Π^1_1-Determinacy and Sharps in Second Order Arithmetic

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August 7, 2015

Contents

Second Order Arithmetic

The language $\mathcal{L}_2$ of SOA is two-sorted and has additonal constants resp. functions $0, 1, +, \cdot, <, =$ and $\in$. Its second-order axioms are

(IA) $\forall X[0 \in X \wedge \forall n[n \in X \rightarrow n+1 \in X] \rightarrow \forall n(n \in X)]$.

(CA) For any formula $\varphi(n)$ such that X is not free in φ:

$$\exists X \forall n[n \in X \leftrightarrow \varphi(n)].$$

(AC) For any $\mathcal{L}_2$-formula $\varphi(n, X)$:

$$\forall n \exists X \varphi(n, X) \rightarrow \exists X \forall n \varphi(n, (X)_n),$$

where $(X)_n = \{m | (n, m) \in X\}$ and $(,)$ denotes Gödel pairing.

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SOA vs. $\mathrm{ZFC}^- + \mathrm{V} = \mathrm{HC}$ (0)

Theorem

SOA *and* $\mathrm{ZFC}^- + \mathrm{V} = \mathrm{HC}$ *bi-interpret each other.*

Clearly, if $M \models \mathrm{ZFC}^- + \mathrm{V} = \mathrm{HC}$, then
$M^2 = (\omega^M, \mathcal{P}^M(\omega), +^M, \cdot^M, =^M, \dots) \models \mathrm{SOA}$.

What about the converse?

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What about the converse?

Trees in SOA

- A **tree** is a set $T \subseteq \text{Seq}$ such that

$$\forall s, t(s \subseteq t \wedge t \in T \rightarrow s \in T).$$

- If $s \in T$, then $T_s = \{t \in \text{Seq} \mid s^\frown t \in T\}$. A tree of the form $T_{\langle n \rangle}$ is called a **direct subtree** of T.
- A function $F : \mathcal{N} \rightarrow \mathcal{N}$ is a **path** through T, if for every n, $\langle F(0), \ldots, F(n-1) \rangle \in T$. We denote the class of paths through T by $[T]$.
- T is **well-founded**, if $[T] = \emptyset$.

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Every well-founded tree codes a hereditarily countable set

$$|T| = \{|T_{\langle n \rangle}| \mid \langle n \rangle \in T\}.$$

Then we can define an equivalence relation on the class of well-founded trees $\mathcal{T}$ by

$$S \simeq T \quad \text{iff} \quad |S| = |T|.$$

Furthermore define $S \tilde{\in} T$ iff $\exists n(\langle n \rangle \in T \wedge S \simeq T_{\langle n \rangle})$.
Then the quotient of $\mathcal{T}$ mod $\simeq$ is a model of $ZFC^{-} + V = HC$.

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The Perfect Subset Property (0)

Definition

A tree T is said to be

- **pruned**, if every element of T has a proper extension.
- **perfect**, if every element of T has at least two incompatible extensions.

Now define a topology on the Baire space $\mathcal{N}$:

$\mathcal{X} \subseteq \mathcal{N}$ is **closed**, if there is a pruned tree such that $\mathcal{X} = [T]$.

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The Perfect Subset Property (1)

Definition

$\mathcal{X} \subseteq \mathcal{N}$ is said to have a **perfect subset**, if there is a perfect tree T with $[T] \subseteq \mathcal{X}$.

Now for Γ a class of formulae, let $\Gamma - \mathrm{PSP}$ state that every uncountable class of reals definable by a formula in Γ has a perfect subset.

The Perfect Subset Property (2)

Theorem (Cantor-Bendixson)

SOA $\vdash \boldsymbol{\Sigma}^1_1 - \mathrm{PSP}$.

Sketch of the proof.

Let $\mathcal{X} = [T]$. We thin out T countably many times by removing at each step its isolated points: Let $T' = \{s \in T \mid \exists t, u \in T(t, u \supseteq s \wedge t \perp u)\}$. Then

- $T_0 = T$
- $T_{\alpha+1} = (T_\alpha)'$
- $T_\alpha = \bigcap_{\beta<\alpha} T_\beta$, if $\alpha \in \mathrm{Lim}$.

