the case  $q + 1$ , where the grading is the induced grading on the boundary and  $E_{q+1} = -\Gamma$ .

For convenience we will write  $\mathbf{D}$  instead of  $\mathbf{D}_t$ . Expanding the supercommutator (2.36) on the other hand gives

$$\begin{aligned} & {}^b\langle [\mathbf{D}, a_0], \dots, [\mathbf{D}, a_k] \rangle \\ & + \sum_{j=1}^k (-1)^{j-1} {}^b\langle a_0, [\mathbf{D}, a_1] \cdots, [\mathbf{D}, a_{j-1}], [\mathbf{D}^2, a_j], \dots, [\mathbf{D}, a_k] \rangle, \end{aligned} \quad (2.38)$$

where we have used  $[\mathbf{D}^2, a_j] = [\mathbf{D}, [\mathbf{D}, a_j]]_{\mathbb{Z}^2}$ .

We can now calculate the effect of  $b$  and  $B$  on  ${}^b\mathrm{Ch}$ .

$$\begin{aligned} & B {}^b\mathrm{Ch}^{k+1}(\mathbf{D})(a_0, \dots, a_k) \\ & = \sum_{j=0}^k (-1)^{kj} {}^b\langle 1, [\mathbf{D}, a_j], \dots, [\mathbf{D}, a_k], [\mathbf{D}, a_0], \dots, [\mathbf{D}, a_{j-1}] \rangle \\ & = \sum_{j=0}^k {}^b\langle [\mathbf{D}, a_0], \dots, [\mathbf{D}, a_{j-1}], 1, [\mathbf{D}, a_j], \dots, [\mathbf{D}, a_k] \rangle \\ & = {}^b\langle [\mathbf{D}, a_0], \dots, [\mathbf{D}, a_k] \rangle, \end{aligned} \quad (2.39)$$

where we used (2.32) twice. Thus, the first summand in (2.38) equals  $B {}^b\mathrm{Ch}^{k+1}(\mathbf{D})(a_0, \dots, a_k)$ .

Furthermore,

$$\begin{aligned}
& b \text{ }^b\text{Ch}^{k-1}(\mathbb{D})(a_0, \dots, a_k) \\
&= {}^b\langle a_0 a_1, [\mathbb{D}, a_2], \dots, [\mathbb{D}, a_k] \rangle \\
&\quad + \sum_{j=1}^{k-1} (-1)^j {}^b\langle a_0, \dots, [\mathbb{D}, a_j a_{j+1}], \dots, [\mathbb{D}, a_k] \rangle \\
&\quad + (-1)^k {}^b\langle a_k a_0, [\mathbb{D}, a_1], \dots, [\mathbb{D}, a_{k-1}] \rangle \\
&= {}^b\langle a_0 a_1, [\mathbb{D}, a_2], \dots, [\mathbb{D}, a_k] \rangle \\
&\quad - {}^b\langle a_0, a_1 [\mathbb{D}, a_2], \dots, [\mathbb{D}, a_k] \rangle \\
&\quad + \sum_{j=1}^{k-2} (-1)^j \left( {}^b\langle a_0, [\mathbb{D}, a_1], \dots, [\mathbb{D}, a_j] a_{j+1}, \dots, [\mathbb{D}, a_k] \rangle \right. \\
&\quad \quad \left. - {}^b\langle a_0, \dots, [\mathbb{D}, a_j], a_{j+1} [\mathbb{D}, a_{j+2}], \dots, [\mathbb{D}, a_k] \rangle \right) \\
&\quad + (-1)^{k-1} {}^b\langle a_0, [\mathbb{D}, a_1], \dots, [\mathbb{D}, a_{k-1}] a_k \rangle \\
&\quad + (-1)^k {}^b\langle a_k a_0, [\mathbb{D}, a_1], \dots, [\mathbb{D}, a_{k-1}] \rangle \\
&= \sum_{j=1}^k (-1)^{j-1} {}^b\langle a_0, [\mathbb{D}, a_1], \dots, [\mathbb{D}^2, a_j], \dots, [\mathbb{D}, a_k] \rangle,
\end{aligned} \tag{2.40}$$

where we have used (2.33) and (2.34). The right hand side of (2.40) equals the sum in the second line of (2.38).

Summing up (2.36), (2.37), (2.38), (2.39), and (2.40) we arrive at Eq. (2.6).

For additional clarity, let us perform two direct checks, for small values of  $k$ .

*Case 1:*  $k = 0$ . In this case (2.38) equals

$${}^b\langle [\mathbb{D}, a_0] \rangle = {}^b\langle 1, [\mathbb{D}, a_0] \rangle = B \text{ }^b\text{Ch}^1(\mathbb{D})(a_0), \tag{2.41}$$

by (2.32) and we are done in this case.

*Case 2:*  $k = 1$ . Then (2.38) equals

$${}^b\langle [\mathbb{D}, a_0], [\mathbb{D}, a_1] \rangle + {}^b\langle a_0, [\mathbb{D}^2, a_1] \rangle. \tag{2.42}$$

The first summand is  $B \text{ }^b\text{Ch}^2(a_0, a_1)$  and the second summand equals in view of (2.34)

$${}^b\langle a_0 a_1 \rangle - {}^b\langle a_1 a_0 \rangle = b \text{ }^b\text{Ch}^0(\mathbb{D})(a_0, a_1). \tag{2.43}$$

**2.3. The transgression formula.** To prove the transgression formula we proceed analogously and start with the supercommutator

$$\sum_{j=0}^k (-1)^j \int_{\Delta_{k+1}} {}^b\text{Str}_q \left( [D_t, a_0 e^{-\sigma_0 D_t^2} [D_t, a_1] \dots \dots [D_t, a_j] e^{-\sigma_j D_t^2} \dot{D} e^{-\sigma_{j+1} D_t^2} \dots [D_t, a_k] e^{-\sigma_{k+1} D_t^2}] \right) d\sigma. \quad (2.44)$$

We compute this supercommutator using Proposition 1.9. Note that  $\mathcal{I}(\dot{D}_t, \lambda)$  is proportional to  $i\Gamma\lambda + D^\partial$ . The summand  $i\Gamma\lambda$  contributes a term proportional to  $\int_{-\infty}^{\infty} \lambda e^{-\lambda^2} d\lambda = 0$ . The remaining summand gives, since  $\int e^{-\lambda^2} d\lambda = \sqrt{\pi}$ ,

$$\begin{aligned} & \sum_{j=0}^k (-1)^j \langle a_{0,\partial}, [D_t^\partial, a_{1,\partial}], \dots, [D_t^\partial, a_{j,\partial}], \dot{D}_t^\partial, \dots, [D_t^\partial, a_{k,\partial}] \rangle \\ & = \not\phi^k(D_\partial, \dot{D}_\partial)(a_{0,\partial}, \dots, a_{k,\partial}). \end{aligned} \quad (2.45)$$

Let us again emphasize that here we are in the case  $q + 1$ , where the grading is the induced grading on the boundary and  $E_{q+1} = -\Gamma$ .

Next we expand the commutator (2.44). However, we will confine ourselves to small  $k$ . The calculation is basically the same as in [GBVF01, p. 451]. The only difference is that on a closed manifold (2.44) is a priori 0 while here it coincides with the transgressed Chern character on the boundary.

*Case 1:  $k = 0$ .* (2.44) expands to

$${}^b\langle [D_t, a_0], \dot{D}_t \rangle + {}^b\langle a_0, [D_t, \dot{D}_t] \rangle. \quad (2.46)$$

On the other hand

$$\begin{aligned} B^b \not\phi^1(D_t, \dot{D}_t)(a_0) &= {}^b\not\phi^1(D_t, \dot{D}_t)(1, a_0) \\ &= {}^b\langle 1, \dot{D}_t, [D_t, a_0] \rangle - {}^b\langle 1, [D_t, a_0], \dot{D}_t \rangle \\ &= -{}^b\langle [D_t, a_0], 1, \dot{D}_t \rangle - {}^b\langle [D_t, a_0], \dot{D}_t, 1 \rangle \\ &= -{}^b\langle [D_t, a_0], \dot{D}_t \rangle, \end{aligned} \quad (2.47)$$

by (2.32). Moreover using the well-known formula

$$\frac{d}{dt} e^{-\sigma D_t^2} = - \int_0^\sigma e^{(\sigma-s) D_t^2} [D_t, \dot{D}_t] e^{-s D_t^2} ds, \quad (2.48)$$

we have

$$\frac{d}{dt} {}^b\text{Ch}^0(D_t)(a_0) = -{}^b\langle a_0, [D_t, \dot{D}_t] \rangle, \quad (2.49)$$

hence altogether

$$\frac{d}{dt} {}^b\text{Ch}^0(D_t) + B^b \not\phi^1(D_t, \dot{D}_t) = -\not\phi^0(D_t^\partial, \dot{D}_t^\partial). \quad (2.50)$$

*Case 2:*  $k = 1$ . To be on the safe side, we also look at an example in the odd case. We will again make repeated use of the formulæ in Proposition 2.7 without further mention. Eq. (2.44) now expands to

$$\begin{aligned} & {}^b\langle [D_t, a_0], \dot{D}_t, [D_t, a_1] \rangle - {}^b\langle [D_t, a_0], [D_t, a_1], \dot{D}_t \rangle \\ & + {}^b\langle a_0, [D_t, \dot{D}_t], [D_t, a_1] \rangle + {}^b\langle a_0, [D_t, a_1], [D_t, \dot{D}_t] \rangle \\ & - {}^b\langle a_0, \dot{D}_t, [D_t^2, a_1] \rangle - {}^b\langle a_0, [D_t^2, a_1], \dot{D}_t \rangle. \end{aligned} \quad (2.51)$$

On the other hand

$$\begin{aligned} B^b\phi^2(D_t, \dot{D}_t)(a_0) &= {}^b\phi^2(D_t, \dot{D}_t)(1, a_0, a_1) - {}^b\phi^2(D_t, \dot{D}_t)(1, a_1, a_0) \\ &= {}^b\langle 1, \dot{D}_t, [D_t, a_0], [D_t, a_1] \rangle - {}^b\langle 1, [D_t, a_0], \dot{D}_t, [D_t, a_1] \rangle \\ & \quad + {}^b\langle 1, [D_t, a_0], [D_t, a_1], \dot{D}_t \rangle - {}^b\langle 1, \dot{D}_t, [D_t, a_1], [D_t, a_0] \rangle \\ & \quad + {}^b\langle 1, [D_t, a_1], \dot{D}_t, [D_t, a_0] \rangle - {}^b\langle 1, [D_t, a_1], [D_t, a_0], \dot{D}_t \rangle \\ &= {}^b\langle [D_t, a_0], [D_t, a_1], 1, \dot{D}_t \rangle + {}^b\langle [D_t, a_0], 1, [D_t, a_1], \dot{D}_t \rangle \\ & \quad + {}^b\langle [D_t, a_0], [D_t, a_1], \dot{D}_t, 1 \rangle - {}^b\langle [D_t, a_0], \dot{D}_t, [D_t, a_1], 1 \rangle \\ & \quad - {}^b\langle [D_t, a_0], \dot{D}_t, 1, [D_t, a_1] \rangle - {}^b\langle [D_t, a_0], 1, \dot{D}_t, [D_t, a_1] \rangle \\ &= {}^b\langle [D_t, a_0], [D_t, a_1], \dot{D}_t \rangle - {}^b\langle [D_t, a_0], \dot{D}_t, [D_t, a_1] \rangle, \end{aligned} \quad (2.52)$$

which equals the negative of the first two summands of (2.51).

Furthermore,

$$b^b\phi^0(D_t, \dot{D}_t)(a_0, a_1) = {}^b\phi^0(D_t, \dot{D}_t)([a_0, a_1]) = {}^b\langle [a_0, a_1], \dot{D}_t \rangle. \quad (2.53)$$

Applying (2.33) and (2.34) to the last two summands of (2.51) we find

$$\begin{aligned} & {}^b\langle a_0, \dot{D}_t, [D_t^2, a_1] \rangle + {}^b\langle a_0, [D_t^2, a_1], \dot{D}_t \rangle \\ &= {}^b\langle a_0, \dot{D}_t a_1 \rangle - {}^b\langle a_1 a_0, \dot{D}_t \rangle + {}^b\langle a_0 a_1, \dot{D}_t \rangle - {}^b\langle a_0, a_1 \dot{D}_t \rangle \\ &= {}^b\langle a_0, [\dot{D}_t, a_1] \rangle + b^b\phi^0(D_t, \dot{D}_t)(a_0, a_1), \end{aligned} \quad (2.54)$$

hence adding  $B^b\phi^2(D_t, \dot{D}_t)(a_0, a_1)$  and  $b^b\phi^0(D_t, \dot{D}_t)(a_0, a_1)$  to the right hand side of (2.51) we obtain

$$\begin{aligned} & \phi^1(D_t^\partial, \dot{D}_t^\partial)(a_{0,\partial}, a_{1,\partial}) + B^b\phi^2(D_t, \dot{D}_t)(a_0, a_1) + b^b\phi^0(D_t, \dot{D}_t)(a_0, a_1) \\ &= - {}^b\langle a_0, [\dot{D}_t, a_1] \rangle + {}^b\langle a_0, [D_t, \dot{D}_t], [D_t, a_1] \rangle + {}^b\langle a_0, [D_t, a_1], [D_t, \dot{D}_t] \rangle \\ &= - \frac{d}{dt} {}^b\text{Ch}^1(D_t)(a_0, a_1) \end{aligned} \quad (2.55)$$

in view of (2.48).

With more effort but in a similar manner, the previous considerations can be extended to arbitrary  $k$ , thus proving Eq. (2.7).

## 3. COMPARISON RESULTS AND ESTIMATES ON THE CYLINDER

This section is devoted to some crucial estimates for the heat kernel of a b-Dirac operator. These estimates will be used to analyze the short and long time behavior of the Chern character. Throughout this section it will be convenient to work in the cylindrical context.

**3.1. Comparison results.** Let  $M_j$ ,  $j = 1, 2$ , be complete Riemannian manifolds with cylindrical ends. Assume that  $M_1$  and  $M_2$  share a common cylinder component  $(-\infty, 0] \times N$ . That is, if  $M_j = (-\infty, 0] \times N_j \cup_{N_j} X_j$ ,  $j = 1, 2$ , then  $N$  is (after a suitable identification) a common (union of) connected component(s) of  $N_1, N_2$ . A typical example will be  $M_2 = \mathbb{R} \times N$ .

Suppose that  $D_j$  are formally self-adjoint Dirac operators (cf. Subsection 1.7) on  $M_j$ ,  $j = 1, 2$  with  $D_{1|(-\infty, 0] \times N} = D_{2|(-\infty, 0] \times N} =: D = c(dx) \frac{d}{dx} + D_\partial$ . The operators  $D_j$  are supposed to act on sections of the Hermitian vector bundles  $W_j$  such that  $W_{1|(-\infty, 0] \times N} = W_{2|(-\infty, 0] \times N}$ . As explained in Appendix A, we identify  $\Gamma^\infty((-\infty, c] \times N; W)$ ,  $c \in \mathbb{R}$ , with the completed tensor product  $\mathcal{C}^\infty((-\infty, c]) \hat{\otimes} \Gamma^\infty(N; W)$  and accordingly  ${}^b\Gamma_{\text{cpt}}^\infty((-\infty, c) \times N; W)$  with  ${}^b\mathcal{C}_{\text{cpt}}^\infty((-\infty, c)) \hat{\otimes} \Gamma^\infty(N; W)$ . We want to compare the resolvents and heat kernels of  $D_j$  on the common cylinder  $(-\infty, 0] \times N$ .

To this end, we will make repeated use of Remark 1.10 without mentioning it every time. Moreover, we also will tacitly use the fact that

$$\inf \text{spec}_{\text{ess}} D^2 = \inf \text{spec } D_\partial^2. \quad (3.1)$$

A proof of this can be found in [MÜL94, Sec. 4]. Concerning notation,  $\|T\|_p$  stands for the  $p$ -th Schatten norm of an operator  $T$  acting on a Hilbert space  $\mathcal{H}$  unless otherwise stated.

**Proposition 3.1.** 1. *Fix an open sector  $\Lambda := \{z \in \mathbb{C}^* \mid \varepsilon \leq \arg z \leq 2\pi - \varepsilon\} \subset \mathbb{C} \setminus \mathbb{R}_+$  where  $\varepsilon > 0$ . Then on  $(-\infty, c] \times N$ ,  $c < 0$ , the difference of the resolvents  $(D_1^2 - \lambda)^{-1} - (D_2^2 - \lambda)^{-1}$ ,  $\lambda \in \Lambda$ , is trace class and*

$$\left\| \left( (D_1^2 - \lambda)^{-1} - (D_2^2 - \lambda)^{-1} \right)_{|(-\infty, c] \times N} \right\|_1 \leq C(c, n) |\lambda|^{-n}, \quad \text{as } \lambda \rightarrow \infty \text{ in } \Lambda, \quad (3.2)$$

with  $n$  arbitrarily large.

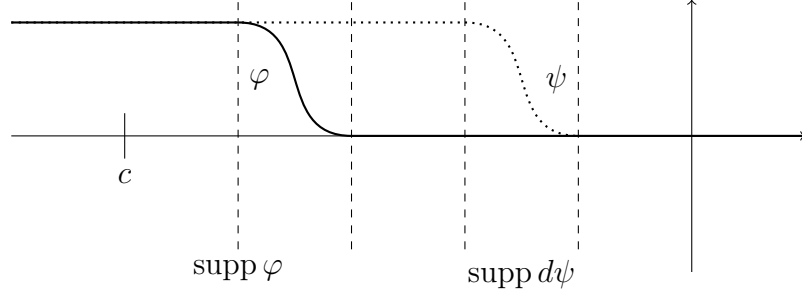
2. *On  $(-\infty, c] \times N$ ,  $c < 0$ , the difference of the heat kernels*

$$\left( D_1^l e^{-tD_1^2} - D_2^l e^{-tD_2^2} \right)_{|(-\infty, c] \times N} \quad (3.3)$$

is trace class for  $t > 0$  and

$$\left\| \left( D_1^l e^{-tD_1^2} - D_2^l e^{-tD_2^2} \right)_{|(-\infty, c] \times N} \right\|_1 \leq C(c, n, t_0) t^n, \quad 0 < t \leq t_0, \quad l \in \mathbb{Z}_+, \quad (3.4)$$

with  $n$  arbitrarily large.

FIGURE 3. The cut-off functions  $\varphi$  and  $\psi$ .

3. Assume in addition that  $D_1, D_2$  are Fredholm operators and denote by  $H_j$  the orthogonal projections onto  $\text{Ker } D_j, j = 1, 2$ . Then for  $0 < \delta < \inf \text{spec}_{\text{ess}} D^2$  there is a constant  $C(c, \delta)$  such that

$$\left\| (D_1^l e^{-tD_1^2} (I - H_1) - D_2^l e^{-tD_2^2} (I - H_2)) \Big|_{(-\infty, c] \times N} \right\|_1 \leq C(c, \delta) e^{-t\delta}, \quad (3.5)$$

for  $0 < t < \infty, l \in \mathbb{Z}_+$ .

*Proof.* 1. We choose cut-off functions  $\varphi, \psi \in C^\infty(\mathbb{R})$  such that

$$\varphi(x) = \begin{cases} 1, & x \leq 4/5c, \\ 0, & x \geq 3/5c, \end{cases} \quad \psi(x) = \begin{cases} 1, & x \leq 2/5c, \\ 0, & x \geq 1/5c, \end{cases} \quad (3.6)$$

see Figure 3. We have  $\psi\varphi = \varphi, \text{supp } d\psi \cap \text{supp } \varphi = \emptyset$ . Consider

$$R_{\psi, \varphi}(\lambda) := \psi((D_1^2 - \lambda)^{-1} - (D_2^2 - \lambda)^{-1})\varphi, \quad (3.7)$$

viewed as an operator acting on sections over  $M_1$ . Then

$$(D_1^2 - \lambda)R_{\psi, \varphi}(\lambda) = [D_1^2, \psi](D_1^2 - \lambda)^{-1}\varphi - [D_2^2, \psi](D_2^2 - \lambda)^{-1}\varphi, \quad (3.8)$$

where again  $[D_2^2, \psi](D_2^2 - \lambda)^{-1}\varphi$  is considered as acting on sections over  $M_1$ . Since the operators  $[D_j^2, \psi], j = 1, 2$  have compact support which is disjoint from the support of  $\varphi$  we may apply Proposition B.2 to the r.h.s. of (3.8) to infer that  $R_{\psi, \varphi}(\lambda)$  is trace class and that the estimate (3.2) holds.

2. This follows from 1. and the contour integral representation (B.12) of the heat kernel. Cf. Proposition B.5.

3. This follows from 1. and (B.12) by taking the contour as in Figure 6, page 78.  $\square$

We refer to Appendix A for the notation  ${}^b\text{Diff}_{\text{cpt}}^a((-\infty, 0) \times N; W)$  (A.16). In what follows, the subscript *cpt* indicates that the objects are supported away from  $\{0\} \times N$ ; it does *not* indicate compact support. The support of objects in  ${}^b\text{Diff}_{\text{cpt}}^a$ , and other spaces having the *cpt* decoration, may be unbounded towards  $\{-\infty\} \times N$ .

**Theorem 3.2.** Let  $A_j \in {}^b\text{Diff}_{\text{cpt}}^{a_j}((-\infty, 0) \times N; W), j = 0, \dots, k$  be given,  $a := \sum_{j=0}^k a_j$ .

1. For  $t_0, n > 0$  there is a constant  $C(t_0, n)$  such that for all  $\sigma \in \Delta_k, \sigma_j > 0$

$$\begin{aligned} & \left\| A_0 e^{-\sigma_0 t D_1^2} \cdot \dots \cdot A_l (e^{-\sigma_l t D_1^2} - e^{-\sigma_l t D_2^2}) A_{l+1} \cdot \dots \cdot A_k e^{-\sigma_k t D_1^2} \right\|_1 \\ & \leq C(t_0, n) \left( \prod_{j=0, j \neq l}^k \sigma_j^{-a_j/2} \right) (\sigma_l t)^n, \quad 0 < t < t_0. \end{aligned} \quad (3.9)$$

2. Assume in addition that  $D_1, D_2$  are Fredholm operators and denote by  $H_j$  the orthogonal projections onto  $\text{Ker } D_j, j = 1, 2$ . Then for  $0 < \delta < \inf \text{spec}_{\text{ess}} D^2$  and all  $\sigma \in \Delta_k, \sigma_j > 0$ ,

$$\begin{aligned} & \left\| A_0 e^{-\sigma_0 t D_1^2} (I - H_1) \cdot \dots \cdot A_l (e^{-\sigma_l t D_1^2} (I - H_1) - e^{-\sigma_l t D_2^2} (I - H_2)) A_{l+1} \cdot \dots \right. \\ & \quad \left. \dots \cdot A_k e^{-\sigma_k t D_1^2} (I - H_1) \right\|_1 \\ & \leq C(\delta) \left( \prod_{j=0, j \neq l}^k \sigma_j^{-a_j/2} \right) e^{-t\delta}, \quad 0 < t < \infty. \end{aligned} \quad (3.10)$$

**Remark 3.3.** With some more efforts one can show that the factors  $\left( \prod_{j=0}^k \sigma_j^{-a_j/2} \right)$  on the right hand sides of the estimates (3.9), (3.10), and also below in (3.34), (3.36) are obsolete. But since this is not needed for our purposes we prefer a less cumbersome presentation.

*Proof.* First note that by Proposition A.2 the operator  $A(i + D)^{-a}$  is bounded for  $A \in {}^b\text{Diff}_{\text{cpt}}^a((-\infty, 0) \times N; W)$  and hence the Spectral Theorem implies that for  $t_0 > 0$  there is a  $C(t_0)$  such that

$$\|Ae^{-tD^2}\|_\infty \leq C(t_0)t^{-a/2}, \quad 0 < t < t_0. \quad (3.11)$$

If  $D$  is Fredholm then for  $0 < \delta < \inf \text{spec}_{\text{ess}} D^2$  and  $t_0 > 0$  there is a  $C(t_0, \delta)$  such that

$$\|Ae^{-tD^2}(I - H)\|_\infty \leq C(\delta)e^{-t\delta}, \quad t_0 < t < \infty. \quad (3.12)$$

(3.11) and (3.12) together imply that for  $0 < \delta < \inf \text{spec}_{\text{ess}} D^2$  there is a  $C(\delta)$  such that

$$\|Ae^{-tD^2}(I - H)\|_\infty \leq C(\delta)t^{-a/2}e^{-t\delta}, \quad 0 < t < \infty. \quad (3.13)$$

The first claim now follows from the previous Theorem 3.1, 2.: with some  $c < 0$  such that  $\text{supp } A_j \subset (-\infty, c] \times N$  we find

$$\begin{aligned} & \left\| A_0 e^{-\sigma_0 t D_1^2} \cdot \dots \cdot A_l (e^{-\sigma_l t D_1^2} - e^{-\sigma_l t D_2^2}) A_{l+1} \cdot \dots \cdot A_k e^{-\sigma_k t D_1^2} \right\|_1 \\ & \leq \left( \prod_{j=0, j \neq l}^k \|A_j e^{-\sigma_j t D_1^2}\|_\infty \right) \|A_l (i + D)^{-a_l}\|_\infty \cdot \\ & \quad \cdot \left\| ((i + D_1)^{a_l} e^{-\sigma_l t D_1^2} - (i + D_2)^{a_l} e^{-\sigma_l t D_2^2}) \Big|_{(-\infty, c] \times N} \right\|_1 \\ & \leq C(t_0, n) \left( \prod_{j=0, j \neq l}^k \sigma_j^{-a_j/2} \right) \sigma_l^n t^{n-a/2}, \quad 0 < t < t_0. \end{aligned} \quad (3.14)$$

Here we have used (3.11). Since  $\sigma_j < 1$  an inequality which is valid for  $0 < \sigma_j t < t_0$  is certainly also valid for  $0 < t < t_0$ . Since  $n$  is arbitrary (3.14) proves the first claim.

