I am no title.

Incompleteness, Undecidability, Undefinability

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Content

Historical background

- Hilbert program: complete and contradiction-free axiomatization of mathematics
- 1889: Axiomatization of arithmetic by G. Peano (Peano-Arithmetic)
- 1908: Axiomatization of set theory by E. Zermelo (ZFC)
- 1910: Principia Mathematica (Whitehead/Russell)
- 1929: Gödel's Completeness Theroem
- 1931: Gödel's Incompleteness Theorems

The Hilbert program

- Assumption: Consistency of the axioms of set theory can be proven
- All of mathematics can be axiomatized
- Consistency can be proven (2nd Hilbert problem) using only finitary methods

The Hilbert program

Example of Hilbert's notion of finitary methods for the equation $x + y = y + x$:

"Eine solche Gleichung [...] wird nicht aufgefasst als eine Aussage über alle Zahlen. Vielmehr wird sie so gedeutet, dass ihr Sinn sich in einem Beweisverfahren erschöpft, bei welchem jeder Schritt eine vollständig aufweisbare Handlung ist, die nach festgesetzten Regeln vollzogen wird."

The liar paradoxon

This statement is false.

In a village, the barber shaves everyone who does not shave himself/herself, but no one else. Who shaves the barber?
B. Russell

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Questions

- What is arithmetic and how can it be “formalized”?
- Can arithmetic be described in a way such that any property is either a consequence of the axioms are it can be shown to be contradictory?
- What is truth and can it be formalized?
- Is truth and provability the same?
- How can one solve the liar paradoxon?

The natural numbers

- What are the natural numbers?
- How can we axiomatize them?
- Is the set of natural numbers absolute (i.e. unique)?

What are the natural numbers?

Leopold Kronecker (1823-1891)

God made the natural numbers; all else is the work of man.

The language of arithmetic

The language $\mathcal{L}_{\mathrm{PA}}$ of Peano Arithmetic consists of

- a constant 0
- a unary function symbol s
- two binary function symbols $+, \cdot$.

Terms and Formulae

Terms

(T1) 0 is a term

(T2) Every variable x is a term

(T3) If t_0, t_1 are terms, then so are $\mathrm{s}t_0$, $t_0 + t_1$ and $t_0 \cdot t_1$.

Formulae

(F1) If t_0, t_1 are formulae, then so is $t_0 = t_1$

(F2) If φ is a formula, then so is $\neg\varphi$

(F3) If φ, ψ are formulae, then so are $\varphi \wedge \psi, \varphi \vee \psi$ and $\varphi \rightarrow \psi$

(F4) If φ is a formula and x is a variable, then $\exists x\varphi$ and $\forall x\varphi$ are formulae.

Logical axioms

For formulae $\varphi, \varphi_0, \varphi_1, \varphi_2$ and ψ we have the following schemes of logical axioms:

(L_1) $\varphi \to (\psi \to \varphi)$

(L_2) $(\psi \to (\varphi_0 \to \varphi_1)) \to ((\psi \to \varphi_0) \to (\psi \to \varphi_1))$

(L_3) $(\varphi \wedge \psi) \to \varphi$

(L_4) $(\varphi \wedge \psi) \to \psi$

(L_5) $\psi \to (\varphi \to (\varphi \wedge \psi))$

(L_6) $\varphi \to (\varphi \vee \psi)$

(L_7) $\psi \to (\varphi \vee \psi)$

(L_8) $(\varphi_0 \to \varphi_2) \to ((\varphi_1 \to \varphi_2) \to ((\varphi_0 \vee \varphi_1) \to \varphi_2))$

(L_9) $(\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi)$

(L_{10}) $\neg\varphi \to (\varphi \to \psi)$

(L_{11}) $\varphi \vee \neg\varphi$.

Logical Axioms

For a formula φ and a term t such that the substitution $\varphi(x/t)$ is admissible we have

(L_{12}) $\forall x \varphi(x) \to \varphi(t)$

(L_{13}) $\varphi(t) \to \exists x \varphi(x)$.