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Theorem

For every real X, $\Pi^1_1[X] - \mathrm{PSP}$ implies $\Sigma^1_2[X] - \mathrm{PSP}$.

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Well-orders in SOA

Definition

- If $X, Y \in \mathrm{WO}$ we write $X \leq_{\mathrm{WO}} Y$ if there is an order-preserving map from the domain of X to the domain of Y.
- $\mathcal{X} \subseteq \mathrm{WO}$ is **order-type bounded**, if there is $X \in \mathrm{WO}$ such that for all $Y \in \mathcal{X}$, $Y \leq_{\mathrm{WO}} X$.

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Every $\boldsymbol{\Sigma}^1_1$-subclass of WO is order-type bounded.

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L[X] in SOA (0)

For any real X we can construct L[X] in a model of SOA.

Observation

For any real X, L[X] $\models$ ZFC$^-$.

Goal

We want to prove that assuming $\Pi^1_1[X]-\mathrm{PSP}$, L[X] $\models$ ZFC.

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L[X] in SOA (1)

Note that the class of reals in L[X] is $\Sigma^1_2[X]$ and the well-ordering $\prec_X$ of L[X] is $\Sigma^1_2[X]$. Define

$$Y \in \mathcal{C}_X \Longleftrightarrow Y \in \mathrm{L}[X] \wedge Y \in \mathrm{WO} \wedge \forall Z(Z \in \mathrm{WO} \wedge Z \prec_X Y \rightarrow Z \neq_{\mathrm{WO}} Y).$$

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$\mathcal{C}_X$ is a Σ^1_2-class of reals.

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L[X] in SOA (2)

Proposition

$\Pi^1_1[X]-\mathsf{PSP}$ *fails in* L[X].

Sketch of the proof.

- $\mathcal{C}_X$ is unbounded in WO $\cap$ L[X].
- $\mathcal{C}_X$ is uncountable in L[X], since otherwise there would exist $Y \in \mathrm{L}[X]$ such that $\mathcal{C}_X = \{(Y)_n \mid n \in \mathbb{N}\}$, i.e. Y bounds $\mathcal{C}_X$ in WO $\cap$ L[X].
- If $\mathcal{X} \subseteq \mathcal{C}_X$ is perfect, then it is $\mathbf{\Sigma}^1_1$ and thus order-type bounded and in particular countable.

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Constructing L[X] in SOA

Theorem

$\Pi^1_1[X] - \mathrm{PSP}$ *implies that* $\mathrm{L}[X] \models \mathrm{ZFC}$.

Sketch of the proof.

- $\mathcal{C}_X$ is countable, thus if Z is an upper bound of $\mathcal{C}_X$ then $\|Z\|$ is uncountable in $\mathrm{L}[X]$. In particular, $\aleph_1^{\mathrm{L}[X]}$ exists.
- Let $\alpha \in \mathrm{Lim}$ and $Y \in \mathrm{L}_{\alpha+1}[X] \setminus \mathrm{L}_\alpha[X]$ with $Y = \{n \in \omega \mid \varphi(n, p_0, \ldots, p_n, X)\}$. Let $\beta < \alpha$ s.t. all $p_i \in \mathrm{L}_\beta[X]$ and n s.t. all formulae defining the p_i in $\mathrm{L}_\beta[X]$ are at most $\Pi_n[X]$. Let H be the $\Sigma_{n+1}[X]$-Skolem hull of all defining parameters of the p_i in $\mathrm{L}_\alpha[X]$. Then $H = \mathrm{L}_{\bar{\alpha}}[X]$ for some $\bar{\alpha} \leq \alpha$. Thus $\alpha = \bar{\alpha}$ is countable. Hence $\mathcal{P}(\omega)^{\mathrm{L}[X]}$ exists in $\mathrm{L}_{\omega_1+1}[X]$.