2. Similarly, using Theorem 3.1, 3. and (3.13)

$$\begin{aligned}
& \left\| A_0 e^{-\sigma_0 t D_1^2} (I - H_1) \cdot \dots \cdot A_l (e^{-\sigma_l t D_1^2} (I - H_1) - e^{-\sigma_l t D_2^2} (I - H_2)) A_{l+1} \cdot \dots \right. \\
& \quad \left. \dots \cdot A_k e^{-\sigma_k t D_1^2} (I - H_1) \right\|_1 \\
& \leq \left( \prod_{j=0, j \neq l}^k \|A_j e^{-\sigma_j t D_1^2} (I - H_1)\|_\infty \right) \|A_l (i + D)^{-a_l}\|_\infty \cdot \\
& \quad \cdot \left\| ((i + D_1)^{a_l} e^{-\sigma_l t D_1^2} (I - H_1) - (i + D_2)^{a_l} e^{-\sigma_l t D_2^2} (I - H_2)) \right\|_{|(-\infty, c] \times N} \\
& \leq C(\delta) \left( \prod_{j=0, j \neq l}^k \sigma_j^{-a_j/2} \right) t^{-a/2} e^{-t\delta}.
\end{aligned} \tag{3.15}$$

Together with the proved short time estimate and the fact that the  $H_j$  are of finite rank and thus of trace class we reach the conclusion.  $\square$

**3.2. Trace estimates for the model heat kernel.** We consider the heat kernel of the Laplacian  $\Delta_{\mathbb{R}}$  on the real line

$$k_t(x, y) = \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t}. \tag{3.16}$$

By slight abuse of notation we will denote the operator of multiplication by  $\text{Id}_{\mathbb{R}}$  by  $X$ . We want to estimate the Schatten norms of  $e^{\alpha|X|} e^{-t\Delta_{\mathbb{R}}} e^{\beta|X|}$ . Before we start with this let us note for future reference:

$$\begin{aligned}
\int_{\mathbb{R}} e^{-\frac{z^2}{\lambda t} - \beta z} dz &= \sqrt{\pi \lambda t} e^{\lambda \beta^2 t/4}, \\
\int_{\mathbb{R}} e^{-\frac{z^2}{\lambda t} + \beta|z|} dz &\leq 2\sqrt{\pi \lambda t} e^{\lambda \beta^2 t/4},
\end{aligned} \quad \beta \in \mathbb{R}; \lambda, t > 0. \tag{3.17}$$

Furthermore, we will need the well-known Schur's test:

**Lemma 3.4** ([HASU78, Thm. 5.2]). *Let  $K$  be an integral operator on a measure space  $(\Omega, \mu)$  with kernel  $k : \Omega \times \Omega \rightarrow \mathbb{C}$ . Assume that there are positive measurable functions  $p, q : \Omega \rightarrow (0, \infty)$  such that*

$$\begin{aligned}
\int_X |k(x, y)| p(y) d\mu(y) &\leq C_p q(x), \\
\int_X |k(x, y)| q(x) d\mu(x) &\leq C_q p(y).
\end{aligned} \tag{3.18}$$

Then  $K$  is bounded in  $L^2(\Omega, \mu)$  with  $\|K\| \leq \sqrt{C_p C_q}$ .

Now we can prove the following estimate.

**Proposition 3.5.** *Let  $\Delta_{\mathbb{R}} = -\frac{d^2}{dx^2}$  be the Laplacian on the real line. Then for  $\alpha > \beta > 0, t > 0, l \in \mathbb{Z}_+$ , the integral operator  $e^{-\alpha|X|} \left(\frac{d}{dx}\right)^l e^{-t\Delta_{\mathbb{R}}} e^{\beta|X|}$  with the (not everywhere smooth) kernel*

$$\frac{1}{\sqrt{4\pi t}} e^{-\alpha|x|} \partial_x^l e^{-(x-y)^2/4t + \beta|y|} \quad (3.19)$$

is  $p$ -summable for  $1 \leq p \leq \infty$  and we have the estimate

$$\begin{aligned} & \left\| e^{-\alpha|X|} \left(\frac{d}{dx}\right)^l e^{-t\Delta_{\mathbb{R}}} e^{\beta|X|} \right\|_p \\ & \leq \left( c_1 t^{-\frac{l}{2} - \frac{1}{2p}} + c_2 t^{-\frac{1}{2p}} \right) (\alpha - \beta)^{-1/p} e^{(\alpha^2 + \beta^2)t}, \quad 0 < t < \infty \end{aligned} \quad (3.20)$$

with (computable) constants  $c_1, c_2$  independent of  $\alpha, \beta, p, t$ .

**Remark 3.6.** We are not striving to make these estimates optimal. We chose to formulate them in such a way that they are sufficient for our purposes and such that the proofs do not become too cumbersome.

*Proof.* We will prove this estimate for  $p = \infty$  using Schur's test Lemma 3.4 and for  $p = 2$  by estimating the  $L^2$ -norm of the kernel. The case  $p = 1$  will then follow from the semigroup property of the heat kernel. The result for general  $p$  follows from the cases  $p = 1$  and  $p = \infty$  and the interpolation inequality (B.15).

The case  $l \geq 2$  can easily be reduced to the case  $l \in \{0, 1\}$  in view of the identity

$$\partial_x^{2k} e^{-t\Delta_{\mathbb{R}}} = (-\Delta_{\mathbb{R}})^k e^{-t\Delta_{\mathbb{R}}} = \partial_t^k e^{-t\Delta_{\mathbb{R}}}. \quad (3.21)$$

$p = \infty$ . We apply Schur's test with  $p(x) = q(x) = 1$ . We will make frequent use of (3.17) without explicitly mentioning it all the time.

$$\begin{aligned} & \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\alpha|x| - (x-y)^2/4t + \beta|y|} dy \\ & \leq \frac{1}{\sqrt{4\pi t}} e^{-\alpha|x|} \int_{\mathbb{R}} e^{-z^2/4t + \beta|x| + \beta|z|} dz \\ & \leq e^{(\beta-\alpha)|x|} 2e^{\beta^2 t} \leq 2e^{\beta^2 t}. \end{aligned} \quad (3.22)$$

Reversing the roles of  $x$  and  $y$  one gets similarly

$$\begin{aligned} & \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{\beta|y| - (x-y)^2/4t - \alpha|x|} dx \\ & \leq \frac{1}{\sqrt{4\pi t}} e^{\beta|y|} \int_{\mathbb{R}} e^{-z^2/4t - \alpha|y| + \alpha|z|} dz \\ & \leq e^{(\beta-\alpha)|y|} 2e^{\alpha^2 t} \leq 2e^{\alpha^2 t}. \end{aligned} \quad (3.23)$$

This proves the result for  $l = 0$  and  $p = \infty$ . In the case  $l = 1$  the integral

$$\frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \frac{|x-y|}{2t} e^{-\alpha|x| - (x-y)^2/4t + \beta|y|} dy \quad (3.24)$$

is estimated similarly.

$p = 2$ . We estimate the  $L^2$ -norm of the kernel on  $\mathbb{R} \times \mathbb{R}$  by:

$$\begin{aligned}
& \frac{1}{4\pi t} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-2\alpha|x| - \frac{(x-y)^2}{2t} + 2\beta|y|} dx dy \\
& \leq \frac{1}{4\pi t} \int_{\mathbb{R}} e^{-2\alpha|x|} \int_{\mathbb{R}} e^{-\frac{z^2}{2t} + 2\beta|z| + 2\beta|x|} dz dx \\
& \leq \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-2(\alpha-\beta)|x|} e^{2\beta^2 t} dx \\
& = \frac{1}{\sqrt{2\pi t}} \frac{1}{\alpha - \beta} e^{2\beta^2 t},
\end{aligned} \tag{3.25}$$

proving the result for  $l = 0$  and  $p = 2$ . Again, the case  $l = 1$  is similar.

$p = 1$ . Put  $c = (\alpha + \beta)/2$ . Then the semigroup property of the heat kernel gives

$$\begin{aligned}
& \|e^{-\alpha|X|} e^{-t\Delta_{\mathbb{R}}} e^{\beta|X|}\|_1 \\
& \leq \|e^{-\alpha|X|} e^{-t/2\Delta_{\mathbb{R}}} e^{c|X|}\|_2 \|e^{-c|X|} e^{-t/2\Delta_{\mathbb{R}}} e^{\beta|X|}\|_2,
\end{aligned} \tag{3.26}$$

and using the proved case  $p = 2$  gives the result for  $p = 1$ .  $\square$

The previous Proposition and standard estimates for the heat kernel on closed manifolds (cf. the Appendix B) immediately give the following result for the heat kernel of the model Dirac operator on the cylinder.

**Proposition 3.7.** *Let  $N$  be a compact closed manifold and  $D = \Gamma\left(\frac{d}{dx} + A\right)$  a Dirac operator on the cylinder  $M = \mathbb{R} \times N$  (cf. Remark 1.10). Furthermore, let  $Q \in {}^b\text{Diff}_{\text{cpt}}^q((-\infty, 0) \times N; W)$  a  $b$ -differential operator of order  $q$  with support in some cylindrical end  $(-\infty, c) \times N$ . Then for  $\alpha > \beta > 0, t > 0$  the integral operator  $e^{-\alpha|X|} Q e^{-tD^2} e^{\beta|X|}$  with kernel*

$$\frac{1}{\sqrt{4\pi t}} e^{-\alpha|x|} Q_x e^{-(x-y)^2/4t + \beta|y|} e^{-tA^2} \tag{3.27}$$

is  $p$ -summable for  $1 \leq p \leq \infty$ . Furthermore, for  $\varepsilon > 0, t_0 > 0$ , there is a constant  $C(\varepsilon, t_0)$ , such that for  $1 \leq p \leq \infty, 0 < t < t_0, 0 < \beta < \alpha$

$$\|e^{-\alpha|X|} Q e^{-tD^2} e^{\beta|X|}\|_p \leq C(\varepsilon, t_0) (\alpha - \beta)^{-1/p} t^{-\frac{\dim M + \varepsilon}{2p} - \frac{q}{2}}. \tag{3.28}$$

If in addition the operator  $A$  is invertible then for  $0 < \delta < \inf \text{spec } A^2$  and  $\varepsilon > 0$  there are constants  $C_j(\delta, \varepsilon), j = 1, 2$  such that for  $1 \leq p \leq \infty, 0 < t < \infty, 0 < \beta < \alpha$  we have the estimate

$$\begin{aligned}
& \|e^{-\alpha|X|} Q e^{-tD^2} e^{\beta|X|}\|_p \\
& \leq (C_1(\delta, \varepsilon) t^{-\frac{q}{2}} + C_2(\delta, \varepsilon)) (\alpha - \beta)^{-1/p} t^{-\frac{\dim M + \varepsilon}{2p}} e^{(\alpha^2 + \beta^2 - \delta)t}.
\end{aligned} \tag{3.29}$$

For the definition of  ${}^b\text{Diff}_{\text{cpt}}^q, {}^b\text{Diff}^q$  see Proposition A.1 and Eq. (A.16).

*Proof.* By Proposition A.1 we may write  $Q$  as a sum of operators of the form

$$f(x, p) P\left(\frac{d}{dx}\right)^l \tag{3.30}$$

with

- $f \in {}^b\Gamma_{\text{cpt}}^\infty((-\infty, c) \times N; \text{End } W)$ ,
- $P \in \text{Diff}^{b-l}(N; W|_N)$  a differential operator of order  $b-l$  on  $N$  which is *constant* in  $x$ -direction.

Note that  $e^{-\alpha|X|}$  commutes with  $f$ . Furthermore,  $f$  is uniformly bounded. Thus

$$\left\| e^{-\alpha|X|} f P \partial_x^l e^{-t\mathbf{D}^2} e^{\beta|X|} \right\|_p \leq \|f\|_\infty \left\| e^{-\alpha|X|} P \partial_x^l e^{-t\mathbf{D}^2} e^{\beta|X|} \right\|_p. \quad (3.31)$$

Inside the  $p$ -norm is now a tensor product of operators

$$e^{-\alpha|X|} P \partial_x^l e^{-t\mathbf{D}^2} e^{\beta|X|} = (e^{-\alpha|X|} \partial_x^l e^{-t\Delta_{\mathbb{R}}} e^{\beta|X|}) \otimes (P e^{-tA^2}). \quad (3.32)$$

Since the  $p$ -norm of a tensor product is the product of the  $p$ -norms the claim follows from Proposition 3.5 and standard elliptic estimates for the closed manifold  $N$  (Propositions B.5, B.6).  $\square$

**3.3. Trace class estimates for the JLO integrand on manifolds with cylindrical ends.** The heat kernel estimate for the Dirac operator on the model cylinder Proposition 3.7 together with the comparison result in Section 3.1 will now be used to obtain trace estimates for  $b$ -differential operators similar to the one in Proposition B.8 if the indicial operator of at least one of the operators  $A_0, \dots, A_k$  vanishes. Let us mention here that in the following we will use the notation introduced in Section 2, in particular Eq. (2.3).

**Proposition 3.8.** *Let  $M = (-\infty, 0] \times N \cup_N X$  be a complete manifold with cylindrical ends and let  $\mathbf{D}$  be a Dirac operator on  $M$ . Let  $A_0, \dots, A_k \in {}^b\text{Diff}(M; W)$  be  $b$ -differential operators of order  $a_0, \dots, a_k$ . Assume that for at least one index  $l \in \{0, \dots, k\}$  the indicial family of  $A_l$  vanishes. Then for  $t > 0$ ,  $\sigma \in \Delta_k$  the operator*

$$A_0 e^{-\sigma_0 t \mathbf{D}^2} A_1 \cdots A_k e^{-\sigma_k t \mathbf{D}^2} \quad (3.33)$$

*is trace class. Furthermore, there are the following estimates:*

1. For  $t_0 > 0$ ,  $\varepsilon > 0$  there is a constant  $C(t_0, \varepsilon)$  such that for all  $\sigma = (\sigma_0, \dots, \sigma_k) \in \Delta_k$ ,  $\sigma_j > 0$ ,

$$\begin{aligned} & \|A_0 e^{-\sigma_0 t \mathbf{D}^2} A_1 \cdots A_k e^{-\sigma_k t \mathbf{D}^2}\|_1 \\ & \leq C(t_0, \varepsilon) \left( \prod_{j=0}^k \sigma_j^{-a_j/2} \right) t^{-a/2 - (\dim M)/2 - \varepsilon}, \quad 0 < t \leq t_0. \end{aligned} \quad (3.34)$$

*In particular, if  $a_j \leq 1$ ,  $j = 0, \dots, k$ , then*

$$\|(A_0, \dots, A_k)_{\sqrt{t}\mathbf{D}}\| = O(t^{-a/2 - (\dim M)/2 - 0}), \quad t \rightarrow 0+. \quad (3.35)$$

2. Assume additionally that  $\mathbf{D}$  is Fredholm and denote by  $H$  the orthogonal projection onto  $\text{Ker } \mathbf{D}$ . Then for  $\varepsilon > 0$  and any  $0 < \delta < \inf \text{spec}_{\text{ess}} \mathbf{D}^2$  there is a constant  $C(\delta, \varepsilon)$  such that for all  $\sigma \in \Delta_k$ ,  $\sigma_j > 0$

$$\begin{aligned} & \|A_0 e^{-\sigma_0 t \mathbf{D}^2} (I - H) A_1 \cdots A_k e^{-\sigma_k t \mathbf{D}^2} (I - H)\|_1 \\ & \leq C(\delta, \varepsilon) \left( \prod_{j=0}^k \sigma_j^{-a_j/2} \right) t^{-a/2 - (\dim M)/2 - \varepsilon} e^{-t\delta}, \quad \text{for all } 0 < t < \infty. \end{aligned} \quad (3.36)$$

In particular, if  $a_j \leq 1, j = 0, \dots, k$ , then

$$\begin{aligned} & \| (A_0(I - H), \dots, A_k(I - H))_{\sqrt{t}\mathbf{D}} \| \\ & \leq \tilde{C}(\varepsilon, \delta) t^{-a/2 - (\dim M)/2 - \varepsilon} e^{-t\delta}, \quad \text{for all } 0 < t < \infty. \end{aligned} \quad (3.37)$$

*Proof.* We first reduce the problem to a problem on the cylinder  $(-\infty, 0] \times N$ .

Write  $A_j = A_j^{(0)} + A_j^{(1)}$  where  $A_j^{(0)}$  has compact support and  $A_j^{(1)}$  is supported on  $(-\infty, c] \times N$  for some  $c < 0$ .

Then we split  $A_0 e^{-\sigma_0 t \mathbf{D}^2} A_1 \dots A_k e^{-\sigma_k t \mathbf{D}^2}$  (resp.  $A_0 e^{-\sigma_0 t \mathbf{D}^2} (I - H) A_1 \dots A_k e^{-\sigma_k t \mathbf{D}^2} (I - H)$ ) into a sum of terms obtained by decomposing  $A_j = A_j^{(0)} + A_j^{(1)}$ .

To the summands involving at least one term  $A_j^{(0)}$  we apply Proposition B.8. To the remaining summand involving only  $A_j^{(1)}$  we first apply the comparison Theorem 3.2 with  $\mathbf{D}_1 = \mathbf{D}$  and  $\mathbf{D}_2 = \Gamma \frac{d}{dx} + \mathbf{D}_\partial = \Gamma \left( \frac{d}{dx} + A \right)$ , where  $\mathbf{D}_2$  acts on sections over the cylinder  $\mathbb{R} \times N$ .

Hence it remains to prove the claim for the cylinder  $M = \mathbb{R} \times N$  where each  $A_j$  is supported on  $(-\infty, c] \times N$  for some  $c < 0$ .

For definiteness it is not a big loss of generality if we assume that the indicial family of  $A_0$  vanishes (I.e.  $l = 0$ ). Write  $A_0 = e^{-|X|} \tilde{A}_0$  with  $A_0 \in {}^b\text{Diff}_{\text{cpt}}^{a_0}((-\infty, 0) \times N; W)$ . Let  $\beta_0, \dots, \beta_{k+1}$  be real numbers with  $1 \geq \beta_0 > \beta_1 > \dots > \beta_k > \beta_{k+1} = 0$ .

Let us assume that  $\mathbf{D}$  is Fredholm and prove the claim 2. The proof of claim 1. is similar and left to the reader. Hölder's inequality yields

$$\begin{aligned} & \| A_0 e^{-\sigma_0 t \mathbf{D}^2} (I - H) A_1 \dots A_k e^{-\sigma_k t \mathbf{D}^2} (I - H) \|_1 \\ & \leq C \| e^{-\beta_0 |X|} \tilde{A}_0 e^{-\sigma_0 t \mathbf{D}^2} (I - H) e^{\beta_1 |X|} \|_{\sigma_0^{-1}} \cdot \\ & \quad \cdot \prod_{j=1}^k \| e^{-\beta_j |X|} A_j e^{-\sigma_j t \mathbf{D}^2} (I - H) e^{\beta_{j+1} |X|} \|_{\sigma_j^{-1}}. \end{aligned} \quad (3.38)$$

The individual factors are estimated by Proposition 3.7 and we obtain for  $0 < t < \infty$ :

$$\begin{aligned} \dots & \leq \prod_{j=0}^k (C_{1,j}(\delta, \varepsilon, \beta) (t \sigma_j)^{-a_j/2} + C_{2,j}(\delta, \varepsilon, \beta)) \cdot \\ & \quad \cdot t^{-\frac{\dim M + \varepsilon}{2} \sigma_j} e^{(\beta_j^2 + \beta_{j+1}^2 - \delta) \sigma_j t} \\ & \leq C(\delta, \varepsilon, \beta, \gamma) \left( \prod_{j=0}^k \sigma_j^{-a_j/2} \right) t^{-a/2 - \frac{\dim M + \varepsilon}{2}} e^{(2 \sum \beta_j^2 + \gamma - \delta)t}, \end{aligned} \quad (3.39)$$

for any  $\gamma > 0$ . The  $\gamma > 0$  is introduced to compensate  $t^{-a/2}$  as  $t \rightarrow \infty$ . Since we may choose  $2 \sum \beta_j^2 + \gamma$  as small as we please, the claim is proved.

The remaining cases are treated similarly.  $\square$

**3.4. Estimates for b-traces.** Now we come to the main result of this section.

**Theorem 3.9.** *In the notation of Proposition 3.8 we now drop the assumption that the indicial family of one of the  $A_l$  vanishes. Then the following estimates hold:*

1. For  $\varepsilon > 0, t_0 > 0$  there is a constant  $C(\varepsilon, t_0)$  such that for all  $\sigma = (\sigma_0, \dots, \sigma_k) \in \Delta_k, \sigma_j > 0$ ,

$$\begin{aligned} & {}^b\mathrm{Tr}(A_0 e^{-\sigma_0 t \mathbf{D}^2} A_1 \dots A_k e^{-\sigma_k t \mathbf{D}^2}) \\ & \leq C(\varepsilon, t_0) \left( \prod_{j=0}^k \sigma_j^{-a_j/2-\varepsilon} \right) t^{-a/2-(\dim M)/2-\varepsilon}, \quad 0 < t \leq t_0. \end{aligned} \quad (3.40)$$

In particular,

$$|{}^b\langle A_0, \dots, A_k \rangle_{\sqrt{t}\mathbf{D}}| = O(t^{-a/2-(\dim M)/2-\varepsilon}), \quad t \rightarrow 0+. \quad (3.41)$$

2. If  $\mathbf{D}$  is Fredholm then for  $\varepsilon > 0$  and any  $0 < \delta < \inf \mathrm{spec}_{\mathrm{ess}} \mathbf{D}^2$  there is a constant  $C(\varepsilon, \delta)$  such that for all  $\sigma \in \Delta_k, \sigma_j > 0$

$$\begin{aligned} & {}^b\mathrm{Tr}(A_0 e^{-\sigma_0 t \mathbf{D}^2} (I - H) A_1 \dots A_k e^{-\sigma_k t \mathbf{D}^2} (I - H)) \\ & \leq C(\varepsilon, \delta) \left( \prod_{j=0}^k \sigma_j^{-a_j/2-\varepsilon} \right) t^{-a/2-(\dim M)/2-\varepsilon} e^{-t\delta}, \quad \text{for all } 0 < t < \infty. \end{aligned} \quad (3.42)$$

In particular,

$$\begin{aligned} & |{}^b\langle A_0(I - H), \dots, A_k(I - H) \rangle_{\sqrt{t}\mathbf{D}}| \\ & \leq \tilde{C}_{\delta, \varepsilon} t^{-a/2-(\dim M)/2-\varepsilon} e^{-t\delta}, \quad \text{for all } 0 < t < \infty. \end{aligned} \quad (3.43)$$

*Proof.* Arguing as in the proof of Proposition 3.8 we may assume that  $\mathbf{D}$  is the model Dirac operator and that  $A_0, \dots, A_k \in {}^b\mathrm{Diff}_{\mathrm{cpt}}((-\infty, 0) \times \partial M; W)$ .

By Proposition 1.6 we have

$$\begin{aligned} & {}^b\mathrm{Tr}(A_0 e^{-\sigma_0 t \mathbf{D}^2} A_1 \dots A_k e^{-\sigma_k t \mathbf{D}^2}) \\ & = - \sum_{j=0}^k \mathrm{Tr}(x A_0 e^{-\sigma_0 t \mathbf{D}^2} \dots [\frac{d}{dx}, A_j] \dots e^{-\sigma_k t \mathbf{D}^2}). \end{aligned} \quad (3.44)$$

Although multiplication by  $x$  is not a  $b$ -differential operator it is easy to see that Proposition 3.8 still holds true for the summands on the right of (3.44). The reason is that for any fixed  $\varepsilon > 0$  the function  $x e^{-\varepsilon|x|}$  is bounded. In fact any  $0 < \varepsilon < 1$  will do since the coefficients of  $[\frac{d}{dx}, A_j]$  are  $O(e^x)$  as  $x \rightarrow -\infty$ .  $\square$

#### 4. ESTIMATES FOR THE COMPONENTS OF THE ENTIRE $b$ -CHARACTER

In this section we continue to work in the setting of a complete Riemannian manifold with cylindrical ends  $M$ , which is equivalent to a compact manifold with boundary with an exact  $b$ -metric, cf. Appendix A. Furthermore, let  $\mathbf{D}$  be a Dirac operator on  $M$ .

##### 4.1. Short time estimates.

**Proposition 4.1.** *The Chern characters  ${}^b\mathrm{Ch}^k$  and  ${}^b\phi^k$  defined in (2.4) and (2.5) satisfy the following estimates for  $k \in \mathbb{Z}_+$ :*

$$\begin{aligned} & {}^b\mathrm{Ch}^k(t\mathbf{D})(a_0, \dots, a_k) = O(t^{k-\dim M-0}), \\ & {}^b\phi^k(t\mathbf{D}, \mathbf{D})(a_0, \dots, a_k) = O(t^{k-\dim M-0}), \end{aligned} \quad t \rightarrow 0+, \quad (4.1)$$

for  $a_j \in {}^b\mathcal{C}^\infty(M)$ ,  $j = 0, \dots, k$ .

In particular,

- (1)  $\lim_{t \rightarrow 0^+} {}^b\text{Ch}^k(t\mathbf{D}) = 0$  for  $k > \dim M$ .
- (2) The function  $t \mapsto {}^b\text{Ch}^k(t\mathbf{D}, \mathbf{D})(a_0, \dots, a_k)$  is integrable on  $[0, T]$  for  $T > 0$  and  $k > \dim M - 1$ .

*Proof.* This follows immediately from Theorem 3.9.  $\square$

**4.2. Large time estimates.** Unless otherwise said we assume in this Subsection that  $\mathbf{D}_\partial$  is invertible. Then  $\inf \text{spec}_{\text{ess}} \mathbf{D}^2 = \inf \text{spec} \mathbf{D}_\partial^2$  (cf. Eq. (3.1)) and hence  $\mathbf{D}$  is a Fredholm operator. We denote by  $H$  the finite rank orthogonal projection onto the kernel of  $\mathbf{D}$ .

**Lemma 4.2.** *Let  $A_j \in {}^b\text{Diff}(M; W)$ ,  $a_j = \text{ord } A_j$ . Furthermore let  $H_j = H$  or  $H_j = I - H$ ,  $j = 0, \dots, k$  and assume that  $H_j = H$  for at least one index  $j$ . Then for each  $0 < \delta < \inf \text{spec}_{\text{ess}} \mathbf{D}^2$*

$$\begin{aligned} & \left\| A_0 H_0 e^{-\sigma_0 t \mathbf{D}^2} A_1 H_1 e^{-\sigma_1 t \mathbf{D}^2} \dots A_k H_k e^{-\sigma_k t \mathbf{D}^2} \right\|_1 \\ & \leq C(\delta) \left( \prod_{l \in \{j_1, \dots, j_q\}} \sigma_l^{-a_l/2} \right) t^{-a/2} e^{-(\sigma_{j_1} + \dots + \sigma_{j_q}) t \delta}, \quad 0 < t < \infty, \end{aligned} \quad (4.2)$$

where  $j_1, \dots, j_q$  are those indices with  $H_j = I - H$ ,  $a = \sum_{l \in \{j_1, \dots, j_q\}} a_l$ .

