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Graduate Seminar on Global Analysis (S4B3)

Fredholm Modules and K-Homology - an Introduction to Noncommutative Geometry

Noncommutative Geometry is an area of mathematics which has been dominated by the work of Alain Connes in the last 20-25 years. The basic idea is that instead of point sets (e.g. manifolds) one studies the coordinate ring of (smooth) functions. This point of view has been around in algebraic geometry for decades but it was Connes' who showed that also manifolds and index theory can be understood from this perspective. Examples of 'noncommutative spaces', where now the coordinate ring is a noncommutative algebra, are abundant and Noncommutative Geometry is an area of active current research.

The purpose of this seminar is modest. We want to study some of the basic material, K-homology, Fredholm modules, Hochschild and cyclic (co)homology, the JLO Chern character, and the Hochschild–Kostant–Rosenberg–Connes Theorem.

For a first reading we will use the first part of the excellent survey by Higson [**12**].

Below I will sketch a breakdown into talks. Some of the topics may need more than one session for a thorough discussion.

Prerequisites: basic knowledge of (at least some of the following) functional analysis, C^*-algebras, homological algebra, K-theory, Global Analysis

Time and Venue: I have reserved room 0.003 (Mathematik Zentrum) Tuesdays 10-12. However, I am flexible.

Preliminary meeting with programme discussion and assignments of talks: Th 04.03.2010, 13:30, Mathematikzentrum, Room 1.033 (my office).

First meeting: Tu, 13.04.2010, 10:15, Mathematikzentrum, Room 0.003.

Talks

1. The trace and the Schatten ideals.

Material: [**17**, Sec. 3.4], [**18**], [**15**, Sec. 2]

Follow the exposition in [**17**, Sec. 3.4] and introduce the trace and trace class operators on a Hilbert space. The trace is a positive linear functional, taking possibly the value $+\infty$, on the cone of nonnegative operators. The theory of the trace in many respects resembles measure theory. This should be emphasized and therefore also the so–called Schatten p–ideals [**17**, E. 3.4.2-E.3.4.4] should be discussed. Another crucial result is the fact that the map $(S,T) \mapsto \operatorname{tr}(ST)$, where S is bounded and T is trace class, implements the duality between the Banach space of trace class operators and the Banach space of bounded linear operators. This generalizes the well–known duality between L^1 and L^∞ in measure theory.

In addition to [**17**] one should briefly address the problem whether for a trace class operator T the sequence of eigenvalues is summable and $\operatorname{tr}(T)$ equals the sum of the eigenvalues. For self–adjoint operators this is easy to see and it should be presented. The general case, the so–called Lidskii–Theorem, is more difficult. Consult [**18**] and give at least a couple of comments.

If time permits the uniqueness problem for the trace may be addressed. See, e.g. [**15**, Sec. 2].

The following simple but important Lemma should be proved:

PROPOSITION. *Let H be a Hilbert space and $T \in \mathcal{B}(H)$ a Fredholm operator. Furthermore let $S \in \mathcal{B}(H)$ be a parametrix of T such that $(I-ST)^k, (I-TS)^k$ are trace class for some integer $k \geq 1$. Then*

$$\operatorname{ind}(T) = \operatorname{tr}((I-ST)^k) - \operatorname{tr}((I-TS)^k).$$

2. Fredholm modules and the index pairing.

Material: [**12**, Sec. 2.1], [**3**, Appendix], [**4**, Appendix IV.A], [**13**, Chap. 8]

The material of [**12**, 2.1] should be covered. However, some more background on the role of Fredholm modules as cycles in K–homology should be given [**13**, Chap. 8]. K–homology is the dual theory to K–theory and there is a natural bilinear pairing

$$\text{index} : K_j(A) \times K^j(A) \longrightarrow \mathbb{Z}.$$

Since Fredholm modules are the cycles in K–homology, every Fredholm module induces naturally a map $K_j(A) \to \mathbb{C}$. This map should be explained in detail in both the even and in the odd case ([**12**, Sec. 2.1], [**4**, Prop. IV.2]).

Explain the difference between algebraic K_j and topological K_j for $j = 0, 1$.

3. The (Chern) character of a finitely summable Fredholm module (tentatively 1 1/2 sessions).

Material: [**12**, Sec. 2.2], [**4**, Sec. IV.1],

Present [**12**, Sec. 2.2] giving full proofs of Theorems 2.10 and Theorem 2.11. For more details consult [**4**, Sec. IV.1], [**3**, p. 53 et seqq.]. Try to give a proof of Theorem 2.7 without becoming too technical about cyclic cohomology.