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Constructing $\mathsf{L}[X]$ in SOA

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Motivation

Start with $M \models \text{ZFC}$. Using a class version of the Lévy collapse, we obtain $M[G] \models \text{ZFC}^- + \text{V} = \text{HC}$ such that in $M[G]$ full projective topological regularity holds, i.e.

- Perfect subset property
- Lebesgue measurability
- Baire property

The class Lévy collapse (0)

Let $M \models$ ZFC be a countable transitive model.

Definition

For $\pi \leq \mathrm{Ord}^M$, the **Lévy collapse** forcing is defined as

$$\mathrm{Col}(\omega, < \pi) = \{p \mid p : \mathrm{dom}(p) \to \pi \wedge \mathrm{dom}(p) \text{ is finite} \wedge \ \mathrm{dom}(p) \subseteq \pi \times \omega \ \wedge \forall \langle \alpha, n \rangle \in \mathrm{dom}(p) : p(\alpha, n) < \alpha\}.$$

Note that for $\pi = \mathrm{Ord}^M$ this is a class forcing.

The class Lévy collapse (1)

$\mathrm{Col}(\omega, < \mathrm{Ord}^M)$ has the following properties:

1. $\mathrm{Col}(\omega, < \mathrm{Ord}^M)$ collapses every M-cardinal, since $f_\alpha^G = \{\langle n, \beta\rangle \mid \exists p \in G(\langle\langle\alpha, n\rangle, \beta\rangle \in p\}$ is a surjection $\omega \twoheadrightarrow \alpha$.
2. $\mathrm{Col}(\omega, < \mathrm{Ord}^M)$ is pretame.
3. If G is generic, then $M[G] = \bigcup_{\pi < \kappa^M} M[G \cap \mathrm{Col}(\omega, < \pi)]$.

Thus we get

Proposition

If G is $\mathrm{Col}(\omega, < \mathrm{Ord}^M)$-generic over M, then $M[G] \models \mathrm{ZFC}^- + \mathrm{V} = \mathrm{HC}$.

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The Lévy collapse and topological regularity

In order to obtain the Solovay model one collapses an inaccessible cardinal and then considers the inner model $\mathrm{L}(\mathbb{R})$. Ord^M is also like an inaccessible, and Solovay's proof can be adapted in order to obtain

Theorem

If G is $\mathrm{Col}(\omega, < \mathrm{Ord}^M)$-generic over M, then $M[G]$ satisfies full projective regularity.

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Conclusion

Theorem

The following theories are equiconsistent:

1. $\mathsf{SOA} + \Pi^1_1 - \mathsf{PSP}$
2. SOA + "full topological regularity"
3. ZFC

Goal

The methods shown in the previous slides will be generalized to shown

Theorem

The following theories are equiconsistent:

- $\mathrm{SOA} + \boldsymbol{\Pi}^1_1 - \mathrm{Det} + \Pi^1_2 - \mathrm{PSP}$
- $\mathrm{SOA} + \boldsymbol{\Pi}^1_1 - \mathrm{Det} +$ "full topological regularity"
- $\mathrm{ZFC} +$ "$\forall a \subseteq \mathrm{Ord}(a^{\#}$ exists)".

Sharps (0)

We briefly recall the definition of sharps. Work in $\mathrm{ZFC}^- + \mathrm{V} = \mathrm{HC}$.

Definition

Let $\mathcal{M}$ be a structure with domain M and $\langle I, < \rangle$ a linearly ordered set with $I \subseteq M$. Then I is a set of **indiscernibles** for $\mathcal{M}$, if for every formula φ in the language of $\mathcal{M}$ and for all $x_1 < \cdots < x_n$, $y_1 < \cdots < y_n$ in I,

$$\mathcal{M} \models \varphi(x_1, \ldots, x_n) \quad \text{iff} \quad \mathcal{M} \models \varphi(y_1, \ldots, y_n).$$

Sharps (1)