*Proof.* We pick an index  $l$  with  $H_l = H$ . Then Hölder's inequality gives

$$\begin{aligned} & \left\| A_0 H_0 e^{-\sigma_0 t \mathbf{D}^2} A_1 H_1 e^{-\sigma_1 t \mathbf{D}^2} \dots A_k H_k e^{-\sigma_k t \mathbf{D}^2} \right\|_1 \\ & \leq \left\| A_l H_l e^{-\sigma_l t \mathbf{D}^2} \right\|_1 \prod_{j \neq l} \left\| A_j H_j e^{-\sigma_j t \mathbf{D}^2} \right\|_\infty. \end{aligned} \quad (4.3)$$

The individual factors are estimated as follows: if  $H_j = H$  then

$$\left\| A_j H e^{-\sigma_j t \mathbf{D}^2} \right\|_p \leq \|A_j H\|_p, \quad \text{for } p \in \{1, \infty\}. \quad (4.4)$$

If  $H_j = I - H$  then by the Spectral Theorem and Proposition A.2 we have for  $0 < \delta < \inf \text{spec}_{\text{ess}} \mathbf{D}^2$

$$\left\| A_j (I - H) e^{-\sigma_j t \mathbf{D}^2} \right\|_\infty \leq C(\delta, A_j) (\sigma_j t)^{-a_j/2} e^{-\sigma_j t \delta}, \quad 0 < t < \infty. \quad \square$$

The next Lemma is extracted from the proof of [CoMo93, Prop. 2].

**Lemma 4.3.** *Let  $f : \mathbb{R}_+^q \rightarrow \mathbb{C}$  be a (continuous) rapidly decreasing function of  $q \leq n$  variables. Then*

$$\begin{aligned} & \lim_{t \rightarrow \infty} t^{2q} \int_{\Delta_n} f(t^2 \sigma_1, \dots, t^2 \sigma_q) d\sigma \\ & = \frac{1}{(n - q)!} \int_{\mathbb{R}_+^q} f(u) du \end{aligned} \quad (4.5)$$

*Proof.* Changing variables  $u_j = t^2 \sigma_j, j = 1, \dots, q; u_j = \sigma_j, j = q + 1, \dots, n$  we find

$$\begin{aligned}
 & t^{2q} \int_{\Delta_n} f(t^2 \sigma_1, \dots, t^2 \sigma_j) d\sigma \\
 &= \int_{\{t^{-2}(u_1 + \dots + u_q) + u_{q+1} + \dots + u_n \leq 1\}} f(u_1, \dots, u_q) du \\
 &= \int_{t^2 \Delta_q} f(u_1, \dots, u_q) \int_{(1-t^{-2}(u_1 + \dots + u_q)) \Delta_{n-q}} du \\
 &= \frac{1}{(n-q)!} \int_{t^2 \Delta_q} (1-t^{-2}(u_1 + \dots + u_q))^{n-q} f(u) du.
 \end{aligned} \tag{4.6}$$

By assumption  $f$  is rapidly decreasing, hence we may apply the Dominated Convergence Theorem to reach the conclusion.  $\square$

4.2.1. *Estimating the transgressed b-Chern character.*

**Proposition 4.4.** *For  $k \geq 1$  and  $a_0, \dots, a_k \in {}^b\mathcal{C}^\infty(M)$  we have*

$${}^b\phi^k(tD, D)(a_0, \dots, a_k) = \begin{cases} O(t^{-2}), & k \text{ even}, \\ O(t^{-3}), & k \text{ odd}, \end{cases} t \rightarrow \infty. \tag{4.7}$$

*Proof.*  $\phi^k(tD, D)(a_0, \dots, a_k)$  is a sum of terms of the form

$$T = {}^b\langle a_0, [tD, a_1], \dots, [tD, a_{i-1}], D, [tD, a_i], \dots, [tD, a_k] \rangle_{tD}. \tag{4.8}$$

Writing  $A_0 = a_0, A_j = [D, a_j], j = 1, \dots, i-1, A_j = [D, a_{j-1}], j = i+1, \dots, k+1, A_i := D$  we find

$$T = t^k \sum_{H_j \in \{H, I-H\}} {}^b\langle A_0 H_0, \dots, A_{k+1} H_{k+1} \rangle_{tD}, \tag{4.9}$$

where the sum runs over all sequences  $H_0, \dots, H_{k+1}$  with  $H_j \in \{H, I-H\}$ . Since  $H[D, a_j]H = 0, HD = DH = 0$  (note  $A_i = D$ !) only terms containing no more than  $[k/2] + 1$  copies of  $H$  can give a non-zero contribution.

Consider such a nonzero summand with at least one index  $j$  with  $H_j = H$  and denote by  $q$  the number of indices  $l$  with  $H_l = I-H$ . Then  $q \geq [\frac{k+1}{2}] + 1$  and we infer from Lemmas 4.2, 4.3

$$t^k {}^b\langle A_0 H_0, \dots, A_{k+1} H_{k+1} \rangle_{tD} = O(t^{k-2q}) = \begin{cases} O(t^{-2}), & k \text{ even}, \\ O(t^{-3}), & k \text{ odd}. \end{cases} \tag{4.10}$$

We infer from Theorem 3.9 that the remaining summand with  $H_j = I-H$  for all  $j$  decays exponentially as  $t \rightarrow \infty$  and we are done.  $\square$

4.2.2. *The limit as  $t \rightarrow \infty$  of the b-Chern character.* As in [CoMo93] we put

$$\varrho_H(A) := HAH, \tag{4.11}$$

and

$$\omega_H(A, B) := \varrho_H(AB) - \varrho_H(A)\varrho_H(B). \tag{4.12}$$

**Proposition 4.5.** *Let  $a_0, \dots, a_k \in {}^b\mathcal{C}^\infty(M)$ . If  $k$  is odd then*

$$\lim_{t \rightarrow \infty} {}^b\text{Ch}^k(tD)(a_0, \dots, a_k) = 0. \quad (4.13)$$

*If  $k = 2q$  is even then*

$$\begin{aligned} & \lim_{t \rightarrow \infty} {}^b\text{Ch}^k(tD)(a_0, \dots, a_{2q}) \\ &= \frac{(-1)^q}{q!} \text{Str}(\varrho_H(a_0)\omega_H(a_1, a_2) \dots \omega_H(a_{2q-1}, a_{2q})) \\ &=: \kappa^{2q}(D)(a_0, \dots, a_{2q}). \end{aligned} \quad (4.14)$$

Of course, since  $H$  is of finite rank  $\varrho_H(a_j)$  and  $\omega_H(a_j, a_{j+1})$  are of trace class.

*Proof.* As in the previous proof we abbreviate  $A_0 = a_0$ ,  $A_j = [D, a_j]$ ,  $j \geq 1$  and decompose

$${}^b\langle A_0, A_1, \dots, A_k \rangle_{tD} = t^k \sum_{H_j \in \{H, I-H\}} {}^b\langle A_0 H_0, \dots, A_k H_k \rangle_{tD}. \quad (4.15)$$

Since  $H[D, a_j]H = 0$  only terms containing no more than  $[k/2] + 1$  copies of  $H$  can give a nonzero contribution.

The term containing no copy of  $H$  decreases exponentially in view of Theorem 3.9.

Consider a term containing  $q$  copies of  $I - H$ . If the number  $k + 1 - q$  of copies of  $H$  is at least one but less than  $k/2 + 1$ , which is always the case if  $k$  is odd, then  $q > k/2$  and hence in view of Lemmas 4.2, 4.3

$$t^k {}^b\langle A_0 H_0, \dots, A_k H_k \rangle_{tD} = O(t^{k-2q}) = O(t^{-1}), \quad t \rightarrow \infty. \quad (4.16)$$

If  $n = 2q$  is even there is exactly one term containing  $k/2 + 1$  copies of  $H$ , namely

$$\begin{aligned} & t^{2q} {}^b\langle A_0 H, A_1(I-H), \dots, A_{2q} H \rangle_{tD} \\ &= t^{2q} \int_{\Delta_{2q}} \text{Tr}(\gamma a_0 H e^{-\sigma_0 t^2 D^2} [D, a_1](I-H) \dots e^{-\sigma_{2q} t^2 D^2} H) d\sigma. \end{aligned} \quad (4.17)$$

The integrand depends only on the  $q = k/2$  variables  $\sigma_1, \sigma_3, \dots, \sigma_{2q-1}$  and so we infer from Lemma 4.3 that the limit as  $t \rightarrow \infty$  equals

$$\frac{1}{q!} \int_{\mathbb{R}_+^q} \text{Tr}(\gamma a_0 H [D, a_1] e^{-u_1 D^2} (I-H) [D, a_2] H \dots e^{-u_{2q-1} D^2} (I-H) [D, a_{2q}] H) du. \quad (4.18)$$

As in [CoMo93, 2.2] one shows that this equals

$$\frac{(-1)^q}{q!} \text{Str}(\varrho_H(a_0)\omega_H(a_1, a_2) \dots \omega_H(a_{2q-1}, a_{2q})). \quad \square$$

## 5. ASYMPTOTIC b-HEAT EXPANSIONS

**5.1. b-Heat expansion.** Let  $M$  be a complete Riemannian manifold with cylindrical ends and let  $D$  be a Dirac operator on  $M$  (cf. Remark 1.10).

Let  $Q \in {}^b\text{Diff}^q(M; W)$  be an auxiliary b-differential operator of order  $q$ . It is well-known (cf. e.g. [GIL95]) that the Schwartz-kernel of the operator  $Qe^{-tD^2}$  has a *pointwise* asymptotic expansion

$$(Qe^{-tD^2})(x, p; x, p) \sim_{t \rightarrow 0^+} \sum_{j=0}^{\infty} a_j(Q, D)(x, p) t^{\frac{j - \dim M - q}{2}}. \quad (5.1)$$

The problem is that in general neither  $Qe^{-tD^2}$  is of trace class nor are the local heat invariants  $a_j(Q, D)$  integrable over the manifold. Nevertheless we have the following theorem, which has been used implicitly by Getzler [GET93A]. However, we could not find a reference where the result is cleanly stated and proved. Therefore, we provide here a proof for the convenience of the reader.

**Theorem 5.1.** *Under the previously stated assumptions the b-heat trace of  $Qe^{-tD^2}$  has the following asymptotic expansion:*

$$\begin{aligned} & {}^b\text{Tr}(Qe^{-tD^2}) \\ & \sim_{t \rightarrow 0^+} \sum_{j=0}^{\infty} \int_{{}^bM} \text{tr}_p(a_j(Q, D)(p)) d \text{vol}(p) t^{\frac{j - \dim M - q}{2}}. \end{aligned} \quad (5.2)$$

The b-integral  $\int_{{}^bM}$  was defined in Subsection 1.6, cf. Definition-Proposition 1.5.

*Proof.* We first write the operator  $Q$  as a sum  $Q = Q^{(0)} + Q^{(1)}$  of differential operators with  $Q^{(0)} \in {}^b\text{Diff}_{\text{cpt}}^q((-\infty, 0) \times \partial M; W)$  and  $Q^{(1)}$  a differential operator supported in the interior. By standard elliptic theory ([GIL95])  $Q^{(1)}e^{-tD^2}$  is trace class and since the asymptotic expansion (5.1) is uniform on compact subsets of  $M$  the claim follows for  $Q^{(1)}$  instead of  $Q$ .

So it remains to prove the claim for an operator  $Q \in {}^b\text{Diff}_{\text{cpt}}^q((-\infty, 0) \times \partial M; W)$ ; for convenience we write from now on again  $Q$  instead of  $Q^{(0)}$ . Next we apply the comparison Theorem 3.1 which allows us to assume that  $M = \mathbb{R} \times \partial M$  is the model cylinder,  $D = c(dx) \frac{d}{dx} + D^\partial$ , and  $Q$  is supported on  $(-\infty, c) \times \partial M$  for some  $c > 0$ .

Furthermore, we may assume that  $Q$  is of the form (3.30). Since the heat kernel of the model operator is explicitly known (cf. (3.16)) we have

$$\begin{aligned} & \left( f(x, p) P \partial_x^l e^{-tD^2} \right)(x, p; y, q) \\ & = \frac{1}{\sqrt{4\pi t}} (\partial_x^l e^{-(x-y)^2/4t}) (P e^{-tA^2})(p, q), \quad A := \Gamma D_\partial. \end{aligned} \quad (5.3)$$

If  $l$  is odd then by induction one easily shows that this kernel vanishes on the diagonal and hence the b-trace  ${}^b\text{Tr}(Qe^{-tD^2})$  as well as all local heat coefficients vanish, proving the Theorem in this case. So let  $l = 2k$  be even. Then using (3.21) and since on the

diagonal  $\partial_t^k e^{-t\Delta_{\mathbb{R}}}(x, x) = \partial_t^k (4\pi t)^{-1/2} =: c_k t^{-1/2-k}$ , we have

$$\begin{aligned} & \left( f(x, p) P \partial_x^j e^{-tD^2} \right) (x, p; x, p) \\ &= f(x, p) (P e^{-tA^2})(p, p) c_k t^{-1/2-k} \\ &\sim_{t \rightarrow 0+} \sum_{j=0}^{\infty} f(x, p) a_j(P, A)(p) c_k t^{\frac{j - \dim M - q}{2}}. \end{aligned} \quad (5.4)$$

Comparing with (5.1) we find for the heat coefficients  $a_j(Q, D)$

$$a_j(Q, D)(x, p) = f(x, p) a_j(P, A)(p) c_k. \quad (5.5)$$

Furthermore, we have using Theorem (1.6)

$$\begin{aligned} & {}^b\mathrm{Tr}(Q e^{-tD^2}) \\ &= \int_{-\infty}^0 \int_{\partial M} \mathrm{tr}_{x,p} \left( x \partial_x f(x, p) (P e^{-tA^2})(p, p) \right) d \mathrm{vol}_{\partial M}(p) dx. \end{aligned} \quad (5.6)$$

$(P e^{-tA^2})(p, p)$  has an  $x$ -independent asymptotic expansion as  $t \rightarrow 0+$ . Since  $x \partial_x f(x, p) = O(e^{(1-\delta)x})$ ,  $x \rightarrow -\infty$ , uniformly in  $p$ , we can plug the asymptotic expansion (5.4) into (5.6) and use (5.5) to find

$$\begin{aligned} & {}^b\mathrm{Tr}(Q e^{-tD^2}) \\ &\sim_{t \rightarrow 0+} \sum_{j=0}^{\infty} \int_{-\infty}^0 \int_{\partial M} \mathrm{tr}_{x,p} \left( x \partial_x f(x, p) a_j(P, A)(p) \right) d \mathrm{vol}_{\partial M}(p) dx \\ &\quad \cdot c_k t^{\frac{j - \dim M - q}{2}} \\ &\sim_{t \rightarrow 0+} \sum_{j=0}^{\infty} \int_{{}^b(-\infty, 0) \times \partial M} \mathrm{tr}_{x,p} (a_j(Q, D)(x, p)) d \mathrm{vol}(x, p) \\ &\quad \cdot t^{\frac{j - \dim M - q}{2}}. \end{aligned} \quad (5.7)$$

The claim is proved.  $\square$

**5.2. The b-trace of the JLO integrand.** To extend Theorem 5.1 to expressions of the form  ${}^b\mathrm{Tr} \left( A_0 e^{-\sigma_0 t D^2} A_1 e^{-\sigma_1 t D^2} \dots A_k e^{-\sigma_k t D^2} \right)$  we use a trick which was already applied successfully in the proof of the local index formula in noncommutative geometry [CoMo95]. Namely, we successively commute  $A_j e^{-\sigma_j t D^2}$  and control the remainder. We will need the estimates proved in Subsections 3.3 and 3.4.

We first need to introduce some notation (cf. [LES99, Lemma 4.2]).

For a b-differential operator  $B \in {}^b\mathrm{Diff}(M; W)$  we put inductively

$$\nabla_D^0 B := B, \quad \nabla_D^{j+1} B := [D^2, \nabla_D^j B]. \quad (5.8)$$

Note that since  $D^2$  has scalar leading symbol we have  $\mathrm{ord}(\nabla_D^j B) \leq j + \mathrm{ord} B$ .

The following formula can easily be shown by induction.

$$\begin{aligned}
 e^{-tD^2} B &= \sum_{j=0}^{n-1} \frac{(-t)^j}{j!} (\nabla_D^j B) e^{-tD^2} + \\
 &\quad + \frac{(-t)^n}{(n-1)!} \int_0^1 (1-s)^{n-1} e^{-stD^2} (\nabla_D^n B) e^{-(1-s)tD^2} ds.
 \end{aligned} \tag{5.9}$$

The identity (5.9) easily allows to prove the following statement about *local heat invariants*, cf. [Wid79], [CoMo90], [BLFo90]:

**Proposition 5.2.** *Let  $A_0, \dots, A_k \in {}^b\text{Diff}(M; W)$  of order  $a_0, \dots, a_k$ . Then the Schwartz kernel of  $A_0 e^{-\sigma_0 t D^2} A_1 e^{-\sigma_1 t D^2} \dots A_k e^{-\sigma_k t D^2}$  has a pointwise asymptotic expansion*

$$\begin{aligned}
 &\left( A_0 e^{-\sigma_0 t D^2} A_1 e^{-\sigma_1 t D^2} \dots A_k e^{-\sigma_k t D^2} \right)(p, p) \\
 &= \sum_{\alpha \in \mathbb{Z}_+^k, |\alpha| \leq n} \frac{(-t)^{|\alpha|}}{\alpha!} \sigma_0^{\alpha_1} (\sigma_0 + \sigma_1)^{\alpha_2} \dots (\sigma_0 + \dots + \sigma_{k-1})^{\alpha_k} \cdot \\
 &\quad \cdot \left( A_0 \nabla_D^{\alpha_1} A_1 \dots \nabla_D^{\alpha_k} A_k e^{-tD^2} \right)(p, p) + O_p(t^{(n+1-a-\dim M)/2}), \\
 &=: \sum_{j=0}^n a_j(A_0, \dots, A_k, D)(p) t^{\frac{j-\dim M-a}{2}} + O_p(t^{(n+1-a-\dim M)/2}),
 \end{aligned} \tag{5.10}$$

where  $a = \sum_{j=0}^k a_j$ . The asymptotic expansion is locally uniformly in  $p$ . Furthermore, it is uniform for  $\sigma \in \Delta_k$ .

Again we are facing the problem explained before Theorem 5.1. Still we will be able to show that one obtains a correct formula by taking the b-trace on the left and partie finie integrals on the right of (5.10):

**Theorem 5.3.** *Under the assumptions of the previous Proposition 5.2 we have an asymptotic expansion*

$$\begin{aligned}
& {}^b\mathrm{Tr}\left(A_0 e^{-\sigma_0 t \mathbb{D}^2} A_1 e^{-\sigma_1 t \mathbb{D}^2} \dots A_k e^{-\sigma_k t \mathbb{D}^2}\right) \\
&= \sum_{\alpha \in \mathbb{Z}_+^k, |\alpha| \leq n} \frac{(-t)^{|\alpha|}}{\alpha!} \sigma_0^{\alpha_1} (\sigma_0 + \sigma_1)^{\alpha_2} \dots (\sigma_0 + \dots + \sigma_{k-1})^{\alpha_k} \cdot \\
&\quad \cdot {}^b\mathrm{Tr}\left(A_0 \nabla_{\mathbb{D}}^{\alpha_1} A_1 \dots \nabla_{\mathbb{D}}^{\alpha_k} A_k e^{-t \mathbb{D}^2}\right) + \\
&\quad + O\left(\left(\prod_{j=1}^k \sigma_j^{-a_j/2}\right) t^{(n+1-a-\dim M)/2}\right), \tag{5.11} \\
&= \sum_{j=0}^n \int_{{}^bM} a_j(A_0, \dots, A_k, \mathbb{D}) d \mathrm{vol} t^{\frac{j-\dim M-a}{2}} + \\
&\quad + O\left(\left(\prod_{j=1}^k \sigma_j^{-a_j/2}\right) t^{(n+1-a-\dim M)/2}\right).
\end{aligned}$$

**Remark 5.4.** The O-constant in (5.11) is independent of  $\sigma \in \Delta_k$ . However, the factor  $(\prod_{j=1}^k \sigma_j^{-a_j/2})$  inside the  $O()$  causes some trouble because it is integrable over the standard simplex  $\Delta_k$  only if  $a_1, \dots, a_k \leq 1$ . We do not claim that this factor is necessarily there. It might be an artifact of the inefficiency of our method. Cf. also Remarks 3.3, 3.6.

*Proof.* The strategy of proof we present here can also be used to prove Proposition 5.2.

Again by the comparison Theorem 3.2 we may assume that  $\mathbb{D}$  is the model Dirac operator and  $A_0, \dots, A_k \in {}^b\mathrm{Diff}_{\mathrm{cpt}}((-\infty, 0) \times \partial M; W)$ .

Using Proposition 1.6 we have

$$\begin{aligned}
& {}^b\mathrm{Tr}\left(A_0 e^{-\sigma_0 t \mathbb{D}^2} A_1 \dots A_k e^{-\sigma_k t \mathbb{D}^2}\right) \\
&= - {}^b\mathrm{Tr}\left(x \left[\frac{d}{dx}, A_0 e^{-\sigma_0 t \mathbb{D}^2} A_1 \dots A_k e^{-\sigma_k t \mathbb{D}^2}\right]\right) \tag{5.12} \\
&= - \sum_{j=0}^k \mathrm{Tr}\left(x A_0 e^{-\sigma_0 t \mathbb{D}^2} A_1 \dots \left[\frac{d}{dx}, A_j\right] \dots A_k e^{-\sigma_k t \mathbb{D}^2}\right).
\end{aligned}$$

$[\frac{d}{dx}, A_j]$  is again in  ${}^b\mathrm{Diff}_{\mathrm{cpt}}((-\infty, 0) \times \partial M; W)$  and its indicial family vanishes. Hence by Proposition 3.8 all summands on the right are of trace class. Cf. also the comment at the end of the proof of Theorem 3.9.

It therefore suffices to prove the claim for the summands on the right of (5.12), i.e. for  $\mathrm{Tr}\left(x A_0 e^{-\sigma_0 t \mathbb{D}^2} A_1 \dots A_k e^{-\sigma_k t \mathbb{D}^2}\right)$  where at least one of the  $A_j$  has vanishing indicial family.

Applying (5.9) to  $A_1$  we get

$$\begin{aligned} e^{-\sigma_0 t D^2} A_1 &= \sum_{j=0}^{n-1} \frac{(-\sigma_0 t)^j}{j!} (\nabla_{\mathbb{D}}^j A_1) e^{-\sigma_0 t D^2} + \\ &\quad + \frac{(-\sigma_0 t)^n}{(n-1)!} \int_0^1 (1-s)^{n-1} e^{-s\sigma_0 t D^2} (\nabla_{\mathbb{D}}^n A_1) e^{-(1-s)\sigma_0 t D^2} ds. \end{aligned} \quad (5.13)$$

Therefore we need to estimate the expression

$$x(\sigma_0 t)^n (1-s)^{n-1} A_0 e^{-s\sigma_0 t D^2} (\nabla_{\mathbb{D}}^n A_1) e^{-(1-s)\sigma_0 t D^2} e^{-\sigma_1 t D^2} \dots A_k e^{-\sigma_k t D^2} \quad (5.14)$$

in the trace norm.

If the index  $l$  for which the indicial family of  $A_l$  vanishes is 0 we write  $A_0$  as  $e^x \tilde{A}_0$  with  $\tilde{A}_0 \in {}^b\text{Diff}_{\text{cpt}}((-\infty, 0) \times \partial M; W)$  and move  $x e^x$  under the trace to the right. This assures that Proposition 3.8 applies to  $(\nabla_{\mathbb{D}}^n A_1) e^{-(1-s)\sigma_0 t D^2} e^{-\sigma_1 t D^2} \dots A_k e^{-\sigma_k t D^2} x e^x$ .

If  $l \geq 1$  we just move  $x$  under the trace to the right. After all w.l.o.g. we may assume that  $l \geq 1$ .

Next we choose an integer  $\beta$  such that  $A_0(D^2 + I)^{-\beta}$  has order  $\in \{0, 1\}$ . Then Hölder yields

$$\begin{aligned} &(\sigma_0 t)^n (1-s)^{n-1} \left\| A_0 (I + D^2)^{-\beta} e^{-s\sigma_0 t D^2} (I + D^2)^\beta (\nabla_{\mathbb{D}}^n A_1) \right. \\ &\quad \left. e^{-(1-s)\sigma_0 t D^2} e^{-\sigma_1 t D^2} \dots A_k e^{-\sigma_k t D^2} x \right\|_1 \\ &\leq (\sigma_0 t)^n (1-s)^{n-1} (s\sigma_0 t)^{-a_0/2+\beta} \left\| (I + D^2)^\beta (\nabla_{\mathbb{D}}^n A_1) \right. \\ &\quad \left. e^{-(1-s)\sigma_0 t D^2} e^{-\sigma_1 t D^2} \dots A_k e^{-\sigma_k t D^2} x \right\|_1 \end{aligned} \quad (5.15)$$

To the remaining trace we apply Proposition 3.8 and obtain

$$\begin{aligned} \dots &\leq \sigma_0^n t^n (1-s)^{n-1} (s\sigma_0 t)^{-a_0/2+\beta} C(t_0, \varepsilon) ((1-s)\sigma_0)^{-\beta-n/2-a_1/2} \\ &\quad \left( \prod_{j=2}^k \sigma_j^{-a_j/2} \right) t^{-a/2+a_0/2-\dim M/2-\varepsilon-n/2-\beta} \\ &\leq C(t_0, \varepsilon) s^{-1/2} (1-s)^{n/2-1-\beta-a_1/2} \sigma_0^{\frac{n-a_0-a_1}{2}} \left( \prod_{j=2}^k \sigma_j^{-a_j/2} \right) t^{\frac{n-a-\dim M}{2}-\varepsilon}. \end{aligned} \quad (5.16)$$

If we choose  $n$  large enough the right hand side is integrable in  $s$  and we obtain the desired estimate.