If ψ is a formula and t is a term with $t \notin \mathrm{free}(\psi)$, then

(L_{14}) $\forall x(\psi \to \varphi(x)) \to (\psi \to \forall x \varphi(x))$

(L_{15}) $\forall x(\varphi(x) \to \psi) \to (\exists x \varphi(x) \to \psi)$.

If $t, t_0, \ldots, t_{n-1}, t'_0, \ldots, t'_{n-1}$ are terms, R is an n-ary relation symbol and F is an n-ary function symbol, the following are logical axioms:

(L_{16}) $t = t$

(L_{17}) $(t_0 = t'_0 \wedge \cdots \wedge t_{n-1} = t'_{n-1}) \to (R(t_0, \ldots, t_{n-1}) \to R(t'_0, \ldots, t'_{n-1}))$

(L_{18}) $(t_0 = t'_0 \wedge \cdots \wedge t_{n-1} = t'_{n-1}) \to (F(t_0, \ldots, t_{n-1}) = F(t'_0, \ldots, t'_{n-1}))$.

Rules of inference

Modus Ponens (MP): $\frac{\varphi \to \psi, \varphi}{\psi}$

Generalization ($\forall$): $\frac{\varphi}{\forall x \varphi}$.

Peano Arithmetic

(PA_1) $\forall x \neg(\mathrm{s}x = 0)$

(PA_2) $\forall x \forall y(\mathrm{s}x = \mathrm{s}y \rightarrow x = y)$

(PA_3) $\forall x(x + 0 = x)$

(PA_4) $\forall x \forall y(x + \mathrm{s}y = \mathrm{s}(x + y))$

(PA_5) $\forall x(x \cdot 0 = 0)$

(PA_6) $\forall x \forall y(x \cdot \mathrm{s}y = (x \cdot y) + x)$

(I_φ) $\forall \vec{y}\big[\varphi(0, \vec{y}) \wedge \forall x\big(\varphi(x, \vec{y}) \rightarrow \varphi(\mathrm{s}x, \vec{y})\big) \rightarrow \forall x \varphi(x, \vec{y})\big]$, where φ is an $\mathcal{L}_{PA}$-formula with $\mathrm{free}(\varphi) = \{x, \vec{y}\}$.

Formal proofs

A **formal proof** is a finite sequence $\varphi_0, \ldots, \varphi_n$ of formulas such that for every $i \leq n$ one of the following holds:

- φ_i is an axiom
- there are $j, k < i$ such that φ_k is $\varphi_j \to \varphi_i$
- there is $j < i$ such that φ_i is $\forall x \varphi_j(x)$ for some variable x.

Models of the natural numbers

Let T be a set of formulae, M a set. Let $\mathrm{s}^M, +^M, \cdot^M$ be unary resp. binary functions on M, and $0^M \in M$. Let $I : Var \to M$ be an assignment. We extend I to terms as follows: $I(0) = 0^M, I(\mathrm{s}t) = \mathrm{s}^M(I(t)), I(t_0 \dot{+} t_1) = I(t_0) \dot{+} I(t_1)$. Then define

- $(M, I) \models s = t \iff I(s) = I(t)$
- $(M, I) \models \varphi \wedge \psi \iff (M, I) \models \varphi$ und $(M, I) \models \psi$
- $(M, I) \models \neg\varphi \iff (M, I) \not\models \varphi$
- $(M, I) \models \exists x \varphi(x) \iff$ there is $a \in M$ with $(M, I\frac{a}{x}) \models \varphi(x)$, where $I\frac{a}{x}$ is the same as I except $I(x) = a$.

M is a **model** of T, if for every assignment I and every $\varphi \in T$, $(M, I) \models \varphi$.

The standard model $\mathbb{N}$

Put $\mathbb{N} = \{0, s0, ss0, sss0, ..., ssssssssss0, ...\}$, i.e.

- 0 is a natural number,
- s...s0 is a natural number, where the symbol s occurs only finitely many times.

But... what does FINITE mean???