Lemma

Assume $\beta \in \mathrm{Lim}$ and I is a set of indiscernibles for L_β. Then for every $\gamma \in \mathrm{Lim}$ there is a model $\mathcal{M}$ of $\mathrm{Th}(\mathrm{L}_\beta)$ and a set $J \subseteq \mathrm{Ord}$ of indiscernibles for $\mathcal{M}$ such that

1. $\mathrm{otp}(J) = \gamma$
2. *For every $\mathcal{L}_\in$-formula $\varphi(\vec{v})$, we have $\mathcal{M} \models \varphi(\vec{y})$ for some $\vec{y} \in J$ iff $\mathrm{L}_\beta \models \varphi(\vec{x})$ for some $\vec{x} \in I$.*
3. *The Skolem hull of J in $\mathcal{M}$ is $\mathcal{M}$.*

*The pair $\langle \mathcal{M}, J \rangle$ is unique up to isomorphism and is called the γ-**model** for $\langle \mathrm{L}_\beta, I \rangle$.*

Sharps (2)

Definition

Let I be a set of indiscernibles for L_β. Then $\mathrm{Th}(L_\beta, I)$ is

1. **well-founded**, if for every $\gamma \in \mathrm{Lim}$, the γ-model is well-founded.
2. **unbounded**, if for every Skolem term $t(v_1, \ldots, v_n)$, $t(c_1, \ldots, c_n) \in \mathrm{Ord} \rightarrow t(c_1, \ldots, c_n) < c_{n+1} \in \mathrm{Th}(L_\beta, I)$.
3. **remarkable**, if it is unbounded and for every Skolem term $t(v_1, \ldots, v_{m+n})$,

$$t(c_1, \cdots, c_{m+n}) \in \mathrm{Ord} \wedge t(c_1, \cdots, c_{m+n}) < c_m$$
$$\rightarrow t(c_1, \cdots, c_{m+n}) = t(c_1, \cdots, c_{m-1}, c_{m+n+1}, \cdots, c_{m+2n+1})$$

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2. **unbounded**, if for every Skolem term $t(v_1, \ldots, v_n)$, $t(c_1, \ldots, c_n) \in \mathrm{Ord} \to t(c_1, \ldots, c_n) < c_{n+1} \in \mathrm{Th}(L_\beta, I)$.
3. **remarkable**, if it is unbounded and for every Skolem term $t(v_1, \ldots, v_{m+n})$,

$$t(c_1, \cdots, c_{m+n}) \in \mathrm{Ord} \wedge t(c_1, \cdots, c_{m+n}) < c_m$$
$$\to t(c_1, \cdots, c_{m+n}) = t(c_1, \cdots, c_{m-1}, c_{m+n+1}, \cdots, c_{m+2n+1})$$

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Sharps (3)

We say that $0^{\#}$ **exists**, if there there is a set $I \subseteq \mathrm{Ord}$ with

1. $\mathrm{otp}(I) = \omega$
2. I is a set of indiscernibles for L_δ, where $\delta = \sup I$
3. The Skolem hull of I in L_δ is L_δ
4. $\mathrm{Th}(\mathrm{L}_\delta, I)$ is remarkable and well-founded.

Significance of $0^{\#}$

Assume $0^{\#}$ exists.

- Well-foundedness implies that the γ-model is of the form L_β for some β.
- Unboundedness means that the indiscernibles of the γ-model L_β are cofinal in $\mathrm{Ord} \cap L_\beta$.
- Remarkability implies that whenever $\gamma \leq \delta$ and $\gamma \in \mathrm{Lim}$ and $\langle x_\xi \mid \xi < \delta \rangle$ are the indiscernibles for the δ-model, then the γ-model is L_{x_γ} and its indiscernibles are $\{x_\xi \mid \xi < \gamma\}$. In particular, $\{x_\xi \mid \xi < \delta\}$ is closed in the ordinals of the δ-model.

Consequence

There is a unique class $\bar{I} = \{x_\xi \mid \xi \in \mathrm{Ord}\}$ of indiscernibles for L which is closed unbounded in Ord and such that the Skolem hull of $\bar{I}$ in L is L and for every $\xi \in \mathrm{Ord}$, $L_{x_\xi} \prec L$.