In the next step we apply (5.9) to  $e^{-(\sigma_0+\sigma_1)tD^2}$  and  $A_2$ . Continuing this way we reach the conclusion after  $k$  steps.  $\square$

## 6. THE CONNES-CHERN CHARACTER OF THE RELATIVE DIRAC CLASS

**6.1. Retracted Connes-Chern character.** In this section we assume that  $\mathbb{D}$  is a Dirac operator on a b-Clifford bundle  $W \rightarrow M$  over the b-manifold  $M$  and  $\mathbb{D}_t = t\mathbb{D}$  is a family of Dirac type operators. We now have all tools to apply the method of [CoMo93]

to convert the entire relative Connes–Chern character, which was constructed using the b-trace, into a finitely supported cocycle.

By integrating Eq. (2.7), one obtains for  $0 < \varepsilon < t$

$$\begin{aligned} {}^b\text{Ch}^k(\varepsilon D) - {}^b\text{Ch}^k(tD) &= b \int_{\varepsilon}^t {}^b\phi h^{k-1}(sD, D) ds \\ &+ B \int_{\varepsilon}^t {}^b\phi h^{k+1}(sD, D) ds + \int_{\varepsilon}^t \phi h^k(sD_{\partial}, D_{\partial}) \circ i^* ds. \end{aligned} \quad (6.1)$$

Of course  $\text{Ch}^{\bullet}(D_{\partial})$  satisfies a cocycle and transgression formula as Eq. (2.6), (2.7) with 0 in the r.h.s. Integrating this we obtain

$$\begin{aligned} \text{Ch}^k(\varepsilon D_{\partial}) - \text{Ch}^k(tD_{\partial}) &= b \int_{\varepsilon}^t \phi h^{k-1}(sD_{\partial}, D_{\partial}) ds \\ &+ B \int_{\varepsilon}^t \phi h^{k+1}(sD_{\partial}, D_{\partial}) ds. \end{aligned} \quad (6.2)$$

By Proposition 4.1 (1), the limit  $\varepsilon \searrow 0$  exists for  $k > \dim M$ , and

$$\begin{aligned} \lim_{\varepsilon \searrow 0} {}^b\text{Ch}^k(\varepsilon D) &= 0, \\ \lim_{\varepsilon \searrow 0} \text{Ch}^{k-1}(\varepsilon D_{\partial}) &= 0, \end{aligned} \quad \text{for all } k > \dim M. \quad (6.3)$$

The second limit statement follows either from an obvious adaption of our calculations to the ordinary trace or from [CoMo93]. Hence one gets for  $k > \dim M$

$$\begin{aligned} -{}^b\text{Ch}^k(tD) &= b {}^b\text{T}\phi h_t^{k-1}(D) + B {}^b\text{T}\phi h_t^{k+1}(D) + \text{T}\phi h_t^k(D_{\partial}) \circ i^*, \\ -\text{Ch}^{k-1}(tD_{\partial}) &= b \text{T}\phi h_t^{k-2}(D_{\partial}) + B \text{T}\phi h_t^k(D_{\partial}), \end{aligned} \quad (6.4)$$

where

$$\begin{aligned} {}^b\text{T}\phi h_t^k(D) &:= \int_0^t \phi h^k(sD, D) ds, \\ \text{T}\phi h_t^{k-1}(D_{\partial}) &:= \int_0^t \phi h^{k-1}(sD_{\partial}, D_{\partial}) ds. \end{aligned} \quad (6.5)$$

The above integrals exist in view of Proposition 4.1 (2) even for  $k \geq \dim M$ . From Eq. (6.2) and Theorem 2.2 we obtain for  $k \geq \dim M$ :

$$\begin{aligned} &b \left( {}^b\text{Ch}^k(tD) + B \int_{\varepsilon}^t {}^b\phi h^{k+1}(sD, D) ds \right) \\ &= -B {}^b\text{Ch}^{k+2}(\varepsilon D) + \text{Ch}^{k+1}(\varepsilon D_{\partial}) \circ i^* - b \int_{\varepsilon}^t \phi h^k(sD_{\partial}, D_{\partial}) ds \\ &\longrightarrow -b \text{T}\phi h_t^k(D_{\partial}) \circ i^*, \quad \varepsilon \rightarrow 0+. \end{aligned} \quad (6.6)$$

Thus

$$\begin{aligned} &b \left( {}^b\text{Ch}^k(tD) + B {}^b\text{T}\phi h_t^{k+1}(D) \right) = -b \text{T}\phi h_t^k(D_{\partial}) \circ i^*, \\ &b \left( \text{Ch}^{k-1}(tD_{\partial}) + B {}^b\text{T}\phi h_t^k(D_{\partial}) \right) = 0, \end{aligned} \quad k \geq \dim M. \quad (6.7)$$

Following Connes-Moscovici [CoMo93], we define for  $k \geq \dim M$  the Chern characters  ${}^b\text{ch}_t^k(\mathbb{D})$ ,  ${}^b\widetilde{\text{ch}}^k(\mathbb{D})$  and  $\text{ch}_t^{k-1}(\mathbb{D}_\partial)$  by

$${}^b\text{ch}_t^k(\mathbb{D}) = \sum_{j \geq 0} {}^b\text{Ch}^{k-2j}(t\mathbb{D}) + B {}^b\text{T}\phi\text{h}_t^{k+1}(\mathbb{D}), \quad (6.8)$$

$$\text{ch}_t^{k-1}(\mathbb{D}_\partial) = \sum_{j \geq 0} \text{Ch}^{k-2j-1}(t\mathbb{D}_\partial) + B \text{T}\phi\text{h}_t^k(\mathbb{D}_\partial), \quad (6.9)$$

$${}^b\widetilde{\text{ch}}_t^k(\mathbb{D}) = {}^b\text{ch}_t^k(\mathbb{D}) + \text{T}\phi\text{h}_t^k(\mathbb{D}_\partial) \circ i^*. \quad (6.10)$$

Let us now compute  $(b + B) {}^b\text{ch}_t^\bullet(\mathbb{D})$ . Using Eq. (2.6) and Eq. (6.7) above, we write

$$\begin{aligned} b {}^b\text{ch}_t^k(\mathbb{D}) + B {}^b\text{ch}_t^k(\mathbb{D}) &= \\ &= \sum_{j \geq 1} \left( b {}^b\text{Ch}^{k-2j}(t\mathbb{D}) + B {}^b\text{Ch}^{k-2j+2}(t\mathbb{D}) \right) \\ &\quad + b \left( {}^b\text{Ch}^k(t\mathbb{D}) + B {}^b\text{T}\phi\text{h}_t^{k+1}(\mathbb{D}) \right) \\ &= \sum_{j \geq 1} \text{Ch}^{k-2j+1}(t\mathbb{D}_\partial) \circ i^* - b \text{T}\phi\text{h}_t^k(\mathbb{D}_\partial) \circ i^* \\ &= \sum_{j \geq 0} \text{Ch}^{k-2j-1}(t\mathbb{D}_\partial) \circ i^* - b \text{T}\phi\text{h}_t^k(\mathbb{D}_\partial) \circ i^* \\ &= \text{ch}_t^{k-1}(\mathbb{D}_\partial) \circ i^* - B \text{T}\phi\text{h}_t^k(\mathbb{D}_\partial) \circ i^* - b \text{T}\phi\text{h}_t^k(\mathbb{D}_\partial) \circ i^* \\ &= \text{ch}_t^{k+1}(\mathbb{D}_\partial) \circ i^*, \end{aligned} \quad (6.11)$$

where the last equality follows from the second line of Eq. (6.4).

In conclusion

$$\begin{aligned} (b + B) {}^b\text{ch}_t^k(\mathbb{D}) &= \text{ch}_t^{k+1}(\mathbb{D}_\partial) \circ i^* \\ (b + B) {}^b\widetilde{\text{ch}}_t^k(\mathbb{D}) &= \text{ch}_t^{k-1}(\mathbb{D}_\partial) \circ i^*. \end{aligned} \quad (6.12)$$

Denoting by  $\widetilde{b}, \widetilde{B}$  the relative Hochschild resp. Connes' coboundaries, *cf.* Eq. (1.5), we thus infer

$$\begin{aligned} (\widetilde{b} + \widetilde{B})({}^b\text{ch}_t^k(\mathbb{D}), \text{ch}_t^{k+1}(\mathbb{D}_\partial)) &= 0, \\ (\widetilde{b} + \widetilde{B})({}^b\widetilde{\text{ch}}_t^k(\mathbb{D}), \text{ch}_t^{k-1}(\mathbb{D}_\partial)) &= 0, \end{aligned} \quad (6.13)$$

*i.e.* the pairs  $({}^b\text{ch}_t^k(\mathbb{D}), \text{ch}_t^{k+1}(\mathbb{D}_\partial))$  and  $({}^b\widetilde{\text{ch}}_t^k(\mathbb{D}), \text{ch}_t^{k-1}(\mathbb{D}_\partial))$  are relative cocycles in the direct sum of total complexes

$$\begin{aligned} \text{Tot}_{\oplus}^k \mathcal{BC}^{\bullet, \bullet}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M)) &:= \\ &:= \text{Tot}_{\oplus}^k \mathcal{BC}^{\bullet, \bullet}(\mathcal{C}^\infty(M)) \oplus \text{Tot}_{\oplus}^{k+1} \mathcal{BC}^{\bullet, \bullet}(\mathcal{C}^\infty(\partial M)). \end{aligned} \quad (6.14)$$

By Eq. (6.11) we have

$$\begin{aligned}
& \left( {}^b\text{ch}_t^k(\mathbb{D}) - {}^b\widetilde{\text{ch}}_t^k(\mathbb{D}), \text{ch}_t^{k+1}(\mathbb{D}_\partial) - \text{ch}_t^{k-1}(\mathbb{D}_\partial) \right) \\
&= \left( -\text{T}\phi_t^k(\mathbb{D}_\partial) \circ i^*, -(b+B)\text{T}\phi_t^k(\mathbb{D}_\partial) \right) \\
&= (\widetilde{b} + \widetilde{B}) \left( 0, \text{T}\phi_t^k(\mathbb{D}_\partial) \right),
\end{aligned} \tag{6.15}$$

hence the two pairs differ only by a coboundary.

Next let us compute  $({}^b\text{ch}_t^{k+2}(\mathbb{D}), \text{ch}_t^{k+3}(\mathbb{D}_\partial)) - S({}^b\text{ch}_t^k(\mathbb{D}), \text{ch}_t^{k+1}(\mathbb{D}_\partial))$  in the above relative cochain complex. Using Eq. (6.4) one checks immediately that

$$\begin{aligned}
{}^b\text{ch}_t^{k+2}(\mathbb{D}) - {}^b\text{ch}_t^k(\mathbb{D}) &= {}^b\text{Ch}^{k+2}(t\mathbb{D}) + B {}^b\text{T}\phi_t^{k+3}(\mathbb{D}) - B \text{T}\phi_t^{k+1}(\mathbb{D}) \\
&= -(b+B)\text{T}\phi_t^{k+1}(\mathbb{D}) - \text{T}\phi_t^{k+2}(\mathbb{D}_\partial) \circ i^*.
\end{aligned}$$

From the second line of Eq. (6.4) (or from [CoMo93, Sec. 2.1])

$$\text{ch}_t^{k+3}(\mathbb{D}_\partial) - \text{ch}_t^{k+1}(\mathbb{D}_\partial) = -(b+B)\text{T}\phi_t^{k+2}(\mathbb{D}_\partial),$$

one thus gets

$$\begin{aligned}
& \left( {}^b\text{ch}_t^{k+2}(\mathbb{D}), \text{ch}_t^{k+3}(\mathbb{D}_\partial) \right) - S\left( {}^b\text{ch}_t^k(\mathbb{D}), \text{ch}_t^{k+1}(\mathbb{D}_\partial) \right) \\
&= (\widetilde{b} + \widetilde{B}) \left( -{}^b\text{T}\phi_t^{k+1}(\mathbb{D}), \text{T}\phi_t^{k+2}(\mathbb{D}_\partial) \right).
\end{aligned} \tag{6.16}$$

Hence, the relative cocycles  $({}^b\text{ch}_t^{k+2}(\mathbb{D}), \text{ch}_t^{k+3}(\mathbb{D}_\partial))$  and  $S({}^b\text{ch}_t^k(\mathbb{D}), \text{ch}_t^{k+1}(\mathbb{D}_\partial))$  are cohomologous. Similarly, one gets

$$\begin{aligned}
{}^b\text{ch}_t^k(\mathbb{D}) - \text{ch}_\tau^k(\mathbb{D}) &= \\
&= \sum_{j \geq 0} ({}^b\text{Ch}^{k-2j}(t\mathbb{D}) - {}^b\text{Ch}^{k-2j}(\tau\mathbb{D})) + B \int_\tau^t {}^b\phi^{k+1}(s\mathbb{D}, \mathbb{D}) ds \\
&= -(b+B) \sum_{j \geq 0} \int_\tau^t {}^b\phi^{k-2j-1}(s\mathbb{D}, \mathbb{D}) ds - \sum_{j \geq 0} \int_\tau^t \phi^{k-2j}(s\mathbb{D}_\partial, \mathbb{D}_\partial) ds,
\end{aligned}$$

resp.

$$\text{ch}_t^{k+1}(\mathbb{D}_\partial) - \text{ch}_\tau^{k+1}(\mathbb{D}_\partial) = -(b+B) \sum_{j \geq 0} \int_\tau^t \phi^{k-2j}(s\mathbb{D}_\partial, \mathbb{D}_\partial) ds,$$

hence  $({}^b\text{ch}_t^k(\mathbb{D}), \text{ch}_t^{k+1}(\mathbb{D}_\partial))$  and  $({}^b\text{ch}_\tau^k(\mathbb{D}), \text{ch}_\tau^{k+1}(\mathbb{D}_\partial))$  are cohomologous in the total relative complex as well. Thus, we have proved (1)-(3) of the following result.

- Theorem 6.1.** (1) *The pairs of retracted cochains  $({}^b\text{ch}_t^k(\mathbb{D}), \text{ch}_t^{k+1}(\mathbb{D}_\partial))$ ,  $({}^b\widetilde{\text{ch}}_t^k(\mathbb{D}), \text{ch}_t^{k-1}(\mathbb{D}_\partial))$ ,  $t > 0$ ,  $t > 0$ ,  $k \geq m = \dim M$ ,  $k - m \in 2\mathbb{Z}$  are cocycles in the relative total complex  $\text{Tot}_{\oplus}^{\bullet} \mathcal{BC}^{\bullet, \bullet}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M))$ .*
- (2) *They represent the same class in  $HC^n(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M))$  which is independent of  $t > 0$ .*
- (3) *They represent the same class in  $HP^\bullet(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M))$  which is independent of  $k$ .*

- (4) Denote by  ${}^b\omega_{\mathbf{D}}, \omega_{\mathbf{D}_\partial}$  the local index forms of  $\mathbf{D}$  resp.  $\mathbf{D}_\partial$  [BGV92, Thm. 4.1], see Eq. (6.17) below. Then one has a pointwise limit

$$\lim_{t \rightarrow 0^+} ({}^b\tilde{\text{ch}}_t^k(\mathbf{D}), \text{ch}_t^{k-1}(\mathbf{D}_\partial)) = \left( \int_{{}^bM} {}^b\omega_{\mathbf{D}} \wedge -, \int_{\partial M} \omega_{\mathbf{D}_\partial} \wedge - \right).$$

Moreover,  $({}^b\tilde{\text{ch}}_t^k(\mathbf{D}), \text{ch}_t^{k-1}(\mathbf{D}_\partial))$  represents the Connes–Chern character of  $[\mathbf{D}] \in KK_m(C_0(M); \mathbb{C}) = K_m(M, \partial M)$ .

The pointwise limit will be explained in the proof below. Up to normalization constants  ${}^b\omega_{\mathbf{D}}$  is the  $\hat{A}$  form  $\hat{A}({}^b\nabla_g^2)$  and  $\omega_{\mathbf{D}_\partial}$  is the  $\hat{A}$  form  $\hat{A}(\nabla_{g_\partial}^2)$  on the boundary. Note also that  $\iota^*\omega_{\mathbf{D}} = \omega_{\mathbf{D}_\partial}$ .

*Proof.* It remains to prove (4). So consider  $a_0, a_1, \dots \in {}^b\mathcal{C}^\infty(M^\circ)$ . Using Getzler’s asymptotic calculus (cf. [GET83], [CoMo90, §3], and [BLFo90, Thm. 4.1]) one shows that the local heat invariants of  $a_0 e^{-\sigma_0 t \mathbf{D}^2} [\mathbf{D}, a_1] e^{-\sigma_1 t \mathbf{D}^2} \dots [\mathbf{D}, a_j] e^{-\sigma_j t \mathbf{D}^2}$  (cf. Proposition 5.2) satisfy

$$\begin{aligned} t^j \int_{\Delta_j} \text{str}_{q, W_p} \left( a_0 e^{-\sigma_0 t \mathbf{D}^2} [\mathbf{D}, a_1] e^{-\sigma_1 t \mathbf{D}^2} \dots [\mathbf{D}, a_j] e^{-\sigma_j t \mathbf{D}^2} \right) (p, p) d \text{vol}_{g_b}(p) \\ = \frac{1}{j!} ({}^b\omega_{\mathbf{D}} \wedge a_0 da_1 \dots \wedge da_j)|_p + O(t^{1/2}), \quad t \rightarrow 0^+. \end{aligned} \quad (6.17)$$

Here  $\text{str}_{q, W_p}$  denotes the fiber supertrace in  $W_p$ ,  $q$  indicates the Clifford degree of  $\mathbf{D}$ , cf. Subsection 1.7, the factor  $\frac{1}{j!}$  is the volume of the simplex  $\Delta_j$ . This statement holds locally on any Riemannian manifold for any choice of a self-adjoint extension of a Dirac operator. So it holds for  $\mathbf{D}$  and accordingly for  $\mathbf{D}_\partial$ .

From Theorem 5.3 and its well-known analogue for closed manifolds, cf. [CoMo93, Sec. 4], we thus infer

$$\begin{aligned} \lim_{t \rightarrow 0^+} {}^b\text{Ch}^j(t\mathbf{D})(a_0, \dots, a_j) \\ = \frac{1}{j!} \int_{{}^bM} {}^b\omega_{\mathbf{D}} \wedge a_0 da_1 \dots \wedge da_j, \quad a_j \in {}^b\mathcal{C}^\infty(M^\circ), \end{aligned} \quad (6.18)$$

resp.

$$\begin{aligned} \lim_{t \rightarrow 0^+} \text{Ch}^{j-1}(t\mathbf{D}_\partial)(a_0, \dots, a_{j-1}) \\ = \frac{1}{(j-1)!} \int_{\partial M} \omega_{\mathbf{D}_\partial} \wedge a_0 da_1 \dots \wedge da_{j-1}, \quad a_{j-1} \in \mathcal{C}^\infty(M). \end{aligned} \quad (6.19)$$

Furthermore, in view of (6.5) we have for  $k \geq \dim M - 1$

$$\begin{aligned} \lim_{t \rightarrow 0^+} {}^b\text{Tch}_t^{k+1}(\mathbf{D})(a_0, \dots, a_{k+1}) = 0, \quad a_j \in {}^b\mathcal{C}^\infty(M^\circ), \\ \lim_{t \rightarrow 0^+} \text{Tch}_t^k(\mathbf{D}_\partial)(a_0, \dots, a_k) = 0, \quad a_j \in \mathcal{C}^\infty(M). \end{aligned} \quad (6.20)$$

To interpret these limit results we briefly recall the relation between de Rham currents and relative cyclic cohomology classes over  $({}^b\mathcal{C}^\infty(M), \mathcal{C}^\infty(M))$ , cf. also [LMP08, Sec. 2.2].

Given a de Rham current  $C$  of degree  $j$  then  $C$  defines naturally a cochain  $\widetilde{C} \in C^j({}^b\mathcal{C}^\infty(M))$  by putting  $\widetilde{C}(a_0, \dots, a_j) := \frac{1}{j!} \langle C, a_0 da_1 \wedge \dots \wedge da_j \rangle$ . One has  $b\widetilde{C} = 0$  and  $B\widetilde{C} = \widetilde{\partial C}$ , where  $\partial$  is the codifferential. Because of this identification we will from now on omit the  $\sim$  from the notation if no confusion is possible.

Given a closed b-differential form  $\omega$  on  $M$  of even degree. By  $C_\omega$  we denote the de Rham current  $\int_{{}^bM} \omega \wedge -$ . There is a natural pullback  $\iota^* \omega$  at  $\infty$  (cf. Definition and Propostion 1.5), which is a closed even degree form on  $\partial M$ . We now find

$$\begin{aligned} \langle \partial C_\omega, \alpha \rangle &= \int_{{}^bM} \omega \wedge d\alpha = \int_{{}^bM} d(\omega \wedge \alpha) = \int_{\partial M} \iota^*(\omega \wedge \alpha) = \\ &= \int_{\partial M} \iota^*(\omega) \wedge \iota^*(\alpha) = \langle C_{\iota^*\omega}, \iota^*(\alpha) \rangle. \end{aligned} \quad (6.21)$$

In view of Subsection 1.2 this means that the pair  $(C_\omega, C_{\iota^*\omega})$  is a relative de Rham cycle or via the above mentioned identification between de Rham currents and cochains that  $(\widetilde{C}_\omega, \widetilde{C}_{\iota^*\omega})$  is a relative cocycle in the relative total complex Eq. (6.14).

If  $\omega = \sum_{j \geq 0} \omega_{2j}$  with closed b-differential forms of degree  $2j$  then the pair  $(\omega, \iota^*\omega)$  still gives rise to a relative cocycle of degree  $\dim M$  in the relative total complex. These considerations certainly apply to the even degree forms  ${}^b\omega_{\mathbb{D}}, \omega_{\mathbb{D}_\partial}$  which satisfy  $\iota^*({}^b\omega) = \omega_{\mathbb{D}_\partial}$ . The limit results (6.18), (6.19), (6.5) can then be summarized as

$$\lim_{t \rightarrow 0^+} ({}^b\widetilde{\text{ch}}_t^k(\mathbb{D}), \text{ch}_t^{k-1}(\mathbb{D}_\partial)) = \left( \int_{{}^bM} {}^b\omega_{\mathbb{D}} \wedge -, \int_{\partial M} \omega_{\mathbb{D}_\partial} \wedge - \right). \quad (6.22)$$

The limit on left is understood pointwise for each component of pure degree.

Finally we need to relate  $({}^b\widetilde{\text{ch}}_t^k(\mathbb{D}), \text{ch}_t^{k-1}(\mathbb{D}_\partial))$  to the Chern character of  $[\mathbb{D}] \in K_m(M, \partial M)$ . First recall from Eq. (1.11) that  $HP^\bullet(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M))$  is naturally isomorphic to  $HP^\bullet(\mathcal{J}^\infty(\partial M, M))$ . Under this isomorphism, the class of the pair  $({}^b\widetilde{\text{ch}}_t^k(\mathbb{D}), \text{ch}_t^{k-1}(\mathbb{D}_\partial))$  is mapped to  ${}^b\widetilde{\text{ch}}_t^k(\mathbb{D})|_{\mathcal{J}^\infty(\partial M, M)}$ , just because elements of  $\mathcal{J}^\infty(\partial M, M)$  vanish on  $\partial M$ . We note in passing that by (A.5) a smooth function  $f$  on  $M^\circ$  lies in  $\mathcal{J}^\infty(\partial M, M)$  iff in cylindrical coordinates one has for all  $l, R$  and every differential operator  $P$  on  $\partial M$

$$\partial_x^l Df(x, p) = O(e^{Rx}), \quad x \mapsto -\infty.$$

In view of (6.17), the class of  $({}^b\widetilde{\text{ch}}_t^k(\mathbb{D}), \text{ch}_t^{k-1}(\mathbb{D}_\partial))$  in  $HP^\bullet(\mathcal{J}^\infty(\partial M, M))$  equals that of  $\widetilde{C}_{{}^b\omega_{\mathbb{D}}}$ . As explained in Section 1.8,  $HP^\bullet(\mathcal{J}^\infty(\partial M, M))$  is naturally isomorphic to  $H_\bullet^{\text{dR}}(M, \partial M; \mathbb{C})$ . Under this isomorphism,  $\widetilde{C}_{{}^b\omega_{\mathbb{D}}}$  corresponds to the relative de Rham cycle  $(C_{{}^b\omega_{\mathbb{D}}}, C_{\omega_{\mathbb{D}_\partial}})$ . Finally, note that under the Poincaré duality isomorphism  $H_\bullet^{\text{dR}}(M, \partial M; \mathbb{C}) \cong H_{\text{dR}}^\bullet(M \setminus \partial M; \mathbb{C})$ , the relative de Rham cycle  $(C_{{}^b\omega_{\mathbb{D}}}, C_{\omega_{\mathbb{D}_\partial}})$  is mapped onto the closed form  ${}^b\omega_{\mathbb{D}}$ . This line of argument shows that the class of  $({}^b\widetilde{\text{ch}}_t^k(\mathbb{D}), \text{ch}_t^{k-1}(\mathbb{D}_\partial))$  depends only on the absolute de Rham cohomology class of the closed form  ${}^b\omega_{\mathbb{D}} = c \cdot \hat{A}({}^b\nabla_g^2)$  on the open manifold  $M \setminus \partial M$ . The transgression formula in Chern–Weil theory shows that the (absolute) de Rham cohomology class of  $\hat{A}(\nabla_g^2)$  is independent of the metric  $g$  on  $M^\circ$ . Thus the de Rham class of  ${}^b\omega_{\mathbb{D}}$  equals that of

$\omega_D = c \cdot \hat{A}(\nabla_{g_0}^2)$  for any smooth metric  $g_0$ . Choosing  $g_0$  to be smooth up to the boundary we infer from Section 1.8 and Proposition 1.12 that the class of  $({}^b\widetilde{\text{ch}}_t^k(D), \text{ch}_t^{k-1}(D_\partial))$  in  $HP^\bullet(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M)) \cong HP^\bullet(\mathcal{J}^\infty(\partial M, M))$  equals that of the Connes–Chern character of  $[D]$ .  $\square$

**6.2. The large time limit and higher  $\eta$ -invariants.** Let us now assume that the boundary Dirac  $D_\partial$  is invertible. In view of Proposition 4.4 and Proposition 4.1 we can now, for  $k \geq \dim M$ , form the transgressed cochain

$${}^b\text{T}\phi_\infty^k(D)(a_0, \dots, a_k) = \int_0^\infty {}^b\phi^k(sD, D) ds. \quad (6.23)$$

In view of Eq. (2.52) we arrive at

$$\begin{aligned} & B {}^b\text{T}\phi_\infty^{k+1}(D)(a_0, \dots, a_k) \\ &= \sum_{j=0}^k (-1)^{j+1} \int_0^\infty s^{k+1} \langle [D, a_0], \dots, [D, a_j], D, [D, a_{j+1}], \dots, [D, a_k] \rangle ds \\ &= \sum_{j=0}^k (-1)^{j+1} \int_0^\infty s^{k+1} \int_{\Delta_{k+1}} {}^b\text{Str}_q([D, a_0]e^{-\sigma_0 s^2 D^2} \dots \\ & \quad [D, a_j]e^{-\sigma_j s^2 D^2} D e^{-\sigma_{j+1} s^2 D^2} \dots [D, a_k]e^{-\sigma_{k+1} s^2 D^2}) d\sigma ds. \end{aligned} \quad (6.24)$$

Together with Proposition 4.5 we have proved the analogue of [CoMo93, Thm. 1] in the relative setting:

**Theorem 6.2.** *Let  $k \geq \dim M$  be of the same parity as  $q$  and assume that  $D_\partial$  is invertible. Then the pair of retracted cochains  $({}^b\text{ch}_t^k(D), \text{ch}_t^{k+1}(D_\partial))$ ,  $t > 0$ , has a limit as  $t \rightarrow \infty$ . For  $k = 2l$  even we have*

$$\begin{aligned} {}^b\text{ch}_\infty^k(D) &= \sum_{j=0}^l \kappa^{2j}(D) + B {}^b\text{T}\phi_\infty^{k+1}(D), \\ \text{ch}_\infty^{k+1}(D_\partial) &= B \text{T}\phi_\infty^{k+2}(D_\partial). \end{aligned} \quad (6.25)$$

If  $k = 2l + 1$  is odd then

$$\begin{aligned} {}^b\text{ch}_\infty^k(D) &= B {}^b\text{T}\phi_\infty^{k+1}(D), \\ \text{ch}_\infty^{k+1}(D_\partial) &= B \text{T}\phi_\infty^{k+2}(D_\partial). \end{aligned} \quad (6.26)$$

## 7. RELATIVE PAIRING FORMULÆ AND GEOMETRIC CONSEQUENCES

Let us briefly recall some facts from the theory of boundary value problems for Dirac operators [BBWo93]. Given a Dirac operator  $D$  acting on sections of the bundle  $W$  on a compact Riemannian manifold with boundary  $(M, g)$ . In contrast to the rest of the paper  $g$  is a "true" Riemannian metric, smooth and non-degenerate up to the boundary, and not a b-metric. We assume that all structures are product near the boundary, that is there is a collar  $U = [0, \varepsilon) \times \partial M$  of the boundary such that  $g|_U = dx^2 \oplus g|_{\partial M}$  is a product metric. In particular the formulæ of Remark 1.10 hold.