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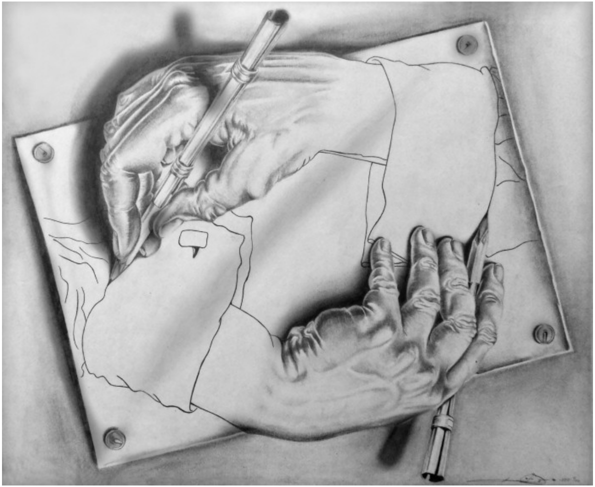
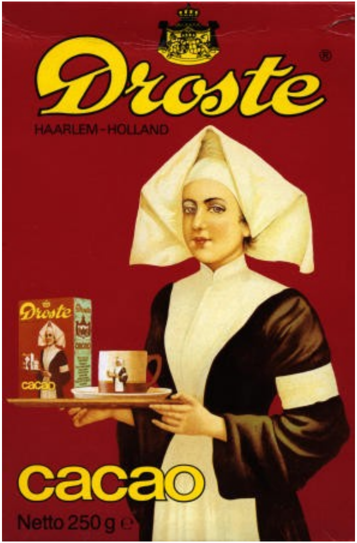


M.C. Escher, *Drawing Hands*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Consistency and completeness

Definition

A set T of formulae is said to be

- **consistent**, if there is no formula φ such that $T \vdash \varphi \wedge \neg\varphi$
- **complete**, if for every formula φ either $T \vdash \varphi$ or $T \vdash \neg\varphi$.

Gödel's Completeness Theorem

Theorem (Completeness Theorem)

A set T of formulae is consistent if and only if it has a model.

Theorem (Compactness Theorem)

A set T of formulae is consistent if and only if every finite subset of T is consistent (resp. has a model).

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Non-standard models

We extend $\mathcal{L}_{\text{PA}}$ to $\mathcal{L}_{\text{PA}}^{+} = \mathcal{L}_{\text{PA}} \cup \{c\}$ by adding a new constant symbol c.

Consider $\text{PA}^{+} = \text{PA} \cup \{c \neq 0, c \neq \text{s}0, c \neq \text{ss}0, \cdots\}$.

Compactness Theorem $\Rightarrow$ PA^{+} has a model N. Then c^{N} is bigger than every standard natural number.

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Is PA complete?

PA is **complete**, if for every $\mathcal{L}_{\mathsf{PA}}$-formula φ either $\mathsf{PA} \vdash \varphi$ or $\mathsf{PA} \vdash \neg\varphi$.

Theorem (First Incompleteness Theorem)

If PA is consistent, thent it is incomplete. More precisely, there is a sentence G_{PA} such that $\mathsf{PA} \nvdash G_{\mathsf{PA}}$ and $\mathsf{PA} \nvdash \neg G_{\mathsf{PA}}$.

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Basic number theory in PA

In PA one can introduce additional

- relations (such as $x \leq y :\leftrightarrow \exists x(x + r = y)$)
- functions (such as $x - y = z :\leftrightarrow (y \leq x \wedge x = z + y) \vee (x < y \wedge z = 0)$).

Furthermore, one can prove elementary number-theoretical results such as

- commutativity of $+$: PA $\vdash \forall x \forall y(x + y = y + x)$
- every number has a prime divisor: $\forall x \exists p(\text{prime}(p) \wedge p \mid x)$
- Chinese Remainder Theorem: f, g unary functions. Then: PA $\vdash \forall k\big[[\forall i < k(\underline{1} < g(i) \wedge f(i) < g(i)) \wedge \forall i \forall j(i < j \wedge j < k \rightarrow \text{coprime}(g(i), g(j))] \rightarrow \exists x \forall i < k(\text{rest}(x, g(i)) = f(i))\big]$.