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Characterizations of $0^{\#}$ (0)

$0^{\#}$ allow to define truth in L by

$$\mathrm{L} \models \varphi(a_1, \dots, a_n) \Longleftrightarrow \forall \delta \in \bar{I}(a_1, \dots, a_n \in \mathrm{L}_\delta \rightarrow \mathrm{L}_\delta \models \varphi(a_0, \dots, a_n)).$$

In particular, we can code $0^{\#}$ as a real by putting

$$0^{\#} = \{\ulcorner \varphi(v_1, \dots, v_n) \urcorner \mid \mathrm{L}_\delta \models \varphi(x_1, \dots, x_n)\}$$

where I is the set of indiscernibles of ordertype ω, $\delta = \sup I$ and $x_1, \dots, x_n \in I$.

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Characterizations of $0^{\#}$ (1)

Theorem (Kunen)

In ZFC, $0^{\#}$ exists if and only if there is a non-trivial elementary embedding $j : \mathrm{L} \rightarrow \mathrm{L}$.

Sketch of the proof.

For ($\Rightarrow$) take any non-trivial order-preserving injection $h : \bar{I} \rightarrow \bar{I}$ and extend it to L by stipulating $h(t^{\mathrm{L}}(x_{\xi_1}, \ldots, x_{\xi_n})) = t^{\mathrm{L}}(h(x_{\xi_1}), \ldots, h(x_{\xi_n}))$ for every Skolem term $t(v_1, \ldots, v_n)$. $\dashv$

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Characterizations of $0^{\#}$ (2)

Theorem (ZFC)

$0^{\#}$ exists if and only if there is an uncountable cardinal κ and a limit ordinal δ such that L_{δ} has a set of indiscernibles of order type κ.

Sharps for sets of ordinals

We can generalize the concept of $0^\#$ to $a^\#$ for any set a of ordinals: We say that $a^\#$ exists if there is a set of indiscernibles I for $\langle \mathrm{L}_\delta[a], a, \xi\rangle_{\xi \leq \sup a}$ with the properties from before.

Let for $T^\#$ denote $T + \forall a \subseteq \mathrm{Ord}(a^\# \text{ exists})$ for a theory T.

Goal

Assume $(\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC})^\#$. We want to construct an inner model of $\mathsf{ZFC}^\#$.

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The Martin-Harrington Theorem

Theorem (Martin-Harrington)

Assume ZFC. *For any real* X, $\Pi^1_1[X]-\mathrm{Det}$ *holds iff* $X^\#$ *exists.*

Harrington's proof that $\Pi^1_1[X]$-determinacy implies the existence of $X^\#$ applies determinacy to a $\Sigma^1_1[X]$-set of reals A_X and shows that Player II cannot have a winning strategy.

Observation (SOA)

$\mathrm{SOA}+\Pi^1_1-\mathrm{Det}+\boldsymbol{\Pi}^1_1-\mathrm{PSP}\vdash 0^\#$ exists.

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Determinacy vs. sharps in SOA

Theorem

SOA $\vdash \boldsymbol{\Pi}^1_1 - \mathrm{Det} \leftrightarrow \forall a \subseteq \mathrm{Ord}(a^{\#} \ \mathit{exists})$.

Proof.

Work in $\mathrm{ZFC}^- + \mathrm{V} = \mathrm{HC}$. Let X be a real with $a \in \mathrm{L}[X]$. By $\Pi^1_1 - \mathrm{Det}$, A_X is determined. If there was a w.s. τ for Player II, then in $\mathrm{L}[\tau]$ there would be a w.s. for Player II. Let σ be a w.s. for Player I, then $\mathrm{L}[X, \sigma] \models \mathrm{ZFC}$ and σ is a w.s. in $\mathrm{L}[X, \sigma]$. Hence $X^{\#}$ exists in $\mathrm{L}[X, \sigma]$. Work in $\mathrm{L}[X, \sigma]$. There is a closed unbounded class $\bar{I}$ of indiscernibles for $\mathrm{L}[X]$ containing all uncountable cardinals. There is a $\kappa > \omega$ with $a \in \mathrm{L}_\kappa[X]$ and $I_\kappa = \bar{I} \cap \kappa$ is a set of indiscernibles of otp κ. Write $a = t^{\mathrm{L}_\kappa[X]}(\vec{x})$, $\vec{x} \in I_\kappa$. Then $J = I_\kappa \setminus \sup \vec{x}$ is a set of indiscernibles for $\mathrm{L}_\kappa[a]$ of otp κ, i.e. $a^{\#}$ exists in $\mathrm{L}[X, \sigma]$ and also in V. $\dashv$