We assume furthermore that we are in the even situation. That is,  $D$  is odd with respect to a  $\mathbb{Z}_2$ -grading. Then in a collar of the boundary  $D$  takes the form

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{d}{dx} + A^+ \\ \frac{d}{dx} + A^+ & 0 \end{bmatrix} = c(dx) \frac{d}{dx} + D_\partial. \quad (7.1)$$

In the matrix notation we have identified  $W^+$  and  $W^-$  via  $c(dx)$  and put  $A^+ := (c(dx)^{-1}D_\partial)|_{W^+}$ .  $A^+$  is a first order self-adjoint elliptic differential operator.

Let  $P \in \Psi^0(\partial M; W^+)$  be a pseudodifferential projection with  $P - 1_{[0,\infty)}(A^+)$  of order  $-1$ . Then we denote by  $D_P$  the operator  $D$  acting on the domain  $\{u \in L_1^2(M; W^+) \mid P(u|_{\partial M}) = 0\}$ .

$D_P$  is a Fredholm operator. The Agranovich–Dynin formula [BBWo93, Prop. 21.4]

$$\text{Ind } D_P^+ - \text{Ind } D_Q^+ = \text{Ind}(P, Q) =: \text{Ind}(P : \text{Im } Q \longrightarrow \text{Im } P) \quad (7.2)$$

expresses the difference of two such indices in terms of the relative index of the two projections  $P, Q$ .

Choosing for  $P$  the positive spectral projection  $P_+(A^+) = 1_{[0,\infty)}(A^+)$  of  $A^+$  we obtain the APS index

$$\text{Ind}_{\text{APS}} D^+ = \text{Ind } D_{P_+(A^+)}^+. \quad (7.3)$$

We shortly comment on the relative index introduced above (cf. [ASS94], [BBWo93, Sec. 15], [BrLe01, Sec. 3]). Two orthogonal projections  $P, Q$  in a Hilbert space are said to form a Fredholm pair if  $PQ : \text{Im } Q \longrightarrow \text{Im } P$  is a Fredholm operator. The index of this Fredholm operator is then called the *relative index*,  $\text{Ind}(P, Q)$ , of  $P$  and  $Q$ . It is easy to see that  $P, Q$  form a Fredholm pair if the difference  $P - Q$  is a compact operator. Furthermore, the relative index is additive

$$\text{Ind}(P, R) = \text{Ind}(P, Q) + \text{Ind}(Q, R) \quad (7.4)$$

if  $P - Q$  or  $P - R$  is compact [BBWo93, Prop. 15.15], [ASS94, Thm. 3.4]. In general, just assuming that all three pairs  $(P, Q)$ ,  $(Q, R)$  and  $(P, R)$  are Fredholm is not sufficient for (7.4) to hold.

Sometimes the spectral flow of a path of self-adjoint operators can be expressed as a relative index. The following proposition is a special case of [LES05, Thm. 3.6]:

**Proposition 7.1.** *Let  $T_s = T_0 + \tilde{T}_s$ ,  $0 \leq s \leq 1$ , be a path of self-adjoint Fredholm operators in the Hilbert space  $H$ . Assume that  $\tilde{T}_s$  is a continuous family of bounded  $T_0$ -compact operators. Then the spectral flow of  $(T_s)_{0 \leq s \leq 1}$  is given by*

$$\text{SF}(T_s)_{0 \leq s \leq 1} = -\text{Ind}(P_+(T_1), P_+(T_0)),$$

where  $P_+(T_s) := 1_{[0,\infty)}(T_s)$ .

**Remark 7.2.** If  $T_0$  is bounded then the condition of  $T_0$ -compactness just means that the  $T_s$  are compact operators. If  $T_0$  is unbounded with compact resolvent then any bounded operator is automatically  $T_0$ -compact. The second case is the one of relevance for us.

In [LES05, Thm. 3.6] Proposition 7.1 is proved for Riesz continuous paths of unbounded Fredholm operators. Since  $s \mapsto \tilde{T}_s$  is continuous the map  $s \mapsto T_0 + \tilde{T}_s$  is automatically Riesz continuous [LES05, Prop. 3.2].

Note that our sign convention for the relative index differs from that of *loc. cit.* Therefore our formulation of Proposition 7.1 differs from [LES05, Thm. 3.6] by a sign, too.

**Proposition 7.3.** *Let  $(D_s)_{0 \leq s \leq 1}$  be a smooth family of self-adjoint  $\mathbb{Z}_2$ -graded (cf. (7.1)) Dirac operators on a compact Riemannian manifold with boundary. We assume that  $D_s$  is in product form near the boundary and that  $D_s = D_0 + \Phi_s$  with a bundle endomorphism  $\Phi_s \in \Gamma^\infty(M; \text{End } V)$ . Then*

$$\text{Ind}_{\text{APS}} D_1^+ - \text{Ind}_{\text{APS}} D_0^+ = -\text{SF}(A_s^+)_{0 \leq s \leq 1}. \quad (7.5)$$

Here, as explained above,  $A_s^+ = (c(dx)^{-1} D_{\partial, s})|_{W^+}$ .

*Proof.* The family  $s \mapsto D_{s, \text{APS}}$  is not necessarily continuous. The reason is that if eigenvalues of  $A_s^+$  cross 0 the family  $P_+(A_s^+)$  of APS projections jump.

However, since  $A_s^+ - A_0^+$  is 0<sup>th</sup> order, the corresponding APS projections  $P_+(A_s^+)$  all have the same leading symbol and hence  $P_+(A_s^+) - P_+(A_{s'}^+)$  is compact for all  $s, s' \in [0, 1]$ .

Hence we can consider the family  $D_{s, P_+(A_0^+)}$ ,  $0 \leq s \leq 1$ . Now the boundary condition is fixed and thus  $s \mapsto D_{s, P_+(A_0^+)}$  is a graph continuous family of Fredholm operators [Nic95, BBF09, BBLZ09]. Therefore, its index is independent of  $s$ .

Applying the Agranovich–Dynin formula (7.2) and Proposition 7.1 we find

$$\begin{aligned} \text{Ind}_{\text{APS}}(D_1^+) &= \text{Ind } D_{1, P_+(A_0^+)}^+ + \text{Ind}(P_+(A_1^+), P_+(A_0^+)) \\ &= \text{Ind } D_{0, P_+(A_0^+)}^+ + \text{Ind}(P_+(A_1^+), P_+(A_0^+)) \\ &= \text{Ind}_{\text{APS}} D_0^+ - \text{SF}(A_s^+)_{0 \leq s \leq 1}. \quad \square \end{aligned} \quad (7.6)$$

Recall that a smooth idempotent  $p : M \rightarrow \text{Mat}_N(\mathbb{C})$  corresponds to a smooth vector bundle  $E \simeq \text{Im } p$  and using the Grassmann connection the twisted Dirac operator  $D^E$  equals  $p(D \otimes \text{Id}_N)p$ . To simplify notation we will write  $pDp$  for  $p(D \otimes \text{Id}_N)p$  whenever confusions are unlikely.

We would like to extend Proposition 7.3 to families of twisted Dirac operators of the form  $D_s = p_s D p_s$ , where  $p_s : M \rightarrow \text{Mat}_N(\mathbb{C})$  is a family of orthogonal projections.

The difficulty is that not only the leading symbol of  $D_s$  but even the Hilbert space of sections, on which the operator acts, varies.

**Proposition 7.4.** *Let  $D$  be a self-adjoint  $\mathbb{Z}_2$ -graded (cf. (7.1)) Dirac operator on a compact Riemannian manifold with boundary  $M$ . Let  $p_s : M \rightarrow \text{Mat}_N(\mathbb{C})$  be a smooth family of orthogonal projections. Assume furthermore, that in a collar neighborhood  $U = [0, \varepsilon) \times \partial M$  of  $\partial M$  we have  $p_s|_U = p_s^\partial$ , i.e.  $p_s|_U$  is independent of the normal variable. Then*

$$\text{Ind}_{\text{APS}} p_1 D^+ p_1 - \text{Ind}_{\text{APS}} p_0 D^+ p_0 = -\text{SF}(p_s A^+ p_s)_{0 \leq s \leq 1}. \quad (7.7)$$

*Proof.* By a standard trick often used in operator  $K$ -theory [BLA86, Prop. 4.3.3] we may choose a smooth path of unitaries  $u : M \rightarrow \text{Mat}_N(\mathbb{C})$  such that  $p_s = u_s p_0 u_s^*$ ,  $u_0 = \text{Id}_N$ . Furthermore, we may assume that  $u|_{[0, \varepsilon) \times \partial M} = u^\partial$  is also independent of the normal variable. Then  $p_s D^+ p_s = u_s (p_0 u_s^* D^+ u_s p_0) u_s^*$  and  $(p_s D^+ p_s)_{\text{APS}} = u_s (p_0 u_s^* D^+ u_s p_0)_{\text{APS}} u_s^*$ .

Since  $u_s^* D^+ u_s = D^+ + u_s^* c(du_s)$  Proposition 7.3 applies to the family  $p_0 u_s^* D u_s p_0$ . Since the spectral flow is invariant under unitary conjugation we reach the conclusion.  $\square$

**Definition 7.5.** In the sequel we will write somewhat more suggestively and for brevity  $\text{SF}(p, D_\partial)$  instead of  $\text{SF}(p_s A^+ p_s)_{0 \leq s \leq 1}$ .

**Theorem 7.6.** *Let  $M$  be a compact manifold with boundary and  $W$  a degree  $q$  Clifford module on  $M$ . Let  $g$  be a smooth Riemannian metric on  $M$ ,  $h$  a Hermitian metric and  $\nabla$  a unitary Clifford connection on  $W$ . Assume that all structures are product near the boundary. Let  $D = D(\nabla, g)$  be the Dirac operator.*

*Let  $[p, q, \gamma] \in K^0(M, \partial M)$  be a relative  $K$ -cycle. That is  $p, q : M \rightarrow \text{Mat}_N(\mathbb{C})$  are orthogonal projections and  $\gamma : [0, 1] \times \partial M \rightarrow \text{Mat}_N(\mathbb{C})$  is a homotopy of orthogonal projections with  $\gamma(0) = p^\partial, \gamma(1) = q^\partial$ . Then*

$$\begin{aligned} \langle [D], [p, q, \gamma] \rangle &= -\text{Ind}_{\text{APS}} p D^+ p + \text{Ind}_{\text{APS}} q D^+ q + \text{SF}(\gamma, D_\partial), \\ &= \int_M \omega_{D(\nabla, g)} \wedge (\text{ch}_\bullet(q) - \text{ch}_\bullet(p)) - \int_{\partial M} \omega_{D_\partial(\nabla, g)} \wedge \text{Tch}_\bullet(h), \end{aligned} \quad (7.8)$$

*in particular the right hand side of (7.8) depends only on the relative  $K$ -theory class  $[p, q, \gamma] \in K^0(M, \partial M)$  and the degree  $q$  Clifford module  $W$ . It is independent of  $\nabla$  and  $g$ .*

*In case all structures are b-structures, and  $D = D({}^b\nabla, g_b)$  is the b-Dirac operator, then we still have*

$$\langle [D], [p, q, \gamma] \rangle = \int_{{}^bM} {}^b\omega_D \wedge (\text{ch}_\bullet(q) - \text{ch}_\bullet(p)) - \int_{\partial M} \omega_{D_\partial} \wedge \text{Tch}_\bullet(h). \quad (7.9)$$

For the fact that relative  $K$ -cycles can be represented by triples  $[p, q, \gamma]$  as above we refer to [BLA86, Thm. 5.4.2], [HIRO00, Sec. 4.3], see also [LMP09, Sec. 1.6].

*Proof.* Denote the right hand side of (7.8) by  $I(p, q, \gamma)$ . We first show that  $I(p, q, \gamma)$  depends indeed only on the relative  $K$ -theory class of  $(p, q, \gamma)$ . By the stability of the Fredholm index we may assume that in a collar neighborhood of  $\partial M$  the projections  $p, q$  do not depend on the normal variable.

After stabilization we need to show the homotopy invariance of  $I(p, q, \gamma)$ . Now consider a homotopy  $(p_t, q_t, \gamma_t)$  of relative  $K$ -cycles. Then by Proposition 7.4 we have (cf. Figure 4, page 63)

$$\begin{aligned} &I(p_1, q_1, \gamma_1) - I(p_0, q_0, \gamma_0) \\ &= \text{Ind}_{\text{APS}}(p_1 D^+ p_1) - \text{Ind}_{\text{APS}}(q_1 D^+ q_1) - \text{SF}(\gamma_1(\cdot), D_\partial) \\ &\quad - \text{Ind}_{\text{APS}}(p_0 D^+ p_0) + \text{Ind}_{\text{APS}}(q_0 D^+ q_0) + \text{SF}(\gamma_0(\cdot), D_\partial) \\ &= -\text{SF}(\gamma_1(0), D_\partial) + \text{SF}(\gamma_1(1), D_\partial) - \text{SF}(\gamma_1(\cdot), D_\partial) + \text{SF}(\gamma_0(\cdot), D_\partial) \\ &= 0, \end{aligned}$$

by the homotopy invariance of the Spectral Flow.

So the l.h.s and the r.h.s. of (7.8) depend only on the relative  $K$ -theory class of  $[p, q, \gamma]$ . By excision in  $K$ -theory (it can of course be shown elementary exploiting

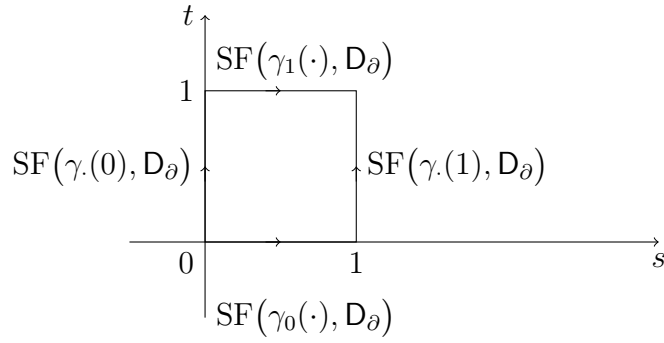


FIGURE 4. By homotopy invariance the signed sum of the four spectral flows adds up to 0.

Swan's Theorem) every relative  $K$ -theory class can even be represented by a triple  $(p, q, p|_{\partial M})$  such that  $p|_{[0,\varepsilon)\times\partial M} = q|_{[0,\varepsilon)\times\partial M}$  and hence  $\gamma(s) = p|_{\partial M}$  is constant.

Then the twisted version of the APS Index Theorem gives

$$\text{Ind}_{\text{APS}} qD^+q - \text{Ind}_{\text{APS}} pD^+p = \int_M \omega_D \wedge (\text{ch}_\bullet(q) - \text{ch}_\bullet(p)), \quad (7.10)$$

where  $\omega_D$  denotes the local index density of  $D$ . Note that since the tangential operators of  $pD^+p$  and of  $qD^+q$  coincide the  $\eta$ -terms cancel.

As outlined in Section 1.8 (*cf.* also the proof of Theorem 6.1) the Connes–Chern character of  $[D]$  in  $HP^\bullet(\mathcal{J}^\infty(\partial M, M)) \simeq H_\bullet^{\text{dR}}(M \setminus \partial M; \mathbb{C})$  is represented by  $\int_M \omega_D$ . By construction, the form  $\text{ch}_\bullet(q) - \text{ch}_\bullet(p)$  is compactly supported in  $M \setminus \partial M$ . Thus the right hand side of (7.10) equals the pairing  $\langle [D], [p, q, p|_{\partial M}] \rangle$  and the first equality in (7.8) is proved.

To prove the second equality in (7.8) we note that it represents the Poincaré duality pairing between the de Rham cohomology class of  $\omega_{D(\nabla, g)}$  (note  $\iota^* \omega_D = \omega_{D_\partial}$ ) and the relative de Rham cohomology class of the pair of forms  $(\text{ch}_\bullet(q) - \text{ch}_\bullet(p), \text{Tch}_\bullet(h))$ . Hence it depends only on the class  $[p, q, h] \in K^0(M, \partial M)$  and on  $[D]$ . In the situation above where  $p$  and  $q$  coincide in a collar of the boundary it equals  $\langle [D], [p, q, \gamma] \rangle$  and hence by homotopy invariance the claim is proved in general up to Eq. (7.9).

For the proof of Eq. (7.9) note first that for a closed even b-differential form  $\omega$  the map (*cf.* Definition and Proposition 1.5)

$$\Omega^k(M) \oplus \Omega^{k-1}(\partial M) \rightarrow \mathbb{C}, \quad (\eta, \tau) \mapsto \int_{\text{b}M} \omega \wedge \eta - \int_{\partial M} \iota^* \omega \wedge \tau$$

descends naturally to a linear form on  $H_{\text{dR}}^k(M, \partial M; \mathbb{C})$ . Hence the right hand side of Eq. (7.9) is well-defined and depends only on the class of  $[p, q, h] \in K^0(M, \partial M)$ . As before we may therefore specialize to  $(p, q, p|_{\partial M})$  such that  $p|_{[0,\varepsilon)\times\partial M} = q|_{[0,\varepsilon)\times\partial M}$ . Then  $\text{ch}_\bullet(q) - \text{ch}_\bullet(p)$  has compact support in  $M \setminus \partial M$  and the remaining claim follows from Theorem 6.1 (4).  $\square$

We now proceed to express the pairing between relative  $K$ -theory classes and the fundamental relative  $K$ -homology class in cohomological terms. We assume here that we are in the b-setting.

Recall that a relative K-theory class in  $K^0(M, \partial M)$  is represented by a pair of bundles  $(E, F)$  over  $M$  whose restrictions  $E_\partial, F_\partial$  to  $\partial M$  are related by a homotopy  $h$ . We will explicitly write the formulæ in the even dimensional case and only point out where the odd dimensional case is different.

The Chern character of  $[E, F, h] \in K^0(M, \partial M)$  is then represented by the relative cyclic homology class

$$\mathrm{ch}_\bullet([E, F, h]) = \left( \mathrm{ch}_\bullet(p_F) - \mathrm{ch}_\bullet(p_E), -\mathrm{Tch}_\bullet(h) \right), \quad (7.11)$$

cf. Eq. (1.17).

By Theorem 6.1 we have for any  $t > 0$

$$\begin{aligned} \langle [D], [E, F, h] \rangle &= \langle (\mathrm{bch}_t^n(D), \mathrm{ch}_t^{n+1}(D_\partial)), \mathrm{ch}_\bullet([E, F, h]) \rangle \\ &= \left\langle \sum_{j \geq 0} \mathrm{bCh}^{n-2j}(tD) + B \mathrm{bTch}_t^{n+1}(D), \mathrm{ch}_\bullet(p_F) - \mathrm{ch}_\bullet(p_E) \right\rangle \\ &\quad - \left\langle \sum_{j \geq 0} \mathrm{Ch}^{n-2j+1}(tD_\partial) + B \mathrm{Tch}_t^{n+2}(D_\partial), \mathrm{Tch}_\bullet(h) \right\rangle. \end{aligned} \quad (7.12)$$

Letting  $t \searrow 0$  yields, again by Theorem 6.1, the local form of the pairing:

$$\begin{aligned} \langle [D], [E, F, h] \rangle &= \int_{\mathrm{b}_M} \omega_D \wedge (\mathrm{ch}_\bullet(p_F) - \mathrm{ch}_\bullet(p_E)) - \int_{\partial M} \omega_{D_\partial}(\partial M) \wedge \mathrm{Tch}_\bullet(h). \end{aligned} \quad (7.13)$$

If  $D_\partial$  is invertible then, at the opposite end, letting  $t \nearrow \infty$  gives in view of Theorem 6.2

$$\begin{aligned} \langle [D], [E, F, h] \rangle &= \left\langle \sum_{0 \leq k \leq \ell} \kappa^{2k}(D) + B \mathrm{bTch}_\infty^{n+1}(D), \mathrm{ch}_\bullet(p_F) - \mathrm{ch}_\bullet(p_E) \right\rangle \\ &\quad - \left\langle B \mathrm{Tch}_\infty^{n+2}(D_\partial), \mathrm{Tch}_\bullet(h) \right\rangle, \end{aligned} \quad (7.14)$$

where  $2\ell = n$ .

By equating the above two limit expressions (7.13) and (7.14) one obtains the following identity:

**Corollary 7.7.** *Let  $n = 2\ell \geq m$  and assume that  $D_\partial$  is invertible. Then*

$$\begin{aligned} &\langle \kappa^0(D), p_F - p_E \rangle + \\ &\quad + \sum_{1 \leq k \leq \ell} (-1)^k \frac{(2k)!}{k!} \left\langle \kappa^{2k}(D), (p_F - \frac{1}{2}) \otimes p_F^{\otimes 2k} - (p_E - \frac{1}{2}) \otimes p_E^{\otimes 2k} \right\rangle \\ &= \int_{\mathrm{b}_M} \omega_D \wedge (\mathrm{ch}_\bullet(p_F) - \mathrm{ch}_\bullet(p_E)) - \int_{\partial M} \omega_{D_\partial} \wedge \mathrm{Tch}_\bullet(h) \\ &\quad - (-1)^{n/2} \frac{n!}{(n/2)!} \left\langle B \mathrm{bTch}_\infty^{n+1}(D), (p_F - \frac{1}{2}) \otimes p_F^{\otimes n} - (p_E - \frac{1}{2}) \otimes p_E^{\otimes n} \right\rangle \\ &\quad + \left\langle B \mathrm{Tch}_\infty^{n+2}(D_\partial), \mathrm{Tch}_{n+1}(h) \right\rangle. \end{aligned} \quad (7.15)$$

The left hand side plays the role of a ‘higher’ relative index, while the right hand side contains local geometric terms and ‘higher’ eta cochains.