Standard natural numbers in PA

For $n \in \mathbb{N}$ define

$$\underline{n} = \underbrace{\mathtt{s} \ldots \mathtt{s}}_{n} 0$$

Then $+, \cdot, \mathtt{s}, =$ and all newly introduced functions and relations are compatible with $n \mapsto \underline{n}$, e.g.

- $\mathrm{PA} \vdash \underline{m} + \underline{n} = \underline{m+n}$ for all $m, n \in \mathbb{N}$
- $m = n \Leftrightarrow \ \mathrm{PA} \vdash \underline{m} = \underline{n}$.

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Gödel's β function

In PA one can define a binary function β such that for every unary function f

$$\mathrm{PA} \vdash \forall k \exists x \forall i < k(\beta(x, i) = f(i)),$$

x encodes the sequence $\langle f(\mathbf{0}), \ldots, f(k - \underline{1})\rangle$, and a pairing function $\langle\, ,\, \rangle$, i.e.

$$\mathrm{PA} \vdash \forall z \exists! x \exists! y(\langle x, y\rangle = z).$$

- $\mathrm{first}(z) = x :\leftrightarrow \exists y(\langle x, y\rangle = z)$,
- $\mathrm{length}(z) = \mathrm{second}(z) = y :\leftrightarrow \exists x(\langle x, y\rangle = z)$
- $(x)_i := \beta(\mathrm{first}(x), i)$
- $\mathrm{seq}(s) :\leftrightarrow \forall x < \mathrm{first}(s) \exists i < \mathrm{length}(s)(\beta(x, i) \neq (s)_i)$.

Encoding finite sequences

Example.

One defines powers in PA as follows:

$$x^k = y :\leftrightarrow \exists s(\mathrm{seq}(s) \wedge \mathrm{length}(s) = \mathrm{s}k \wedge (s)_0 = \underline{1} \wedge \\ \forall i < \mathrm{s}k((s)_{\mathrm{s}i} = x \cdot (s)_i) \wedge (s)_k = y).$$

e.g. $x^{\underline{2}} = y :\leftrightarrow \exists s(\mathrm{seq}(s) \wedge \mathrm{length}(s) = \underline{3} \wedge (s)_0 = \underline{1} \wedge \\ \forall i < \underline{3}((s)_{\mathrm{s}i} = x \cdot (s)_i) \wedge (s)_{\underline{2}} = y)$

$(s)_0 = \underline{1}$

$(s)_{\underline{1}} = x \cdot \underline{1} = x$

$x^{\underline{2}} = y = (s)_{\underline{2}} = x \cdot x$

s codes the sequence $\langle \underline{1}, x, x \cdot x \rangle$.

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Gödelization

Symbol ζ	Gödel number $\#\zeta$
0	2
$\mathtt{s}$	4
$+$	6
$\cdot$	8
$=$	10
$\neg$	12
$\wedge$	14
$\exists$	16
$\vee$	18
$\forall$	20
$\rightarrow$	22
x_n	$2n+1$

Term τ	Gödel number $\#\tau$
0	2
x_n	$2n+1$
$\mathtt{s}\tau$	$2^{\#\mathtt{s}} \cdot 3^{\#\tau}$
$\tau_1 + \tau_2$	$2^{\#+} \cdot 3^{\#\tau_1} \cdot 5^{\#\tau_2}$
$\tau_1 \cdot \tau_2$	$2^{\#\cdot} \cdot 3^{\#\tau_1} \cdot 5^{\#\tau_2}$

Gödelization

Formula φ	Gödel number $\#\varphi$
$\tau_1 = \tau_2$	$2^{\#=} \cdot 3^{\#\tau_1} \cdot 5^{\#\tau_2}$
$\neg\psi$	$2^{\#\neg} \cdot 3^{\#\varphi}$
$\psi_1 \wedge \psi_2$	$2^{\#\wedge} \cdot 3^{\#\psi_1} \cdot 5^{\#\psi_2}$
$\psi_1 \vee \psi_2$	$2^{\#\vee} \cdot 3^{\#\psi_1} \cdot 5^{\#\psi_2}$
$\psi_1 \rightarrow \psi_2$	$2^{\#\rightarrow} \cdot 3^{\#\psi_1} \cdot 5^{\#\psi_2}$
$\exists x\psi$	$2^{\#\exists} \cdot 3^{\#x} \cdot 5^{\#\psi_2}$
$\forall x\psi$	$2^{\#\forall} \cdot 3^{\#x} \cdot 5^{\#\psi_2}$