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Suppose that $\forall a \subseteq \mathrm{Ord}(a^{\#}$ exists). Let B be a $\Pi^1_1[X]-$class of reals for some real X. Then $X^{\#}$ and $(X^{\#})^{\#}$ exist. Then $\mathrm{L}[X^{\#}] \models \mathrm{ZFC} + X^{\#}$ exists. So in $\mathrm{L}[X^{\#}]$, there is wlog a w.s. σ for Player I. By absoluteness, σ is also a w.s. in V. $\dashv$

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Getting an inner model of ZFC$^{\#}$ (0)

Assume SOA + Π^1_1-Det resp. $(\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC})^{\#}$.

$$^{\#}\alpha = (\bigcup_{\beta<\alpha} {}^{\#}\beta)^{\#}, \alpha \in \mathsf{Ord}$$

$$\# = \bigcup_{\alpha \in \mathsf{Ord}} {}^{\#}\alpha.$$

$$\mathsf{L}^{\#}_{\alpha} = \mathsf{L}_{\alpha}[\#] \text{ and } \mathsf{L}^{\#} = \mathsf{L}[\#].$$

In particular, this means that $\mathsf{L}^{\#} \models \mathsf{ZFC}^-$.

Question

What do we need for $\mathsf{L}^{\#}$ *to be a model of* $\mathsf{ZFC}^{\#}$*?*

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Sketch of the proof.

Let $a \in \mathrm{L}^{\#}_{\alpha}$ a set of ordinals s.t. $\#$ is cofinal in α. Then $A := \# \cap \alpha = \bigcup_{\xi<\gamma} {}^{\#}\xi$ for some γ. Then $A^{\#} = {}^{\#}\gamma \in \mathrm{L}^{\#}$. Then $\mathrm{L}[A^{\#}] \models \mathrm{ZFC} + A^{\#}$ exists. Work in $\mathrm{L}[A^{\#}]$. The ω_1-model is of the form $\mathrm{L}_{\mu}[A]$ with a set I of indiscernibles of otp ω_1. $a \in \mathrm{L}_{\mu}[A]$, so $a = t^{\mathrm{L}_{\mu}[A]}(\vec{x})$ for some Skolem term t and $\vec{x} \in I$. Then $J = I \setminus \sup \vec{x}$ is a set of indiscernibles of otp ω_1 for $\mathrm{L}_{\mu}[a]$. So $a^{\#}$ exists in $\mathrm{L}[A^{\#}]$ and thus in V. $\dashv$

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Getting an inner model of ZFC$^{\#}$ (2)

Observation

Th classes $\mathcal{N} \cap \mathrm{L}^{\#}$ and $<_{\#} \cap \mathcal{N} \times \mathcal{N}$ are Σ^1_3.

Lemma

Assume $(\mathrm{ZFC}^- + \mathrm{V} = \mathrm{HC})^{\#} + \Sigma^1_3 - \mathrm{PSP}$. *Then* $\mathrm{L}^{\#} \models \mathrm{ZFC}^{\#}$.

Sketch of the proof.