The pairing formula acquires a simpler form if one chooses special representatives for the class  $[E, F, h]$ . For example, one can always assume that  $E_{\partial} = F_{\partial}$ , in which case one obtains

$$\begin{aligned} \langle [D], [E, F, h_0] \rangle &= \left\langle \sum_{0 \leq k \leq \ell} \kappa^{2k}(\mathbb{D}) + B^b \text{Tch}_{\infty}^{n+1}(\mathbb{D}), \text{ch}_{\bullet}(p_F) \right\rangle \\ &\quad - \left\langle \sum_{0 \leq k \leq \ell} \kappa^{2k}(\mathbb{D}) + B^b \text{Tch}_{\infty}^{n+1}(\mathbb{D}), \text{ch}_{\bullet}(p_E) \right\rangle. \end{aligned} \quad (7.16)$$

Specializing even more, one can assume  $F = \mathbb{C}^N$ . Then the pairing formula becomes

$$\begin{aligned} \langle [D], [E, \mathbb{C}^N, h_0] \rangle &= - \left\langle \sum_{0 \leq k \leq \ell} \kappa^{2k}(\mathbb{D}) + B^b \text{Tch}_{\infty}^{n+1}(\mathbb{D}), \text{ch}_{\bullet}(p_E) \right\rangle \\ &\quad + N(\dim \text{Ker } D^+ - \dim \text{Ker } D^-). \end{aligned} \quad (7.17)$$

On the other hand, applying Theorem 7.6

$$\langle [D], [E, \mathbb{C}^N, h_0] \rangle = - \text{Ind}_{\text{APS}}(p_E D^+ p_E) + N \text{Ind}_{\text{APS}} D^+,$$

one obtains an index formula for the  $b$ -Dirac operator which is the direct analogue of Eq. (3.4) in [CoMo93]:

**Corollary 7.8.** *Let  $E$  be a vector bundle on  $M$  whose restriction to  $\partial M$  is trivial and assume  $D_{\partial}$  to be invertible. Then for any  $n = 2\ell \geq m$*

$$\text{Ind}_{\text{APS}}(p_E D^+ p_E) = \left\langle \sum_{0 \leq k \leq \ell} \kappa^{2k}(\mathbb{D}) + B^b \text{Tch}_{\infty}^{n+1}(\mathbb{D}), \text{ch}_{\bullet}(p_E) \right\rangle. \quad (7.18)$$

The expression  $\text{Ind}_{\text{APS}}(p_E D^+ p_E)$  is to be understood as follows: if  $p_E^{\partial} D_{\partial} p_E^{\partial}$  is invertible, then it is the Fredholm index of  $p_E D^+ p_E$ . If  $p_E D^+ p_E$  is not Fredholm, then chose a metric  $\tilde{g}$  smooth up to the boundary and construct on the Clifford module of  $\mathbb{D}$  the Dirac operator  $\tilde{D}$  to the Riemannian metric  $\tilde{g}$ . Then, by Theorem. 6.1 the Connes–Chern characters of  $\tilde{D}$  and  $D$  coincide and thus

$$\langle [D], [E, \mathbb{C}^N, h_0] \rangle = - \text{Ind}_{\text{APS}} p_E \tilde{D}^+ p_E + N \text{Ind}_{\text{APS}} \tilde{D}^+.$$

As a by-product of the above considerations, we can now establish the following generalization of the Atiyah–Patodi–Singer odd-index theorem for trivialized flat bundles (comp. [APS76, Prop. 6.2, Eq. (6.3)].

**Corollary 7.9.** *Let  $N$  be a closed odd dimensional spin manifold, and let  $E', F'$  be two vector bundles which are equivalent in  $K$ -theory via a homotopy  $h$ . With  $D_{g'}$  denoting the Dirac operator associated to a Riemannian metric  $g'$  on  $N$ , one has*

$$\xi(D_{g'}^{F'}) - \xi(D_{g'}^{E'}) = \int_N \hat{A}(\nabla_{g'}^2) \wedge \text{Tch}_{\bullet}(h) + \text{SF}(h, D_{g'}), \quad (7.19)$$

or equivalently,

$$\int_0^1 \frac{1}{2} \frac{d}{dt} (\eta(p_{h(t)} D_{g'} p_{h(t)})) dt = \int_N \hat{A}(\nabla_{g'}^2) \wedge \text{Tch}_{\bullet}(h), \quad (7.20)$$

where  $p_{h(t)}$  is the path of projections joining  $E'$  and  $F'$ .

*Proof.* This follows from equating the local pairing (7.8) and the relative APS index formula (0.17) after the following modifications. We recall that Eq. (7.8) holds in complete generality without invertibility hypothesis on  $D_\partial$ . First by passing to a multiple one can assume that  $N = \partial M$ . Then by adding a complement  $G'$  we can replace  $F'$  by a trivial bundle. Then both  $E' \oplus G'$  and  $F' \oplus G'$  extend to  $M$ . It remains to notice that both sides of the formula (7.19) are additive.

The alternative formulation (7.20) follows immediately from the known relation (see e.g. [KIL04, Lemma 3.4])

$$\xi(D_{g'}^{F'}) - \xi(D_{g'}^{E'}) = \text{SF}(h, D_\partial) + \int_0^1 \frac{1}{2} \frac{d}{dt} (\eta(p_{h(t)} D_{g'} p_{h(t)})) dt. \quad \square$$

In the odd dimensional case the pairing formulæ are similar, except that the contribution from the kernel of  $D$  does not occur. Let  $(U, V, h)$  be a representative of an odd relative K-theory class where  $U, V : M \rightarrow U(N)$  are unitaries and  $h$  is a homotopy between  $U_{\partial M}$  and  $V_{\partial M}$ . Then

$$\begin{aligned} \langle [D], [U, V, h] \rangle &= \langle B^b \text{Tch}_\infty^{n+1}(D), \text{ch}_n(U) - \text{ch}_n(V) \rangle \\ &\quad - \langle B \text{Tch}_\infty^{n+2}(D_\partial), \text{Tch}_{n+1}(h) \rangle. \end{aligned} \quad (7.21)$$

Choosing a representative of the class with  $U_{\partial M} = V_{\partial M}$ , the above formula simplifies to

$$\langle [D], [U, V, h_0] \rangle = \langle B^b \text{Tch}_\infty^{n+1}(D), \text{ch}_n(U) - \text{ch}_n(V) \rangle, \quad (7.22)$$

and if moreover one takes  $V = \text{Id}$ , it reduces to

$$\langle [D], [U, \text{Id}, h_0] \rangle = \langle B^b \text{Tch}_\infty^{n+1}(D), \text{ch}_n(U) \rangle. \quad (7.23)$$

Finally, the equality between the local form of the pairing (7.13) and the expression (7.21) gives the following odd analogue of Corollary 7.7.

**Corollary 7.10.** *Let  $n \geq m$ , both odd and assume that  $D_\partial$  is invertible. Then*

$$\begin{aligned} &\langle B^b \text{Tch}_\infty^{n+1}(D), \text{ch}_n(U) \rangle - \langle B^b \text{Tch}_\infty^{n+1}(D), \text{ch}_n(V) \rangle = \\ &= \int_{b_M} \hat{A}({}^b \nabla_g^2) \wedge (\text{ch}_\bullet(U) - \text{ch}_\bullet(V)) - \int_{\partial M} \hat{A}(\nabla_{g'}^2) \wedge \text{Tch}_\bullet(h) \\ &\quad - \langle B \text{Tch}_\infty^{n+2}(D_\partial), \text{Tch}_{n+1}(h) \rangle. \end{aligned} \quad (7.24)$$

**7.1. Relation with the generalized APS pairing.** Wu [WU93] showed that the full cochain  $\eta^\bullet(D_\partial)$  has a finite radius of convergence, proportional to the lowest eigenvalue of  $|D_\partial|$  (assumed to be invertible). Both Wu and Getzler [GET93A] proved, by different methods, the following generalized Atiyah-Patodi-Singer index formula :

$$\text{Ind}_{\text{APS}} D^E = \int_{b_M} \hat{A}({}^b \nabla_g^2) \wedge \text{ch}_\bullet(p_E) + \langle \eta^\bullet(D_\partial) \circ i^*, \text{ch}_\bullet(p_E) \rangle, \quad (7.25)$$

for any vector bundle  $E = \text{Im } p_E$  over  $M$  whose restriction to the boundary satisfies the *almost  $\partial$ -flatness* condition  $\| [D_\partial, r_\partial(p_E)] \| < \lambda_1(|D_\partial|)$ . Their result does provide a decoupled index pairing, but only for those classes in  $K^m(M, \partial M)$  which can be

represented by pairs of almost  $\partial$ -flat bundles. Furthermore, if  $(E, F, h)$  is such a triple, on applying (7.25) and Theorem 7.6 one obtains

$$\begin{aligned} \langle [D], [E, F, h] \rangle &= \int_{\mathfrak{b}_M} \hat{A}(\mathfrak{b}\nabla_g^2) \wedge (\text{ch}_\bullet(p_F) - \text{ch}_\bullet(p_E)) \\ &\quad + \langle \eta^\bullet(D_\partial), i^*(\text{ch}_\bullet(p_F) - \text{ch}_\bullet(p_E)) \rangle + \text{SF}(h, D_\partial), \end{aligned} \quad (7.26)$$

where  $\text{SF}(h, D_\partial)$  is an abbreviation for  $\text{SF}(h(s)A^+h(s))_{0 \leq s \leq 1}$ , cf. also Eq. (7.1). By [Wu93, Proof of Thm. 3.1],

$$(b + B)\eta^\bullet(D_\partial) = - \int_{\partial M} \hat{A}(\nabla_{g'}^2) \wedge -.$$

At the formal level

$$\begin{aligned} \langle \eta^\bullet(D_\partial) \circ i^*, \text{ch}_\bullet(p_F) - \text{ch}_\bullet(p_E) \rangle &= \langle \eta^\bullet(D_\partial), (b + B) \text{Tch}_\bullet(h) \rangle \\ &= \langle (b + B)\eta^\bullet(D_\partial), \text{Tch}_\bullet(h) \rangle = - \int_{\partial M} \hat{A}(\nabla_{g'}^2) \wedge \text{Tch}_\bullet(h). \end{aligned} \quad (7.27)$$

However, to ensure that the pairing  $\langle \eta^\bullet(D_\partial), (b + B) \text{Tch}_\bullet(h) \rangle$  makes sense one has to assume that  $p_{h(t)}$  satisfy the same almost  $\partial$ -flatness condition. Then

$$\text{Ker}(p_{h(t)}D_\partial p_{h(t)}) = 0,$$

hence there is no spectral flow along the path.

Thus, the total eta cochain disappears and (7.27) together with (7.26) just lead to the known local pairing formula, cf. Eq. (7.8),

$$\begin{aligned} \langle [D], [E, F, h] \rangle &= \int_{\mathfrak{b}_M} \hat{A}(\mathfrak{b}\nabla_g^2) \wedge (\text{ch}_\bullet(p_F) - \text{ch}_\bullet(p_E)) - \int_{\partial M} \hat{A}(\nabla_{g'}^2) \wedge \text{Tch}_\bullet(h). \end{aligned}$$

**Remark 7.11.** The fact that the total eta cochain is not entire is not an artifact of the method used to estimate its radius of convergence. If it was entire, then  $\text{SF}(h, D) = 0$  for any  $h(t)$ , which is easy to disprove by a counterexample.

## APPENDIX A. THE $\mathfrak{b}$ -CALCULUS FOR MANIFOLDS WITH BOUNDARIES

For the convenience of the reader we provide here a brief review of the basic elements of the  $\mathfrak{b}$ -calculus for manifolds with boundaries invented by Melrose. For further details we refer the reader to the monograph [MEL93] and the article [LOY05].

**A.1. Exact  $\mathfrak{b}$ -metrics and  $\mathfrak{b}$ -functions on cylinders.** Let  $M$  be a compact manifold with boundary of dimension  $m$ , let  $\partial M$  be its boundary, and denote by  $M^\circ$  its interior  $M \setminus \partial M$ . Then choose a collar for  $M$  which means a diffeomorphism of the form  $(r, \eta) : Y \rightarrow [0, 2) \times \partial M$ , where  $Y \subset M$  is an open neighborhood of  $\partial M = r^{-1}(0)$ . The map  $r : Y \rightarrow [0, 2)$  is called the *boundary defining function* of the collar, the submersion  $\eta : Y \rightarrow \partial M$  its *boundary projection*. For  $s \in (0, 2)$  denote by  $Y^s$  the open subset  $r^{-1}([0, s))$ , put  $Y^{os} := r^{-1}((0, s))$  and finally let  $M^s := M \setminus Y^s$  and  $\overline{M^s} := M \setminus Y^{os}$ ; likewise  $Y^\circ := Y \setminus \partial M$ . Next, let  $x : Y \rightarrow \mathbb{R}$  be the smooth function  $x := \ln \circ r$ . Then  $(x, \eta) : Y^{\circ 3/2} \rightarrow (-\infty, \ln \frac{3}{2}) \times \partial M$  is a diffeomorphism of  $Y^{3/2}$  onto a cylinder.

After having fixed these data for  $M$ , we choose the most essential ingredient for the b-calculus, namely an exact b-metric for  $M$ . Following [MEL93], one understands by this a Riemannian metric  $g_b$  on  $M^\circ$  such that on  $Y^\circ$ , the metric can be written in the form

$$g_{b|Y^\circ} = \frac{1}{r_{|Y^\circ}^2} (dr \otimes dr)_{|Y^\circ} + \eta_{|Y^\circ}^* g_\partial, \quad (\text{A.1})$$

where  $g_\partial$  is a Riemannian metric on the boundary  $\partial M$ . Clearly, one then has in the cylindrical coordinates  $(x, \eta)$

$$g_{b|Y^\circ} = (dx \otimes dx)_{|Y^\circ} + \eta_{|Y^\circ}^* g_\partial. \quad (\text{A.2})$$

Now consider the cylinder  $\mathbb{R} \times \partial M := \mathbb{R} \times \partial M$  together with the product metric

$$g_{\text{cyl}} = dx \otimes dx + \text{pr}_2^* g_\partial, \quad (\text{A.3})$$

where here (with a slight abuse of language),  $x$  denotes the first coordinate of the cylinder, and  $\text{pr}_2 : \mathbb{R} \times \partial M \rightarrow \partial M$  the projection onto the second factor.

Next we introduce various algebras of so-called b-functions on  $\mathbb{R} \times \partial M$ . For  $c \in \mathbb{R}$  define  ${}^b\mathcal{C}^\infty((-\infty, c) \times \partial M)$  resp.  ${}^b\mathcal{C}^\infty((c, \infty) \times \partial M)$  as the algebra of smooth functions  $f$  on  $(-\infty, c) \times \partial M$  resp. on  $(c, \infty) \times \partial M$  for which there exist functions  $f_0^-, f_1^-, f_2^-, \dots \in \mathcal{C}^\infty(\partial M)$  resp.  $f_0^+, f_1^+, f_2^+, \dots \in \mathcal{C}^\infty(\partial M)$  such that the following asymptotic expansions hold true:

$$\begin{aligned} f &\sim_{x \rightarrow -\infty} f_0^- + f_1^- e^x + f_2^- e^{2x} + \dots \quad \text{resp.} \\ f &\sim_{x \rightarrow \infty} f_0^+ + f_1^+ e^{-x} + f_2^+ e^{-2x} + \dots \end{aligned} \quad (\text{A.4})$$

More precisely, this means that there exists for every  $k, l \in \mathbb{N}$  and every differential operator  $D$  on  $\partial M$  a constant  $C > 0$  such that

$$\begin{aligned} \left| \partial_x^l Df(x, p) - 0^l Df_0^-(p) - \dots - k^l Df_k^-(p) e^{kx} \right| &\leq C e^{(k+1)x} \\ &\text{for all } x \leq c - 1 \text{ and } p \in \partial M \text{ resp.} \\ \left| \partial_x^l Df(x, p) - 0^l Df_0^+(p) - \dots - (-k)^l Df_k^+(p) e^{-kx} \right| &\leq C e^{-(k+1)x} \\ &\text{for all } x \geq c + 1 \text{ and } p \in \partial M. \end{aligned} \quad (\text{A.5})$$

The asymptotic expansion guarantees that  $f \in {}^b\mathcal{C}^\infty((-\infty, c) \times \partial M)$  if and only if the transformed function  $[0, e^c[ \times \partial M \ni (r, p) \mapsto f(\ln r, p)$  is a smooth function on the collar  $[0, e^c[ \times \partial M$ .

The algebra  ${}^b\mathcal{C}^\infty(\mathbb{R} \times \partial M)$  of b-functions on the full cylinder consists of all smooth functions  $f$  on  $\mathbb{R} \times \partial M$  such that

$$f_{|(-\infty, 0) \times \partial M} \in {}^b\mathcal{C}^\infty((-\infty, 0) \times \partial M) \quad \text{and} \quad f_{|(0, \infty) \times \partial M} \in {}^b\mathcal{C}^\infty((0, \infty) \times \partial M).$$

Next, we define the algebras of b-functions with compact support on the cylindrical ends by  ${}^b\mathcal{C}_{\text{cpt}}^\infty((-\infty, c) \times \partial M) :=$

$$\{f \in {}^b\mathcal{C}^\infty((-\infty, c) \times \partial M) \mid f(x, p) = 0 \text{ for } x \geq c - \varepsilon, p \in \partial M \text{ and some } \varepsilon > 0\}$$

resp. by  ${}^b\mathcal{C}_{\text{cpt}}^\infty((c, \infty) \times \partial M) :=$

$$\{f \in {}^b\mathcal{C}^\infty((c, \infty) \times \partial M) \mid f(x, p) = 0 \text{ for } x \leq c + \varepsilon, p \in \partial M \text{ and some } \varepsilon > 0\}.$$

The essential property of the thus defined algebras of b-functions is that the coordinate system  $(x, \eta) : Y^{\circ 3/2} \rightarrow (-\infty, \ln 3/2) \times \partial M$  induces an isomorphism

$$(x, \eta)^* : {}^b\mathcal{C}^\infty((-\infty, \ln 3/2) \times \partial M) \rightarrow \mathcal{C}^\infty(Y^{\frac{3}{2}}), \quad (\text{A.6})$$

which is defined by putting

$$(x, \eta)^* f := \left( Y^{\frac{3}{2}} \ni p \mapsto \begin{cases} f(x(p), \eta(p)), & \text{if } p \notin \partial M, \\ f_0^-(\eta(p)), & \text{if } p \in \partial M. \end{cases} \right),$$

for all  $f \in {}^b\mathcal{C}^\infty((-\infty, \ln 3/2) \times \partial M)$ . Under this isomorphism,  ${}^b\mathcal{C}_{\text{cpt}}^\infty((-\infty, 3/2) \times \partial M)$  is mapped onto  $\mathcal{C}_{\text{cpt}}^\infty(Y^{\frac{3}{2}})$ . In this article, we will use the isomorphism  $(x, \eta)^*$  to obtain essential information about solutions of boundary value problems on  $M$  by transforming the problem to the cylinder over the boundary and then performing computations there with b-functions on the cylinder.

The final class of b-functions used in this work is the algebra  ${}^b\mathcal{S}(\mathbb{R} \times \partial M)$  of *exponentially fast decreasing functions* or *b-Schwartz test functions* on the cylinder defined as the space of all smooth functions  $f \in \mathcal{C}^\infty(\mathbb{R} \times \partial M)$  such that for all  $l, n \in \mathbb{N}$ ,  $D \in \text{Diff}(\partial M)$  there exists a  $C_{l,D,n} > 0$  such that

$$|\partial_x^l D f(x, p)| \leq C_{l,D,n} e^{-n|x|} \quad \text{for all } x \in \mathbb{R} \text{ and } p \in \partial M. \quad (\text{A.7})$$

Obviously,  $(x, y)^*$  maps  ${}^b\mathcal{S}(\mathbb{R} \times \partial M) \cap {}^b\mathcal{C}_{\text{cpt}}^\infty((-\infty, \ln 3/2) \times \partial M)$  onto the function space  $\mathcal{J}^\infty(\partial M, Y^{\frac{3}{2}}) \cap \mathcal{C}_{\text{cpt}}^\infty(Y^{\frac{3}{2}})$ .

**A.2. Global symbol calculus for pseudodifferential operators.** In this section, we briefly recall the global symbol for pseudodifferential operators which was introduced by Widom in [WID80] (see also [FUK88, PFL98]). We assume that  $(M_0, g)$  is a Riemannian manifold (without boundary), and that  $\pi_E : E \rightarrow M_0$  and  $\pi_F : F \rightarrow M_0$  are smooth vector bundle carrying a Hermitian metric  $\mu_E$  resp.  $\mu_F$ . In later applications,  $M_0$  will be the interior of a given manifold with boundary  $M$ .

Recall that there exists an open neighborhood  $\Omega_0$  of the diagonal in  $M_0 \times M_0$  such that each two points  $p, q \in M_0$  can be joined by a unique geodesic. Let  $\alpha_0$  be a *cut-off function* for  $\Omega_0$  which means a smooth map  $M_0 \times M_0 \rightarrow [0, 1]$  which has support in  $\Omega_0$  and is equal to 1 on a neighborhood of the diagonal. These data give rise to the map

$$\Phi : M \times M \rightarrow TM, \quad (p, q) \mapsto \begin{cases} \alpha_0(p, q) \exp_p^{-1}(q) & \text{if } (p, q) \in \Omega_0, \\ 0 & \text{else,} \end{cases} \quad (\text{A.8})$$

which is called a *connection-induced linearization* (cf. [FUK88]). Next denote for  $(p, q) \in \Omega_0$  by  $\tau_{p,q}^E : E_p \rightarrow E_q$  the parallel transport in  $E$  along the geodesic joining  $p$  and  $q$ . This gives rise to the map

$$\tau^E : E \times M \rightarrow E, \quad (e, q) \mapsto \begin{cases} \alpha_0(\pi_E(e), q) \tau_{\pi_E(e), q}^E(e), & \text{if } (\pi_E(e), q) \in \Omega_0, \\ 0 & \text{else,} \end{cases} \quad (\text{A.9})$$

which is called a *connection-induced local transport* on  $E$ . (cf. [FUK88]).

Next let us define the symbol spaces  $\mathcal{S}^m(T^*M_0; \pi_{T^*M}^*E)$ . For fixed  $m \in \mathbb{R}$  this space consists of all smooth sections  $a : T^*M_0 \rightarrow \pi_{T^*M_0}^*E$  such that in local coordinates  $x : U \rightarrow \mathbb{R}^{\dim M_0}$  of over  $U \subset M_0$  open and vector bundle coordinates  $(x, \eta) : E|_U \rightarrow \mathbb{R}^{\dim M_0 + \dim_{\mathbb{R}} E}$  the following estimate holds true for each compact  $K \subset U$  and appropriate  $C_K > 0$  depending on  $K$ :

$$\left\| \partial_x^\alpha \partial_\xi^\beta (\eta \circ a)(\xi) \right\| \leq C_K (1 + \|T^*x(\xi)\|)^{m-|\beta|} \quad \text{for all } \xi \in T_{|K}^*M_0. \quad (\text{A.10})$$

Given a symbol  $a \in \mathcal{S}^m(T^*M_0; \pi_{T^*M}^* \text{Hom}(E, F))$  one defines now a pseudodifferential operator  $\text{Op}(a) \in \Psi^m(M_0; E, F)$  by

$$\begin{aligned} (\text{Op}(a) u)(p) &:= \\ &:= \frac{1}{(2\pi)^{\dim M}} \int_{T_p M_0 \times T_p^* M_0} \alpha_0(p, \exp v) e^{-i\langle v, \xi \rangle} a(p, \xi) \tau^E(u(\exp v), p) dv d\xi, \end{aligned} \quad (\text{A.11})$$

where  $u \in \Gamma_{\text{cpt}}^\infty(E)$  and  $p \in M_0$ . Moreover, there is a quasi-inverse, the symbol map  $\sigma : \Psi^m(M_0; E, F) \rightarrow \mathcal{S}^m(T^*M_0; \pi_{T^*M}^* \text{Hom}(E, F))$  which is defined by

$$\sigma(A)(\xi)(e) := A\left(\alpha_0(p, -) \tau^E(e, -) e^{i\langle \xi, \Phi(p, -) \rangle}\right)(\pi(\xi)), \quad (\text{A.12})$$

where  $p \in M_0$ ,  $\xi \in T_p^*M_0$ ,  $e \in E_p$ . It is a well-known result from global symbol calculus (cf. [WID80, FUK88, PFL98]) that the map  $\text{Op}$  maps  $\mathcal{S}^{-\infty}(T^*M_0; \pi_{T^*M}^* \text{Hom}(E, F))$  onto  $\Psi^{-\infty}(M_0; E, F)$  and that up to these spaces,  $\text{Op}$  and  $\sigma$  are inverse to each other.

**A.3. Classical b-pseudodifferential operators.** Let us explain in the following the basics of the (small) calculus of b-pseudodifferential operators on a manifold with boundary  $M$ . In our presentation, we lean on the approach [LOY05]. For more details on the original approach confer [MEL93].

In this section, we assume that  $M$  carries a b-metric denoted by  $g_b$ . Furthermore, let  $\pi_E : E \rightarrow M$  and  $\pi_F : F \rightarrow M$  be two smooth Hermitian vector bundles over  $M$ , and fix metric connections  $\nabla^E$  and  $\nabla^F$ . Then observe that by the Schwartz Kernel Theorem there is an isomorphism between bounded linear maps

$$A : \mathcal{J}^\infty(\partial M, M; E) \rightarrow \mathcal{J}^\infty(\partial M, M; F)'$$

and the strong dual  $\mathcal{J}^\infty(\partial(M \times M), M \times M; E \boxtimes F)'$ , where  $\mathcal{J}^\infty(\partial M, M; E) := \mathcal{J}^\infty(\partial M, M) \cdot \mathcal{C}^\infty(M; E)$ . This isomorphism is given by

$$A \mapsto K_A := \left( \mathcal{J}^\infty(\partial M, M; E) \hat{\otimes} \mathcal{J}^\infty(\partial M, M; F) \ni (u \otimes v) \mapsto \langle Au, v \rangle \right), \quad (\text{A.13})$$

where we have used that

$$\mathcal{J}^\infty(\partial(M \times M), M \times M; E \boxtimes F) = \mathcal{J}^\infty(\partial M, M; E) \hat{\otimes} \mathcal{J}^\infty(\partial M, M; F)$$

with  $\hat{\otimes}$  denoting the completed bornological tensor product. The b-volume form  $\mu_b$  associated to  $g_b$  gives rise to an embedding

$$\begin{aligned} \mathcal{M}^\infty(\partial M, M; E' \otimes F) &\hookrightarrow \mathcal{J}^\infty(\partial(M \times M), M \times M; E \boxtimes F)', \\ k &\mapsto \left( u \otimes v \mapsto \int_{M \times M} \langle k(p, q), u(p) \otimes v(q) \rangle d(\mu_b \otimes \mu_b)(p, q) \right), \end{aligned} \quad (\text{A.14})$$

which we use implicitly throughout this work. In the formula for the embedding,  $\langle -, - \rangle$  denotes the natural pairing of an element of a vector bundle with an element of the dual bundle over the same base point,  $u, v$  are elements of  $\mathcal{J}^\infty(\partial M, M; E)$  and  $\mathcal{J}^\infty(\partial M, M; F')$  respectively, and  $\mathcal{M}^\infty(X, M; E)$  denotes for  $X \subset M$  closed the space of all sections  $u \in \Gamma^\infty(M \setminus X; E)$  such that in local coordinates  $(y, \eta) : \pi_E^{-1}(U) \rightarrow \mathbb{R}^{\dim M + \dim E}$  with  $U \subset M$  open one has for every compact  $K \subset U$ ,  $p \in K \setminus X$ , and  $\alpha \in \mathbb{N}^{\dim M}$  an estimate of the form

$$\|\partial_y^\alpha(\eta \circ u)(p)\| \leq C \frac{1}{\left(d(y(p), y(X \cap U))\right)^\lambda},$$

where  $C > 0$  and  $\lambda > 0$  depend only on the local coordinate system,  $K$ , and  $\alpha$ . The fundamental property of  $\mathcal{M}^\infty(X, M; E)$  is that

$$\mathcal{J}^\infty(X, M) \cdot \mathcal{M}^\infty(X, M; E) \subset \mathcal{J}^\infty(X, M; E).$$

Note that the vector bundle  $E$  (and likewise the vector bundle  $F$ ) gives rise to a pull-back vector bundle  $\text{pr}_{\partial M}^* E|_{\partial M}$  on the cylinder, where  $\text{pr}_{\partial M} : \mathbb{R} \times \partial M \rightarrow \partial M$  is the canonical projection. This pull-back vector bundle will be denoted by  $E$  (resp.  $F$ ), too. As further preparation we introduce two auxiliary functions  $\psi : M \rightarrow [0, 1]$  and  $\varphi : M \rightarrow [0, 1]$  on  $M$  which are smooth and satisfy  $\text{supp } \psi \subset\subset M^1$ ,  $\psi(p) = 1$  for  $p \in M^{3/2}$ ,  $\text{supp } \varphi \subset\subset Y^1$ , and finally  $\varphi(p) = 1$  for  $p \in Y^{1/2}$ . Such a pair of functions will be called a *pair of auxiliary cut-off functions*.