Gödelization

Example (Gödelization of $s0 + 0$)

- $\#(s0) = 2^{\#s} \cdot 3^{\#0} = 2^4 \cdot 3^2 = 144$
- $\#(s0+0) = 2^{\#+} \cdot 3^{\#s0} \cdot 5^{\#0} = 2^6 \cdot 3^{144} \cdot 3^2 = 8.120460577 \cdot 10^{71}$

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Gödelization

Gödelization in PA:

$$\ulcorner \zeta \urcorner := \underline{\#\zeta}$$

for symbols, terms or formulae ζ. Then we can define

- $\mathrm{var}(v) \;:\leftrightarrow \exists n(v = \underline{2} \cdot n + \underline{1})$,
- $\mathrm{term}(t) :\leftrightarrow \exists c\big[\mathrm{seq}(c) \wedge (c)_{\mathrm{length}(c)-\underline{1}} = t \wedge \forall k < \mathrm{length}(c)$ $\big(\mathrm{var}((c)_k) \vee (c)_k = \ulcorner 0 \urcorner \vee \exists \mathtt{i} < \mathtt{k} \exists \mathtt{j} < \mathtt{k}((\mathtt{c})_\mathtt{k} = \underline{2}^{\ulcorner \mathtt{s} \urcorner} \cdot \underline{3}^{(\mathtt{c})_\mathtt{i}}$ $\vee (\mathtt{c})_\mathtt{k} = \underline{2}^{\ulcorner + \urcorner} \cdot \underline{3}^{(\mathtt{c})_\mathtt{i}} \cdot \underline{5}^{(\mathtt{c})_\mathtt{j}} \vee (\mathtt{c})_\mathtt{k} = \underline{2}^{\ulcorner \cdot \urcorner} \cdot \underline{3}^{(\mathtt{c})_\mathtt{i}} \cdot \underline{5}^{(\mathtt{c})_\mathtt{j}}))\big]$.

In a similar way one defines predicates
$\mathrm{formula}(f), \mathrm{axiom}(a), \mathrm{proof}(x, y), \mathrm{provable}(x)$ and so on.

Gödelization

Proposition

Let φ be an $\mathcal{L}_{\mathrm{PA}}$-formula without free variables.

1. If $\mathrm{PA} \vdash \varphi$ then there is $n \in \mathbb{N}$ such that $\mathrm{PA} \vdash \mathrm{proof}(\underline{n}, \ulcorner\varphi\urcorner)$.
2. If $\mathrm{PA} \nvdash \varphi$ then $\mathrm{PA} \vdash \neg\mathrm{proof}(\underline{n}, \ulcorner\varphi\urcorner)$ for every $n \in \mathbb{N}$.

The Diagonalization Lemma

Theorem (Diagonalization Lemma)

For every $\mathcal{L}_{\mathrm{PA}}$-formula $\varphi(x_0)$ with exactly one free variable there is an $\mathcal{L}_{\mathrm{PA}}$-formula σ without free variables such that

$$\sigma \equiv_{\mathrm{PA}} \varphi(x_0/\ulcorner \sigma \urcorner).$$

The First Incompleteness Theorem

Theorem (First Incompleteness Theorem)

If PA *is consistent then it is incomplete.*

Idea of the proof: Construct G_{PA} such that G_{PA} says

"I am unprovable".

- If $\mathrm{PA} \vdash G_{\mathrm{PA}}$, then G_{PA} is provable!
- If $\mathrm{PA} \vdash \neg G_{\mathrm{PA}}$ then $\neg G_{\mathrm{PA}}$ is provable, but $\neg G_{\mathrm{PA}}$ says "I am unprovable"!

Thus $\mathrm{PA} \nvdash G_{\mathrm{PA}}$ and $\mathrm{PA} \nvdash G_{\mathrm{PA}}$.