4. Hochschild (co)homology.

Material: [**16**, Chap I], [**12**, Sec. 2.3], [**8**], [**20**]

Concentrate on Sections 1.0, 1.1, 1.5 and 1.6 of Loday's book. You may use the framework of simplicial modules but you should avoid a too high level of abstraction. The main examples of simplicial modules are the Hochschild complex and co-complex.

The bar resolution Prop. 1.1.12 and the normalized Hochschild complex (1.1.14) should be discussed thoroughly. Loday uses a spectral sequence argument. Also for later purposes it might be helpful to give a concise presentation of the spectral sequence of a double complex [**16**, Appendix D].

It will help to demystify the spectral sequence argument if you give a short direct proof of the acyclicity of the normalized Hochschild complex using the displayed formula in the proof of [**16**, Lemma 1.6.6].

5./6. Cyclic (co)homology I and II.

Material: [**16**, Chap II], [**12**, Sec. 2.3], [**4**, Sec. III.1, III.3], [**8**], [**20**]

These talks are rather crucial. Connes' original presentation [**4**, Sec. III.1,III.3], [**3**, Part II] is a nice read. Loday's book is more streamlined but maybe a bit dry. I leave it to the speaker which source she prefers. In any case he should present the various complexes which calculate cyclic (co)homology (cyclic bicomplex, Connes' complex, Connes' bB–bicomplex). Connes' SBI–sequence and the periodicity operator as well as period cyclic (co)homology should be covered, too. Note that in Loday's book the periodic theory is discussed much later in Chap. V.

Another topic which should be discussed is the notion of the character of a cycle [**4**, Sec. III.1.α]. This gives a natural way to construct cyclic cocycles. This sheds new light on the 3. talk. Finally present Theorems 2.19 and 2.21 in [**12**, Sec. 2.3].

7. The Hochschild–Kostant–Rosenberg–Connes Theorem.

Material: [**12**, Sec. 2.4], [**3**, Theorem 46], [**11**, Sec. 8.5]

For the algebra of smooth functions on a manifold the (continuous) Hochschild homology is canonically isomorphic to the space of differential forms and the (continuous) periodic cyclic homology is canonically isomorphic to the $\mathbb{Z}_2$–graded de Rham cohomology. This is the content of the so–called HKRC–Theorem. As a consequence Hochschild homology may be viewed as the noncommutative substitute to differential forms and periodic cyclic homology may be viewed as the noncommutative substitute for de Rham cohomology.

A proof of this important theorem can be found in [**3**] and in [**11**]. There exists a relatively new proof due to Teleman (see the references in [**11**]).

Explain the commutative diagram on page 52 of [**3**].

8./9. The (entire) Chern character of a θ–summable Fredholm module.

Material: [**12**, Sec. 2.5], [**9**], [**10**], [**11**, Chap. 10]

For compact manifolds the Chern character is a natural transformation from K–cohomology to de Rham cohomology. The noncommutative Chern character is a natural transformation from K–theory to cyclic homology. Discuss and prove [**12**, Thm. 2.27, 2.7].

An elliptic operator D on a closed manifold naturally induces a finitely summable Fredholm module over the algebra of smooth functions $C^\infty(M)$. From a computational point of view, however, it is rather awkward to deal with the phase $F = D/|D|$ of D. Jaffe–Lesniewski–Osterwalder invented a different approach to the Chern character which uses the heat operator e^{-tD^2} and which has some computational advantages. In these two talks we will study this JLO Chern character in detail. It pairs naturally with the JLO Chern character of a spectral triple. The presentation should follow the original papers by Getzler and Getzler–Szenes.

10. Transgression of the Chern character.

Material: [**7**]

The JLO Chern character a priori lives in entire cyclic cohomology. In [**7**] it is shown how the JLO Chern character can be related to the Chern character in periodic cyclic cohomology by analyzing short and large time limits of the heat operator. This talk should basically present the content of [**7**].

11. Further topics. There are various options for further topics (if time permits, the SS is short). Highlights to select from are

- Connes–Moscovici's proof of a version of the Novikov conjecture [**4**]
- The Quantum Hall effect [**4**]
- The Dixmier trace and the Hochschild character theorem [**12**, Sec. 3]
- The Local Index Theorem of Connes and Moscovici [**6**]

References

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