By a b-pseudodifferential operator of order  $m \in \mathbb{R}$  we now understand a continuous operator  $A : \mathcal{J}^\infty(\partial M, M; E) \rightarrow \mathcal{J}^\infty(\partial M, M; F)'$  such that for one (and hence for all) pair(s) of auxiliary cut-off functions the following is satisfied:

- (<sup>b</sup>Ψ1) The operator  $(1 - \varphi)A(1 - \varphi)$  is a compactly supported pseudodifferential operator of order  $m$  in the interior  $M^\circ$ .
- (<sup>b</sup>Ψ2) The operator  $\varphi A \psi$  is smoothing. Its integral kernel  $K_{\varphi A \psi}$  has support in  $\text{supp } \varphi \times M$  and lies in  $\mathcal{J}^\infty(\partial(M \times M), M \times M; E' \boxtimes F)$ .
- (<sup>b</sup>Ψ3) The operator  $\psi A \varphi$  is smoothing. Its integral kernel  $K_{\psi A \varphi}$  has support in  $M \times \text{supp } \varphi$  and lies in  $\mathcal{J}^\infty(\partial(M \times M), M \times M; E' \boxtimes F)$ .
- (<sup>b</sup>Ψ4) Consider the induced operator on the cylinder

$$\begin{aligned} \tilde{A} : {}^b\mathcal{S}(\mathbb{R} \times \partial M; E) &\rightarrow {}^b\mathcal{S}(\mathbb{R} \times \partial M; F)', \\ u &\mapsto \left( (t, p) \mapsto [(1 - \psi)A(1 - \psi)((x, \eta)^* u)]((x, \eta)^{-1}(t, p)) \right), \end{aligned}$$

where  ${}^b\mathcal{S}(\mathbb{R} \times \partial M; E) := {}^b\mathcal{S}(\mathbb{R}) \hat{\otimes} \mathcal{C}^\infty(\partial M; E)$ . Denote by  $\tilde{a} := \sigma(\tilde{A}) \in \mathcal{S}^m(T^*(\mathbb{R} \times \partial M); \pi^* \text{Hom}(E, F))$  the complete symbol of  $\tilde{A}$  defined by Eq. (A.12)

with respect to the product metric on  $\mathbb{R} \times \partial M$ . Then the following conditions hold true:

- (i) Let  $y$  denote local coordinates of  $\partial M$ ,  $(y, \xi)$  the corresponding local coordinates of  $T^*\partial M$ , and  $\tau$  the cotangent variable of the cylinder variable  $t \in \mathbb{R}$ . Then the symbol  $\tilde{a}(t, \tau, y, \xi)$  can be (uniquely) extended to an entire function in  $\tau \in \mathbb{C}$  such that uniformly in  $t$ , uniformly in a strip  $|\operatorname{Im} \tau| \leq R$  with  $R > 0$  and locally uniformly in  $y$

$$\left\| \partial_t^k \partial_\tau^l \partial_y^\alpha \partial_\xi^\beta \tilde{a} \right\| \leq C_{k,l,\alpha,\beta} (1 + |\tau| + \|\xi\|)^{m-l-|\beta|}$$

for  $l \in \mathbb{N}$ ,  $\beta \in \mathbb{N}^{\dim M-1}$ .

- (ii) There exist symbols  $\tilde{a}_k(\tau, y, \xi) \in \mathcal{S}^m(\mathbb{C} \times T^*\partial M; \pi^* \operatorname{Hom}(E, F))$ ,  $k \in \mathbb{N}$ , and  $r_n(t, \tau, y, \xi) \in \mathcal{S}^m(\mathbb{R} \times \mathbb{C} \times T^*\partial M; \pi^* \operatorname{Hom}(E, F))$ ,  $n \in \mathbb{N}$ , which all are entire in  $\tau$  and fulfill growth conditions as in (i) such that for every  $n \in \mathbb{N}$  the following asymptotic expansion holds:

$$\tilde{a}(t, \tau, y, \xi) = \sum_{k=0}^n e^{kt} \tilde{a}_k(\tau, y, \xi) + e^{(n+1)t} r_n(t, \tau, y, \xi).$$

- (iii) The Schwartz kernel  $K_{\tilde{B}}$  of the operator  $\tilde{B} := \tilde{A} - \operatorname{Op}(\tilde{a})$  with  $\operatorname{Op}(\tilde{a})$  defined by Eq. (A.11) can be represented in the form

$$K_{\tilde{B}}(t, p, t', p') = \int_{\mathbb{R}} e^{i(t-t')\tau} \tilde{b}(t, \tau, p, p') d\tau$$

with a symbol

$$\tilde{b}(t, \tau, p, p') \in \mathcal{S}^{-\infty}(T^*\mathbb{R} \times \partial M \times \partial M; \pi^* \operatorname{Hom}(E, F))$$

which is entire in  $\tau$  and which for every  $\tilde{m} \in \mathbb{N}$ ,  $k, l \in \mathbb{N}$  and every pair of differential operators  $D_p$  and  $D'_{p'}$  on  $\partial M$  (acting on the variable  $p$  resp.  $p'$ ) satisfies the following estimate uniformly in  $t, p, p'$  and uniformly in a strip  $|\operatorname{Im} \tau| \leq R$  with  $R > 0$

$$\left\| \partial_t^k \partial_\tau^l D_p D'_{p'} \tilde{b} \right\| \leq C_{\tilde{m},k,l,D,D'} (1 + |\tau|)^{\tilde{m}}.$$

- (iv) There exist symbols

$$\tilde{b}_k(\tau, p, p') \in \mathcal{S}^{-\infty}(\mathbb{C} \times \partial M \times \partial M; \pi^* \operatorname{Hom}(E, F)),$$

for  $k \in \mathbb{N}$  and symbols

$$r_n(t, \tau, p, p') \in \mathcal{S}^m(\mathbb{R} \times \mathbb{C} \times \partial M \times \partial M; \pi^* \operatorname{Hom}(E, F)),$$

for  $n \in \mathbb{N}$  which all are entire in  $\tau$  and fulfill growth conditions as in (iii) such that for every  $n \in \mathbb{N}$  the following asymptotic expansion holds:

$$\tilde{b}(t, \tau, p, p') = \sum_{k=0}^n e^{kt} \tilde{b}_k(\tau, p, p') + e^{(n+1)t} r_n(t, \tau, p, p').$$

If in addition to the above conditions the operators  $(1 - \varphi)A(1 - \varphi)$  and  $\tilde{A}$  are both classical pseudodifferential operators, then  $A$  is a classical b-pseudodifferential operator of order  $m$ . We denote the space of classical b-pseudodifferential operators on  $(M, g_b)$  of order  $m$  between  $E$  and  $F$  by  ${}^b\Psi^m(M; E, F)$ , and put as usual  ${}^b\Psi^\infty(M; E, F) := \bigcup_{m \in \mathbb{Z}} {}^b\Psi^m(M; E, F)$ . It is straightforward (though somewhat tedious) to check that  ${}^b\Psi^\infty(M; E) := {}^b\Psi^\infty(M; E, E)$  even forms an algebra. Obviously,  ${}^b\Psi^\infty(M; E, F)$  contains as a natural subspace the space  ${}^b\text{Diff}(M; E, F)$  of all b-differential operators on  $M$  from  $E$  to  $F$  which means of all local classical b-pseudodifferential operators. The following is immediate to check.

**Proposition A.1.** *Let  $A \in {}^b\Psi^\infty(M; E, F)$ . Using the notation from above the following propositions are then equivalent:*

- (1)  $A \in {}^b\text{Diff}(M; E, F)$ .
- (2) *The operators  $(1 - \varphi)A(1 - \varphi)$  and  $\tilde{A}$  are differential operators, and both the operators  $\varphi A \psi$  and  $\psi A \varphi$  vanish.*
- (3) *The operator  $A$  acts as a differential operator over the interior, i.e. as a local operator on  $\Gamma^\infty(E|_{M^\circ})$ . In addition, over the cylinder  $(-\infty, 0) \times \partial M$  the operator  $\tilde{A}$  can be written locally in the form*

$$\tilde{A} = \sum_{j+|\alpha| \leq \text{ord } A} a_{j,\alpha} \partial_y^\alpha \partial_x^j, \quad (\text{A.15})$$

where  $a_{j,\alpha} \in {}^b\mathcal{C}^\infty((-\infty, 0) \times U)$ ,  $U \subset \partial M$  open and  $y : U \rightarrow \mathbb{R}^{\dim M - 1}$  are local coordinates of  $\partial M$ .

Over a cylinder  $(-\infty, c) \times N$  with  $c \in \mathbb{R}$  and  $N$  a compact manifold, we sometimes use the notation  ${}^b\Psi_{\text{cpt}}^\infty((-\infty, c) \times N; E, F)$  to denote the space of all pseudodifferential operators in  ${}^b\Psi^\infty((-\infty, c) \times N; E, F)$  having support in some cylinder  $(-\infty, c - \varepsilon] \times N$  with  $\varepsilon > 0$ . We also put

$$\begin{aligned} {}^b\text{Diff}_{\text{cpt}}((-\infty, c) \times N; E, F) := \\ {}^b\text{Diff}((-\infty, c) \times N; E, F) \cap {}^b\Psi_{\text{cpt}}^\infty((-\infty, c) \times N; E, F). \end{aligned} \quad (\text{A.16})$$

Note that in condition (bΨ4) above, the operator  $\tilde{A}$  is an element of  ${}^b\Psi_{\text{cpt}}^\infty((-\infty, 3/2) \times \partial M; E, F)$ .

Throughout this work, we also need the b-versions of Sobolev-spaces. For  $m \in \mathbb{N}$ , the b-Sobolev space  ${}^bH^m(M; E)$  is defined by

$${}^bH^m(M, E) := \{u \in L^2(M, E) \mid Du \in L^2(M, E) \text{ for all } D \in {}^b\text{Diff}^m(M, E)\}. \quad (\text{A.17})$$

For the definition of  ${}^bH^m(M, E)$  for arbitrary  $m \in \mathbb{R}$  we refer the reader to [MEL93]. The following result is straightforward.

**Proposition A.2.** *Let  $A \in {}^b\Psi^l(M; E, F)$  be a b-pseudodifferential operator. Then the following holds true:*

- (1) *A has a natural extension*

$$A : {}^bH^m(M; E) \rightarrow {}^bH^{m-l}(M; F), \quad (\text{A.18})$$

which we denote by the same symbol like the original operator.

- (2) The  $b$ -Sobolev-space  ${}^bH^1(M, E)$  is the natural domain of any elliptic first order  $b$ -pseudodifferential operator acting on sections of  $E$ .
- (3) If  $A$  has order  $l = 0$ , then  $A$  is bounded.

**A.4. Indicial family.** Assume  $A \in {}^b\Psi^m(M; E, F)$ . Denote by  $\tilde{A}$  and  $\tilde{a}$  the induced operator and its complete symbol on the cylinder  $\mathbb{R} \times \partial M$  as above in condition  $({}^b\Psi 4)$ . Consider the zeroth order term  $\tilde{a}_0$  in the asymptotic expansion  $({}^b\Psi 4)(ii)$  and put for  $\tau \in \mathbb{C}$ ,  $u \in \Gamma^\infty(\partial M; E)$  and  $p \in \partial M$

$$\begin{aligned} \mathcal{I}(A)(\tau)u(p) &:= \text{Op}(\tilde{a}_0(\tau, -))u(p) = \\ &= \frac{1}{(2\pi)^{\dim M - 1}} \int_{T_p \partial M \times T_p^* \partial M} \alpha_0(p, \exp v) e^{-i\langle v, \xi \rangle} \tilde{a}_0(\tau, p, \xi) \tau^E(u(\exp v), p) dv d\xi, \end{aligned} \quad (\text{A.19})$$

where, as explained in Section A.2,  $\alpha_0 : M \times M \rightarrow [0, 1]$  is a cut-off function vanishing outside the injectivity radius and  $\tau^E$  is a connection induced parallel transport on  $E$ . One thus obtains an entire family  $\mathcal{I}(A)$  of pseudodifferential operators on the boundary  $\partial M$  which is called the *indicial family* of  $A$ . The indicial family plays a crucial role in deriving the Atiyah–Patodi–Singer index formula within the  $b$ -calculus (cf. [MEL93]).

## APPENDIX B. BASIC RESOLVENT AND HEAT KERNEL ESTIMATES

For the convenience of the reader we are going to summarize some basic estimates for the resolvent and the heat operator associated to an elliptic operator. These estimates are an essential tool for the investigation of the asymptotics of the Chern character in Section 3.

During the whole section  $M$  will be a Riemannian manifold without boundary and  $D_0 : \Gamma^\infty(M; W) \rightarrow \Gamma^\infty(M; W)$  will denote a first order formally self-adjoint elliptic differential operator acting between sections of the Hermitian vector bundle  $W$ . We assume that there exists a self-adjoint extension,  $D$ , of  $D_0$ . E.g. if  $M$  is complete and  $D_0$  is of Dirac type then  $D_0$  is essentially self-adjoint; if  $M$  is the interior of a compact manifold with boundary then  $D$  can be obtained by imposing an appropriate boundary condition. For the following considerations it is irrelevant which self-adjoint extension is chosen. We just fix one.

**B.1. Resolvent estimates.** We fix an open sector  $\Lambda := \{z \in \mathbb{C} \setminus \{0\} \mid 0 < \varepsilon \leq \arg z \leq 2\pi - \varepsilon\} \subset \mathbb{C} \setminus \mathbb{R}_+$  in the complex plane.

We introduce the following notation: for a function  $f : \Lambda \rightarrow \mathbb{C}$  we write  $f(\lambda) = O(|\lambda|^{\alpha+0})$ ,  $\lambda \rightarrow \infty$ ,  $\lambda \in \Lambda$  if for every  $\delta > 0$ ,  $\lambda_0 \in \Lambda$ , there is a constant  $C_{\delta, \lambda_0}$  such that  $|f(\lambda)| \leq C_{\delta} |\lambda|^{\alpha+\delta}$  for  $\lambda \in \Lambda$ ,  $|\lambda| \geq |\lambda_0|$ .

We write  $f(\lambda) = O(|\lambda|^{-\infty})$ ,  $\lambda \rightarrow \infty$ ,  $\lambda \in \Lambda$  if  $f(\lambda) = O(|\lambda|^{-N})$  for every  $N$ ; the  $O$ -constant may depend on  $N$ .

$L_s^2(M; W)$  denotes the Hilbert space of sections of  $W$  which are of Sobolev class  $s$ . The Sobolev norm of an element  $f \in L_s^2(M; W)$  is denoted by  $\|f\|_s$ . For a linear operator  $T : L_s^2(M; W) \rightarrow L_t^2(M; W)$  its operator norm is denoted by  $\|T\|_{s,t}$ .

For an operator  $T$  in a Hilbert space  $\mathcal{H}$  we denote by  $\|T\|_p$  the  $p$ -th Schatten norm. To avoid confusions the letter  $p$  will not be used for Sobolev orders. Note that the operator norm of  $T$  in  $\mathcal{H}$  coincides with  $\|T\|_\infty$ .

**Proposition B.1.** *Let  $A, B \in \Psi^\bullet(M, W)$  be pseudodifferential operators of order  $a, b$  with compact support.<sup>1</sup>*

1. *If  $k > (\dim M)/4 + a/2$  then  $A(D^2 - \lambda)^{-k}, (D^2 - \lambda)^{-k}A$  are Hilbert–Schmidt operators for  $\lambda \notin \text{spec } D^2$  and we have*

$$\|A(D^2 - \lambda)^{-k}\|_2 = O(|\lambda|^{a/2+(\dim M)/4-k+0}), \quad \text{as } \lambda \rightarrow \infty \text{ in } \Lambda. \quad (\text{B.1})$$

*The same estimate holds for  $\|(D^2 - \lambda)^{-k}A\|_2$ .*

2. *If  $k > (\dim M + a + b)/2$  then  $A(D^2 - \lambda)^{-k}B$  is of trace class for  $\lambda \notin \text{spec } D^2$  and*

$$\|A(D^2 - \lambda)^{-k}B\|_1 = O(|\lambda|^{(\dim M + a + b)/2-k+0}), \quad \text{as } \lambda \rightarrow \infty \text{ in } \Lambda. \quad (\text{B.2})$$

3. *Denote by  $\pi_1, \pi_2 : M \times M \rightarrow M$  the projection onto the first resp. second factor and assume that  $\pi_2(\text{supp } A) \cap \pi_1(\text{supp } B) = \emptyset$ . Then  $A(D^2 - \lambda)^{-k}B$  is a trace class operator for any  $k \geq 1$  and*

$$\|A(D^2 - \lambda)^{-k}B\|_1 = O(|\lambda|^{-\infty}), \quad \text{as } \lambda \rightarrow \infty \text{ in } \Lambda. \quad (\text{B.3})$$

*Proof.* 1. Sobolev embedding and elliptic regularity implies that for  $f \in L^2(M; W)$  the section  $A(D^2 - \lambda)^{-k}f$  is continuous. Moreover, for  $r > \dim M/2, |\lambda| \geq |\lambda_0|$ , and  $x$  in the compact set  $\text{supp } A =: K$

$$\begin{aligned} \|(A(D^2 - \lambda)^{-k}f)(x)\| &\leq C\|(D^2 - \lambda)^{-k}\|_{a+r, K} \\ &\leq C\|(D^2 + I)^{(a+r)/2}(D^2 - \lambda)^{-k}f\|_0 \\ &\leq C|\lambda|^{-k+(a+r)/2}\|f\|. \end{aligned} \quad (\text{B.4})$$

For the Schwartz–kernel this implies the estimate

$$\sup_{x \in \text{supp } A} \int_M \|A(D^2 - \lambda)^{-k}(x, y)\|^2 d \text{vol}(y) \leq C|\lambda|^{-2k+a+r}, \quad (\text{B.5})$$

and since  $A$  has compact support, integration over  $x$  yields

$$\begin{aligned} \|A(D^2 - \lambda)^{-k}\|_2^2 &\leq \int_{\text{supp } A} \int_M \|A(D^2 - \lambda)^{-k}(x, y)\|^2 d \text{vol}(x) d \text{vol}(y) \\ &\leq C|\lambda|^{-2k+a+r}, \end{aligned} \quad (\text{B.6})$$

proving the estimate (B.1). The estimate for  $(D^2 - \lambda)^{-k}A$  follows by taking the adjoint.

2. The second claim follows from the first one using the Hölder inequality.

3. To prove the third claim we choose cut–off functions  $\varphi, \psi \in \mathcal{C}_0^\infty(M)$  with  $\varphi = 1$  on  $\pi_2(\text{supp } A)$ ,  $\psi = 1$  on  $\pi_1(\text{supp } B)$  and  $\text{supp } \varphi \cap \text{supp } \psi = \emptyset$ .

<sup>1</sup>This means that their Schwartz kernels are compactly supported in  $M \times M$ .

Then  $A(D^2 - \lambda)^{-k}B = A\varphi(D^2 - \lambda)^{-k}\psi B$  and  $\varphi(D^2 - \lambda)^{-k}\psi$  is a smoothing operator in the *parameter dependent calculus* (cf. Shubin [SHU01, Chap. II]). Hence for any real numbers  $s, t, N$  we have

$$\|\varphi(D^2 - \lambda)^{-k}\psi\|_{s,t} \leq C(s, t, N) |\lambda|^{-N}, \quad \text{as } \lambda \rightarrow \infty \text{ in } \Lambda. \quad (\text{B.7})$$

Since the Sobolev orders  $s, t$  are arbitrary this implies the claim.  $\square$

**Proposition B.2.** *Let  $A \in \Psi^a(M, W)$  be a pseudodifferential operator with compact support.*

1. *Let  $\varphi \in C^\infty(M)$  be a smooth function such that  $\text{supp } d\varphi$  is compact, i.e. outside a compact set  $\varphi$  is locally constant. Moreover suppose that  $\text{supp } \varphi \cap \pi_1(\text{supp } A) = \emptyset$ . Then  $\varphi(D^2 - \lambda)^{-k}A$  is a trace class operator for any  $k \geq 1$  and the estimate (B.3) holds for  $\varphi(D^2 - \lambda)^{-k}A$ .*

2. *If  $k > (\dim M + a)/2$  then  $A(D^2 - \lambda)^{-k}, (D^2 - \lambda)^{-k}A$  are trace class operators and the estimate (B.2) holds with  $B = I$ .*

*Proof.* From

$$(D^2 - \lambda)\varphi(D^2 - \lambda)^{-k}A = [D^2, \varphi](D^2 - \lambda)^{-k}A + \varphi(D^2 - \lambda)^{-k+1}A \quad (\text{B.8})$$

we infer since  $\varphi A = 0$

$$\begin{aligned} & \varphi(D^2 - \lambda)^{-k}A \\ &= (D^2 - \lambda)^{-1} \begin{cases} [D^2, \varphi](D^2 - \lambda)^{-k}A, & k = 1, \\ [D^2, \varphi](D^2 - \lambda)^{-k}A + \varphi(D^2 - \lambda)^{-k+1}A, & k > 1. \end{cases} \end{aligned} \quad (\text{B.9})$$

Applying Proposition B.1.3 to the right hand side we inductively obtain the first assertion.

To prove the second assertion we choose a cut-off function  $\varphi \in C_0^\infty(M)$  with  $\varphi = 1$  on  $\pi_1(\text{supp } A)$ . Then we apply Proposition B.1.2 to  $\varphi(D^2 - \lambda)^{-k}A$  and the proved first assertion to  $(1 - \varphi)(D^2 - \lambda)^{-k}A$  to reach the conclusion.  $\square$

For the following Proposition it is crucial that we are precise about domains of operators:

**Definition B.3.** Given a differential operator  $A \in \text{Diff}^a(M, W)$  we say that the commutator  $[D^2, A]$  has compact support if

- (1)  $A$  and  $A^t$  map  $\text{dom}(D^k)$  into  $\text{dom}(D^{k-a})$  for  $k \geq a$  and
- (2) the differential expression  $[D^2, A]$  has compact support.

The main example we have in mind is where  $D$  is a Dirac type operator on a complete manifold and  $A$  is multiplication by a smooth function  $\varphi$  such that  $d\varphi$  has compact support. Then  $[D^2, \varphi]$  has compact support in the above sense.

**Proposition B.4.** *Let  $A \in \text{Diff}^a(M, W)$  be a differential operator such that  $[D^2, A]$  has compact support and is of order  $\leq a + 1$ . Then for  $k > \dim M + a$  the commutator  $[A, (D^2 - \lambda)^{-k}]$  is trace class and*

$$\|[A, (D^2 - \lambda)^{-k}]\|_1 = O(|\lambda|^{(\dim M + a - 1)/2 - k + 0}), \quad \text{as } \lambda \rightarrow \infty \text{ in } \Lambda. \quad (\text{B.10})$$

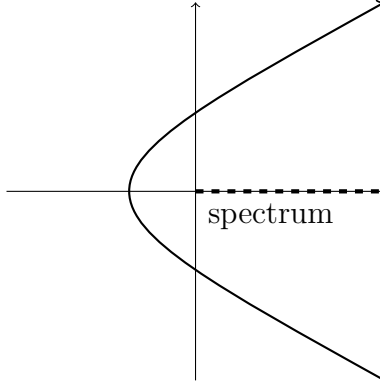


FIGURE 5. Contour of integration for calculating  $e^{-tD^2}$  from the resolvent.

*Proof.* Note first that since  $A$  and  $A^t$  map  $\text{dom}(D^k)$  into  $\text{dom}(D^{k-a})$  the commutator  $[A, (D^2 - \lambda)^{-k}]$  is defined as a linear operator on  $L^2(M; W)$  and we have the identity

$$[A, (D^2 - \lambda)^{-k}] = \sum_{j=1}^k (D^2 - \lambda)^{-j} [D^2, A] (D^2 - \lambda)^{-k+j-1}. \quad (\text{B.11})$$

Since  $k > \dim M + a$  we have in each summand  $j > (\dim M + a + 1)/2$  or  $k - j + 1 > (\dim M + 1 + 1)/2$ . Say in the first case we apply Proposition B.2 to  $\|(D^2 - \lambda)^{-j} [D^2, A]\|_1$  and the Spectral Theorem to estimate  $\|(D^2 - \lambda)^{-k+j-1}\|$  and find

$$\begin{aligned} \|(D^2 - \lambda)^{-j} [D^2, A] (D^2 - \lambda)^{-k+j-1}\|_1 & \\ & \leq \|(D^2 - \lambda)^{-j} [D^2, A]\|_1 \|(D^2 - \lambda)^{-k+j-1}\|_\infty \\ & \leq O(|\lambda|^{(\dim M + a + 1)/2 - j + 0}) \cdot O(|\lambda|^{-k+j-1}) \\ & = O(|\lambda|^{(\dim M + a - 1)/2 - k + 0}). \quad \square \end{aligned}$$

**B.2. Heat kernel estimates.** From Propositions B.1, B.2 we can derive short and large times estimates for the heat operator  $e^{-tD^2}$ . We write

$$\begin{aligned} e^{-tD^2} &= \frac{1}{2\pi i} \int_{\gamma} e^{-t\lambda} (D^2 - \lambda)^{-1} d\lambda \\ &= \frac{t^{-k} k!}{2\pi i} \int_{\gamma} e^{-t\lambda} (D^2 - \lambda)^{-k-1} d\lambda, \end{aligned} \quad (\text{B.12})$$

where integration is over the contour sketched in Figure 5. The notation  $O(t^{\alpha-0})$ ,  $O(t^\infty)$  as  $t \rightarrow 0+$  resp.  $O(t^{\alpha-0})$ ,  $O(t^{-\infty})$  as  $t \rightarrow \infty$  is defined analogously to the corresponding notation for  $\lambda \in \Lambda$  in the previous Subsection.