As in the proof of $\mathrm{L} \models \mathrm{ZFC}$: Consider the Σ^1_3-class

$$X \in \mathcal{C} \leftrightarrow X \in \mathrm{L}^{\#} \wedge X \in \mathrm{WO} \wedge \forall Y (Y \in \mathrm{WO} \\ \wedge Y \prec_{\#} X \rightarrow Y \neq_{\mathrm{WO}} X).$$

Then $\mathcal{C}$ is unbounded in $\mathrm{L}^{\#}$, uncountable in $\mathrm{L}^{\#}$ and contains no perfect subset. So $\mathcal{C}$ is countable and $\aleph_1^{\mathrm{L}^{\#}}$ exists. ⊣

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We can reduce the complexity to Π^1_2 by adapting the proof of Solovay(?) that if $\omega_1^L = \omega_1$ the $\Pi^1_1 - \mathrm{PSP}$ fails.

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$\Pi^1_2 - \mathrm{PSP}$ *fails in* $\mathrm{L}^{\#}$.

Sketch of the proof.

For an ordinal α countable in $\mathrm{L}^{\#}$ there is $X_0 \in \mathcal{N} \cap \mathrm{L}^{\#}$ of otp α. To specify the minimal such X_0 is not Π^1_2 but it is Π^1_2 to say that some $X_1 \in \mathcal{N} \cap \mathrm{L}^{\#}$ codes the least $\mathrm{L}^{\#}_{\delta}$ such that $\# \cap \delta$ is cofinal in δ and $X_0 \in \mathrm{L}^{\#}_{\delta}$, again to specify the minimal such X_1 is not Π^1_2 and so on. Then $X = \langle X_i \mid i \in I \rangle$ codes α uniquely. The class of such reals X is Π^1_2, unbounded in $\mathrm{L}^{\#}$ and contains no perfect subset. $\dashv$

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Thus we have shown

Theorem

Assume $\mathrm{SOA} + \boldsymbol{\Pi}^1_1 - \mathrm{Det} + \Pi^1_2 - \mathrm{PSP}$. *Then* $\mathrm{L}^\# \models \mathrm{ZFC}^\#$.

Furthermore, the complexity is optimal: Otherwise, start with $\mathrm{V} \models \mathrm{ZFC}^\#$ and take the analytical part $\mathrm{V}^2 = (\omega^{\mathrm{V}}, \mathcal{P}^{\mathrm{V}}(\omega), \dots)$ which satisfies $\mathrm{SOA} + \boldsymbol{\Pi}^1_1 - \mathrm{Det}$, thus $(\mathrm{L}^\#)^{\mathrm{V}^2} \models \mathrm{ZFC}^\#$ and so on. Thus we obtain a decreasing $\in$-sequence of models of $\mathrm{SOA} + \boldsymbol{\Pi}^1_1 - \mathrm{Det}$.

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Collapsing a model of ZFC$^\#$

Theorem

If $M \models \mathrm{ZFC}^\#$ *and* G *is* $\mathrm{Col}(\omega, < \mathrm{Ord}^M)$*-generic over* M*, then* $M[G] \models (\mathrm{ZFC}^- + \mathrm{V} = \mathrm{HC})^\# +$ *projective topological regularity.*

Sketch of the proof.

Idea: show that sharps are preserved in intermediate extensions and then use absoluteness. Assume $\pi \in \mathrm{Ord}$ and G $\mathrm{Col}(\omega, < \pi)$-generic. If $\dot{A} \in M$ is s.t. $\dot{A}^G$ is a set of ordinals in $M[G]$, then code $\dot{A}$ as a set of ordinals. Since $M \models \mathrm{ZFC}^\#$ there is an elementary embedding $j : \mathrm{L}[\dot{A}] \to \mathrm{L}[\dot{A}]$. Lift j to an elementary embedding $\bar{j} : \mathrm{L}[A] \to \mathrm{L}[A]$ in $M[G]$. $\dashv$

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Sketch of the proof.

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Conclusion

Thus we have shown

Theorem

The following theories are equiconsistent:

- $\mathrm{SOA} + \boldsymbol{\Pi}^1_1 - \mathrm{Det} + \Pi^1_2 - \mathrm{PSP}$
- $\mathrm{SOA} + \boldsymbol{\Pi}^1_1 - \mathrm{Det} +$ projective topological regularity
- $\mathrm{ZFC}^{\#}$.