We infer from Propositions B.1, B.2

**Proposition B.5.** *Let  $A, B \in \Psi^\bullet(M, W)$  be pseudodifferential operators of order  $a, b$  with compact support.*

1. *For  $t > 0$  the operators  $Ae^{-tD^2}$ ,  $e^{-tD^2}A$  are trace class operators. For  $t_0, \varepsilon > 0$  there is a constant  $C(t_0, \varepsilon) > 0$  such that for all  $1 \leq p \leq \infty$  we have the following*

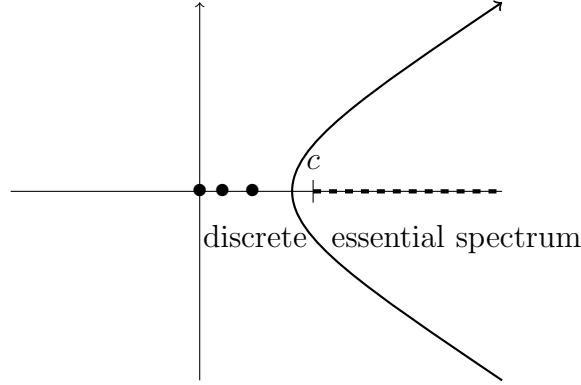


FIGURE 6. Contour of integration if bottom of the essential spectrum of  $D^2$  is  $c$ .

estimate in the Schatten  $p$ -norm

$$\|Ae^{-tD^2}\|_p \leq C(t_0, \varepsilon) t^{-a/2 - \frac{\dim M + \varepsilon}{2p}}, \quad 0 < t \leq t_0. \quad (\text{B.13})$$

Note that  $C(t_0, \varepsilon)$  is independent of  $p$ . The same estimate holds for  $\|e^{-tD^2}A\|_p$ .

2. Denote by  $\pi_1, \pi_2 : M \times M \rightarrow M$  the projection onto the first resp. second factor and assume that  $\pi_2(\text{supp } A) \cap \pi_1(\text{supp } B) = \emptyset$ . Then

$$\|Ae^{-tD^2}B\|_1 = O(t^\infty), \quad 0 < t < t_0, \quad (\text{B.14})$$

with  $N$  arbitrarily large.

3. Let  $\varphi \in C^\infty(M)$  be a smooth function such that  $\text{supp } \varphi$  is compact. Moreover suppose that  $\text{supp } \varphi \cap \pi_1(\text{supp } A) = \emptyset$ . Then the estimate (B.14) also holds for  $\varphi e^{-tD^2}A$ .

*Proof.* 1. From Proposition B.2 and the contour integral (B.12) we infer the inequality (B.13) for  $p = 1$ . For  $p = \infty$  it follows from the Spectral Theorem. The Hölder inequality implies the following interpolation inequality for Schatten norms

$$\|T\|_p = \text{Tr}(|T|^p)^{1/p} \leq \|T\|_\infty^{1-1/p} \|T\|_1^{1/p}, \quad 1 \leq p \leq \infty. \quad (\text{B.15})$$

From this we infer (B.13).

The remaining claims follow immediately from the contour integral (B.12) and the corresponding resolvent estimates.  $\square$

For the next result we assume additionally that  $D$  is a Fredholm operator and we denote by  $H$  the orthogonal projection onto  $\text{Ker } D$ .  $H$  is a finite rank smoothing operator. Put  $c := \inf \text{spec}_{\text{ess}} D^2$ . Then  $e^{-tD^2}(I - H) = e^{-tD^2} - H$  can again be expressed in terms of a contour integral as in (B.12) where the contour is now depicted in Figure 6.

This allows to make large time estimates. The result is as follows:

**Proposition B.6.** *Assume that  $D$  is Fredholm and let  $A \in \Psi^a(M, W)$  be a pseudodifferential operator with compact support. Then for any  $0 < \delta < \inf \text{spec}_{\text{ess}} D^2$  and any  $\varepsilon > 0$  there is a constant  $C(\delta, \varepsilon)$  such that for  $1 \leq p \leq \infty$*

$$\|Ae^{-tD^2}(I - H)\|_p \leq C(\delta, \varepsilon) t^{-a/2 - \frac{\dim M + \varepsilon}{2p}} e^{-t\delta}, \quad 0 < t < \infty. \quad (\text{B.16})$$

*Proof.* For  $t \rightarrow 0+$  the estimate follows from Proposition B.5.1.

For  $t \rightarrow \infty$  and  $p = 1$  the estimate follows from Proposition B.1 and (B.12) by taking the contour as in Figure 6. For  $p = \infty$  the estimate is a simple consequence of the Spectral Theorem. The general case then follows again from the interpolation inequality (B.15).  $\square$

Finally we state the analogue of Proposition B.4 for the heat kernel.

**Proposition B.7.** *Let  $A \in \text{Diff}^a(M, W)$  be a differential operator such that  $[D^2, A]$  has compact support (in the sense of Definition B.3) and is of order  $\leq a + 1$ .*

*Then for  $t > 0$  the operator  $[A, e^{-tD^2}]$  is of trace class. For  $t_0, \varepsilon > 0$  there is a constant  $C(t_0, \varepsilon)$  such that for all  $1 \leq p \leq \infty$  we have the following estimate in the Schatten  $p$ -norm*

$$\|[A, e^{-tD^2}]\|_p \leq C(t_0, \varepsilon) t^{-a/2 - \frac{\dim M - 1 + \varepsilon}{2p}}, \quad 0 < t \leq t_0; \quad (\text{B.17})$$

$C(t_0, \varepsilon)$  is independent of  $p$ .

*If  $D$  is a Fredholm operator then for any  $0 < \delta < \inf \text{spec}_{\text{ess}} D^2$  and any  $\varepsilon > 0$  there is a constant  $C(\delta, \varepsilon)$  such that for  $1 \leq p \leq \infty$*

$$\|[A, e^{-tD^2}(I - H)]\|_p \leq C(\delta, \varepsilon) t^{-a/2 - \frac{\dim M - 1 + \varepsilon}{2p}} e^{-t\delta}, \quad 0 < t < \infty. \quad (\text{B.18})$$

*Proof.* For  $p = 1$  this follows from Proposition B.4 and the contour integral representation (B.12) by taking the contours as in Figure 5 for  $t \rightarrow 0+$  and as in Figure 6 in the Fredholm case as  $t \rightarrow \infty$ . For  $p = \infty$  the estimates are a simple consequence of the Spectral Theorem. The general case then follows from the interpolation inequality (B.15).  $\square$

**B.3. Estimates for the JLO integrand.** Recall that we denote the standard  $k$ -simplex by  $\Delta_k := \{(\sigma_0, \dots, \sigma_k) \in \mathbb{R}^{k+1} \mid \sigma_j \geq 0, \sigma_0 + \dots + \sigma_k = 1\}$ . Furthermore, recall the notation (2.3).

**Proposition B.8.** *Let  $A_j \in \text{Diff}^{a_j}(M; W)$ ,  $j = 0, \dots, k$ , be  $D^{a_j}$ -bounded differential operators on  $M$ ,  $a := \sum_{j=0}^k a_j$ . Furthermore, assume that  $\text{supp } A_{j_0}$  is compact for at least one index  $j_0$ .*

1. *For  $t_0, \varepsilon > 0$  there is a constant  $C(t_0, \varepsilon)$  such that for all  $\sigma = (\sigma_0, \dots, \sigma_k) \in \Delta_k$ ,  $\sigma_j > 0$ ,*

$$\begin{aligned} & \|A_0 e^{-\sigma_0 t D^2} A_1 \cdots A_k e^{-\sigma_k t D^2}\|_1 \\ & \leq C(t_0, \varepsilon) \left( \prod_{j=0}^k \sigma_j^{-a_j/2} \right) t^{-a/2 - (\dim M)/2 - \varepsilon}, \quad 0 < t \leq t_0. \end{aligned} \quad (\text{B.19})$$

*In particular, if  $a_j \leq 1, j = 0, \dots, k$ , then*

$$\|(A_0, \dots, A_k)_{\sqrt{t}D}\| = O(t^{-a/2 - (\dim M)/2 - 0}), \quad t \rightarrow 0+. \quad (\text{B.20})$$

2. *Assume additionally that  $D$  is Fredholm and denote by  $H$  the orthogonal projection onto  $\text{Ker } D$ . Then for  $\varepsilon > 0$  and any  $0 < \delta < \inf \text{spec}_{\text{ess}} D^2$  there is a constant  $C(\delta, \varepsilon)$*

such that for all  $\sigma \in \Delta_k$ ,  $\sigma_j > 0$

$$\begin{aligned} & \|A_0 e^{-\sigma_0 t D^2} (I - H) A_1 \cdots \cdots A_k e^{-\sigma_k t D^2} (I - H)\|_1 \\ & \leq C(\delta, \varepsilon) \left( \prod_{j=0}^k \sigma_j^{-a_j/2} \right) t^{-a/2 - (\dim M)/2 - \varepsilon} e^{-t\delta}, \quad \text{for all } 0 < t < \infty. \end{aligned} \quad (\text{B.21})$$

In particular,  $a_j \leq 1$ ,  $j = 0, \dots, k$ , then

$$\begin{aligned} & \|(A_0(I - H), \dots, A_k(I - H))_{\sqrt{t}D}\| \\ & = O(t^{-a/2 - (\dim M)/2 - 0} e^{-t\delta}), \quad \text{for all } 0 < t < \infty. \end{aligned} \quad (\text{B.22})$$

*Proof.* We first reduce the problem to the case that all  $A_j$  are compactly supported. To this end choose  $\varphi_{j_0-1}, \varphi_{j_0} \in C_0^\infty(M)$  such that  $\text{supp } \varphi_{j_0} \cap \text{supp}(1 - \varphi_{j_0-1}) = \emptyset$  and such that  $\varphi_{j_0} A_{j_0} = A_{j_0} \varphi_{j_0} = A_{j_0}$ . Decompose  $A_{j_0-1} = A_{j_0-1} \varphi_{j_0-1} + A_{j_0-1} (1 - \varphi_{j_0-1})$ .

First we show that the estimates (B.19), (B.21) hold if we replace  $A_{j_0-1}$  by  $A_{j_0-1}(1 - \varphi_{j_0-1})$ :

Case 1. Proposition B.5.3 gives

$$\|A_{j_0-1}(1 - \varphi_{j_0-1}) e^{-\sigma_{j_0-1} t D^2} \varphi_{j_0}\|_1 \leq C_{t_0} \sigma_{j_0-1}^N t^N, \quad \text{for } \sigma_{j_0-1} t \leq t_0. \quad (\text{B.23})$$

The operator norm of the other factors can be estimated using the Spectral Theorem, taking into account the  $D^{a_j}$ -boundedness of  $A_j$ :

$$\|A_j e^{-\sigma_j t D^2}\| \leq C_{t_0} (\sigma_j t)^{-a_j/2}, \quad \text{for } \sigma_j t \leq t_0. \quad (\text{B.24})$$

Hence by the Hölder inequality

$$\begin{aligned} & \|A_0 e^{-\sigma_0 t D^2} A_1 \cdots \cdots A_{j_0-1} (1 - \varphi_{j_0-1}) e^{-\sigma_{j_0-1} t D^2} \cdot A_k e^{-\sigma_k t D^2}\|_1 \\ & \leq C_{t_0} \left( \prod_{j=0}^k \sigma_j^{-a_j/2} \right) t^{-\sum_{j=0}^k a_j + N}, \quad 0 < t \leq t_0, \end{aligned} \quad (\text{B.25})$$

which is even better than (B.19).

Case 2 ( $D$  Fredholm). From (B.23), Proposition B.6 and the fact that  $H$  is a finite rank operator with  $e^{-\xi D^2} H = H$  we infer

$$\begin{aligned} & \|A_{j_0-1} (1 - \varphi_{j_0-1}) e^{-\sigma_{j_0-1} t D^2} (I - H) \varphi_{j_0}\|_1 \\ & \leq C_\delta e^{-\sigma_{j_0-1} t \delta}, \quad \text{for all } 0 < t < \infty. \end{aligned} \quad (\text{B.26})$$

To the other factors we apply Proposition B.6 with  $p = \infty$ :

$$\|A_j e^{-\sigma_j t D^2} (I - H)\| \leq C_\delta (\sigma_j t)^{-a_j/2} e^{-\sigma_j t \delta}, \quad 0 < t < \infty. \quad (\text{B.27})$$

The Hölder inequality combined with (B.26), (B.27) gives (B.21).

Altogether we are left to consider  $A_0, \dots, A_{j_0-1} \varphi_{j_0-1}, A_{j_0}, \dots, A_k$  where now  $A_{j_0-1} \varphi_{j_0-1}$  and  $A_{j_0}$  are compactly supported. Continuing this way, also to the right of  $j_0$ , it remains to treat the case where *each*  $A_j$  has compact support.

Case 1. We apply Hölder's inequality for Schatten norms and Proposition B.5:

$$\begin{aligned}
 & \|A_0 e^{-\sigma_0 t D^2} A_1 \cdots A_k e^{-\sigma_k t D^2}\|_1 \\
 & \leq \prod_{j=0}^k \|A_j e^{-\sigma_j t D^2}\|_{\sigma_j^{-1}} \\
 & \leq C(t_0, \varepsilon) \prod_{j=0}^k (\sigma_j t)^{-a_j/2 - \frac{\dim M + \varepsilon}{2} \sigma_j} \\
 & \leq C(t_0, \varepsilon) \left( \prod_{j=0}^k \sigma_j^{-a_j/2} \right) t^{-a/2 - \frac{\dim M + \varepsilon}{2}}, \quad 0 < t \leq t_0,
 \end{aligned} \tag{B.28}$$

thanks to the fact that  $\sigma \mapsto \sigma^{-\frac{\dim M + \varepsilon}{2}} \sigma$  is bounded as  $\sigma \rightarrow 0$ .

Case 2. If  $D$  is Fredholm we estimate

$$\|A_0 e^{-\sigma_0 t D^2} (I - H) A_1 \cdots A_k e^{-\sigma_k t D^2} (I - H)\|_1$$

using Hölder as in (B.28) and apply Proposition B.6 to the individual factors:

$$\|A_j e^{-\sigma_j t D^2} (I - H)\|_{\sigma_j^{-1}} \leq C_\delta (\sigma_j t)^{-a_j/2 - \frac{\dim M + \varepsilon}{2} \sigma_j} e^{-\sigma_j t \delta}, \quad 0 < t < \infty, \tag{B.29}$$

to reach the conclusion.

Finally we remark that the inequalities (B.20), (B.22) follow by integrating the inequalities (B.19), (B.21) over the standard simplex  $\Delta_k$ . Note that  $\int_{\Delta_k} (\sigma_0 \cdots \sigma_k)^{-1/2} d\sigma < \infty$ .  $\square$

## REFERENCES

- [APS75] M. F. ATIYAH, V. K. PATODI, and I. M. SINGER, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69. MR 0397797 (53 #1655a)
- [APS76] ———, *Spectral asymmetry and Riemannian geometry. III*, Math. Proc. Cambridge Philos. Soc. **79** (1976), no. 1, 71–99. MR 0397799 (53 #1655c)
- [ASS94] J. AVRON, R. SEILER, and B. SIMON, *The index of a pair of projections*, J. Funct. Anal. **120** (1994), no. 1, 220–237. MR 1262254 (95b:47012)
- [BBFu98] B. BOOSS-BAVNBEK and K. FURUTANI, *The Maslov index: a functional analytical definition and the spectral flow formula*, Tokyo J. Math. **21** (1998), no. 1, 1–34. MR 1630119 (99e:58172)
- [BBLZ09] B. BOOSS-BAVNBEK, M. LESCH, and C. ZHU, *The Calderón projection: new definition and applications*, J. Geom. Phys. **59** (2009), no. 7, 784–826. arXiv:0803.4160v1 [math.DG], DOI: 10.1016/j.geomphys.2009.03.012, <http://dx.doi.org/10.1016/j.geomphys.2009.03.012>, MR 2536846
- [BBWo93] B. BOOSS-BAVNBEK and K. P. WOJCIECHOWSKI, *Elliptic boundary problems for Dirac operators*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1993. MR 1233386 (94h:58168)
- [BDT89] P. BAUM, R. G. DOUGLAS, and M. E. TAYLOR, *Cycles and relative cycles in analytic K-homology*, J. Differential Geom. **30** (1989), no. 3, 761–804. MR 1021372 (91b:58244)
- [BGV92] N. BERLINE, E. GETZLER, and M. VERGNE, *Heat kernels and Dirac operators*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 298, Springer-Verlag, Berlin, 1992. MR 1215720 (94e:58130)

- [BLA86] B. BLACKADAR, *K-theory for operator algebras*, Mathematical Sciences Research Institute Publications, vol. 5, Springer-Verlag, New York, 1986. MR 859867 (88g:46082)
- [BLFO90] J. BLOCK and J. FOX, *Asymptotic pseudodifferential operators and index theory*, Geometric and topological invariants of elliptic operators (Brunswick, ME, 1988), Contemp. Math., vol. 105, Amer. Math. Soc., Providence, RI, 1990, pp. 1–32. MR 1047274 (91e:58186)
- [BRLE01] J. BRÜNING and M. LESCH, *On boundary value problems for Dirac type operators. I. Regularity and self-adjointness*, J. Funct. Anal. **185** (2001), no. 1, 1–62. arXiv:math/9905181 [math.FA], MR 1853751 (2002g:58034)
- [BRPF08] J.-P. BRASSELET and M. J. PFLAUM, *On the homology of algebras of Whitney functions over subanalytic sets*, Ann. of Math. (2) **167** (2008), no. 1, 1–52. MR 2373151
- [CoMo90] A. CONNES and H. MOSCOVICI, *Cyclic cohomology, the Novikov conjecture and hyperbolic groups*, Topology **29** (1990), no. 3, 345–388. MR 1066176 (92a:58137)
- [CoMo93] ———, *Transgression and the Chern character of finite-dimensional K-cycles*, Comm. Math. Phys. **155** (1993), no. 1, 103–122. MR 1228528 (95a:46091)
- [CoMo95] A. CONNES and H. MOSCOVICI, *The local index formula in noncommutative geometry*, Geom. Funct. Anal. **5** (1995), no. 2, 174–243. MR 1334867 (96e:58149)
- [CON85] A. CONNES, *Noncommutative differential geometry*, Inst. Hautes Études Sci. Publ. Math. **62** (1985), 257–360. MR 823176 (87i:58162)
- [CON88] ———, *Entire cyclic cohomology of Banach algebras and characters of  $\theta$ -summable Fredholm modules*, K-Theory **1** (1988), no. 6, 519–548. MR MR953915 (90c:46094)
- [CON94] ———, *Noncommutative geometry*, Academic Press Inc., San Diego, CA, 1994. MR 1303779 (95j:46063)
- [CUQU93] J. CUNTZ and D. QUILLEN, *On excision in periodic cyclic cohomology*, C. R. Acad. Sci. Paris Sér. I Math. **317** (1993), no. 10, 917–922. MR 1249360 (94i:19002)
- [CUQU94] ———, *On excision in periodic cyclic cohomology. II. The general case*, C. R. Acad. Sci. Paris Sér. I Math. **318** (1994), no. 1, 11–12. MR 1260526 (94m:19001)
- [FUKE88] S. A. FULLING and G. KENNEDY, *The resolvent parametrix of the general elliptic linear differential operator: a closed form for the intrinsic symbol*, Trans. Amer. Math. Soc. **310** (1988), no. 2, 583–617. MR 973171 (90b:58260)
- [GBVF01] J. M. GRACIA-BONDÍA, J. C. VÁRILLY, and H. FIGUEROA, *Elements of noncommutative geometry*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Boston Inc., Boston, MA, 2001. MR 1789831 (2001h:58038)
- [GESZ89] E. GETZLER and A. SZENES, *On the Chern character of a theta-summable Fredholm module*, J. Funct. Anal. **84** (1989), no. 2, 343–357. MR 1001465 (91g:19007)
- [GET83] E. GETZLER, *Pseudodifferential operators on supermanifolds and the Atiyah-Singer index theorem*, Comm. Math. Phys. **92** (1983), no. 2, 163–178. MR 728863 (86a:58104)
- [GET93A] ———, *Cyclic homology and the Atiyah-Patodi-Singer index theorem*, Index theory and operator algebras (Boulder, CO, 1991), Contemp. Math., vol. 148, Amer. Math. Soc., Providence, RI, 1993, pp. 19–45. MR 1228498 (94j:58162)
- [GET93B] ———, *The odd Chern character in cyclic homology and spectral flow*, Topology **32** (1993), no. 3, 489–507. MR 1231957 (95c:46118)
- [GIL95] P. B. GILKEY, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, second ed., Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR 1396308 (98b:58156)
- [GRSE95] G. GRUBB and R. T. SEELEY, *Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer boundary problems*, Invent. Math. **121** (1995), no. 3, 481–529. MR 1353307 (96k:58216)
- [HASU78] P. R. HALMOS and V. S. SUNDER, *Bounded integral operators on  $L^2$  spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. 96, Springer-Verlag, Berlin, 1978. MR 517709 (80g:47036)

- [HiRo00] N. HIGSON and J. ROE, *Analytic K-homology*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000, Oxford Science Publications. MR 1817560 (2002c:58036)
- [JLO88] A. JAFFE, A. LESNIEWSKI, and K. OSTERWALDER, *Quantum K-theory. I. The Chern character*, Comm. Math. Phys. **118** (1988), no. 1, 1–14. MR 954672 (90a:58170)
- [KiLE04] P. KIRK and M. LESCH, *The  $\eta$ -invariant, Maslov index, and spectral flow for Dirac-type operators on manifolds with boundary*, Forum Math. **16** (2004), no. 4, 553–629. arXiv:math/0012123 [math.DG], MR 2044028 (2005b:58029)
- [LES99] M. LESCH, *On the noncommutative residue for pseudodifferential operators with log-polyhomogeneous symbols*, Ann. Global Anal. Geom. **17** (1999), no. 2, 151–187. arXiv:dg-ga/9708010, MR 1675408 (2000b:58050)
- [LES05] ———, *The uniqueness of the spectral flow on spaces of unbounded self-adjoint Fredholm operators*, Spectral geometry of manifolds with boundary and decomposition of manifolds, Contemp. Math., vol. 366, Amer. Math. Soc., Providence, RI, 2005, pp. 193–224. arXiv:math/0401411 [math.FA], MR 2114489 (2005m:58049)
- [LMP08] M. LESCH, H. MOSCOVICI, and M. J. PFLAUM, *Regularized traces and K-theory invariants of parametric pseudodifferential operators*, Traces in number theory, geometry and quantum fields, Aspects Math., E38, Friedr. Vieweg, Wiesbaden, 2008, pp. 161–177. www.matthiaslesch.de publications, MR 2427595
- [LMP09] ———, *Relative pairing in cyclic cohomology and divisor flows*, J. K-Theory **3** (2009), no. 2, 359–407. arXiv:math/0603500 [math.KT], MR 2496452
- [LOY05] P. LOYA, *Dirac operators, boundary value problems, and the b-calculus*, Spectral geometry of manifolds with boundary and decomposition of manifolds, Contemp. Math., vol. 366, Amer. Math. Soc., Providence, RI, 2005, pp. 241–280. MR 2114491 (2005k:58038)
- [MAL67] B. MALGRANGE, *Ideals of differentiable functions*, Tata Institute of Fundamental Research Studies in Mathematics, No. 3, Tata Institute of Fundamental Research, Bombay, 1967. MR 0212575 (35 #3446)
- [MEL93] R. B. MELROSE, *The Atiyah-Patodi-Singer index theorem*, Research Notes in Mathematics, vol. 4, A K Peters Ltd., Wellesley, MA, 1993. MR 1348401 (96g:58180)
- [MÜL94] W. MÜLLER, *Eta invariants and manifolds with boundary*, J. Differential Geom. **40** (1994), no. 2, 311–377. MR 1293657 (96c:58165)
- [NIC95] L. I. NICOLAESCU, *The Maslov index, the spectral flow, and decompositions of manifolds*, Duke Math. J. **80** (1995), no. 2, 485–533. MR 1369400 (96k:58208)
- [PFL98] M. J. PFLAUM, *The normal symbol on Riemannian manifolds*, New York J. Math. **4** (1998), 97–125 (electronic). MR 1640055 (99e:58181)
- [SHU01] M. A. SHUBIN, *Pseudodifferential operators and spectral theory*, second ed., Springer-Verlag, Berlin, 2001, Translated from the 1978 Russian original by Stig I. Andersson. MR 1852334 (2002d:47073)
- [TAY96] M. E. TAYLOR, *Partial differential equations. II*, Applied Mathematical Sciences, vol. 116, Springer-Verlag, New York, 1996, Qualitative studies of linear equations. MR 1395149 (98b:35003)
- [TOU72] J.-C. TOUGERON, *Idéaux de fonctions différentiables*, Springer-Verlag, Berlin, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 71. MR 0440598 (55 #13472)
- [WID79] H. WIDOM, *Szegő's theorem and a complete symbolic calculus for pseudodifferential operators*, Seminar on Singularities of Solutions of Linear Partial Differential Equations (Inst. Adv. Study, Princeton, N.J., 1977/78), Ann. of Math. Stud., vol. 91, Princeton Univ. Press, Princeton, N.J., 1979, pp. 261–283. MR 547022 (81b:58043)
- [WID80] ———, *A complete symbolic calculus for pseudodifferential operators*, Bull. Sci. Math. (2) **104** (1980), no. 1, 19–63. MR 560744 (81m:58078)
- [WU93] F. WU, *The Chern-Connes character for the Dirac operator on manifolds with boundary*, K-Theory **7** (1993), no. 2, 145–174. MR 1235286 (95f:58076)

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