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- If $\mathrm{PA} \vdash \neg G_{\mathrm{PA}}$ then $\neg G_{\mathrm{PA}}$ is provable, but $\neg G_{\mathrm{PA}}$ says "I am unprovable"!

Thus $\mathrm{PA} \nvdash G_{\mathrm{PA}}$ and $\mathrm{PA} \nvdash G_{\mathrm{PA}}$.

The First Incompleteness Theorem

Sqetch of the proof.

Additional assumption: ω-**consistence**, i.e. if $\mathrm{PA} \vdash \exists x \varphi(x)$, then there is $n \in \mathbb{N}$ such that $\mathrm{PA} \nvdash \neg\varphi(\underline{n})$.
Diagonalization Lemma $\Rightarrow$ there is a formula G_{PA} with

$$G_{\mathrm{PA}} \equiv_{\mathrm{PA}} \neg \mathrm{provable}(\ulcorner G_{\mathrm{PA}} \urcorner) \equiv_{\mathrm{PA}} \neg \exists c \, \mathrm{proof}(c, \ulcorner G_{\mathrm{PA}} \urcorner).$$

- If $\mathrm{PA} \vdash G_{\mathrm{PA}}$ then there is $n \in \mathbb{N}$ with $\mathrm{PA} \vdash \mathrm{proof}(\underline{n}, \ulcorner G_{\mathrm{PA}} \urcorner)$.
- If $\mathrm{PA} \vdash \neg G_{\mathrm{PA}}$ there is $m \in \mathbb{N}$ with $\mathrm{PA} \vdash \mathrm{proof}(\underline{m}, \ulcorner \neg G_{\mathrm{PA}} \urcorner)$. But $\neg G_{\mathrm{PA}} \equiv_{\mathrm{PA}} \exists c \, \mathrm{proof}(c, \ulcorner G_{\mathrm{PA}} \urcorner)$ i.e. $\mathrm{PA} \vdash \exists d \, \mathrm{proof}(d, \ulcorner G_{\mathrm{PA}} \wedge \neg G_{\mathrm{PA}} \urcorner)$.

$\dashv$

The First Incompleteness Theorem

Question.

If $\mathrm{PA} \nvdash G_{\mathrm{PA}}$ could one not just consider instead $\mathrm{PA} + G_{\mathrm{PA}}$?

Reactions to the First Incompleteness Theorem

However complicated a machine we construct, it will, if it is a machine, correspond to a formal system, which in turn will be liable to the Gödel procedure for finding a formula unprovable in that system. This formula the machine will be unable to produce as true, although a mind can see that it is true. And so the machine will not be an adequate model of the mind.

J.R. Lucas, in "Minds, Machines and Gödel".

Reactions to the First Incompleteness Theorem

Either mathematics is incompletable [...], that its evident axioms can never be comprised in a finite rule, that is to say, the human mind (even within the realm of pure mathematics) inifinitely surpasses the powers of any finite machine, or there exist absolutely unsolvable diophantine equations

K. Gödel.

Undecidability

Definition

A theory T is **decidable**, if there is an algorithm with decides whether φ follows from the axioms of T or not.

Complete theories are always decidable.

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Recursion Theory

Definition

A predicate P is said to be **representable** if there is a formula $\varphi(x)$ such that

1. if $\mathbb{N} \models P(n)$ then $\mathrm{PA} \vdash \varphi(\underline{n})$
2. if $\mathbb{N} \models \neg P(n)$ then $\mathrm{PA} \vdash \neg\varphi(\underline{n})$.

Example

"n is even" can be represented by $\exists y(x = \underline{2} \cdot y + \underline{1})$.

Theorem (Representation Theorem)

All recursive functions and predicates are representable in PA.

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Theorem (Representation Theorem)

All recursive functions and predicates are representable in PA.

Undecidability

Theorem (Church's Theorem)

PA *ist undecidable.*

Sqetch of the proof.

If not, then PA $\vdash \alpha$ is recursive, i.e. there is $\varphi(x)$ such that

1. PA $\vdash \alpha \Rightarrow$ PA $\vdash \varphi(\ulcorner \alpha \urcorner)$
2. PA $\nvdash \alpha \Rightarrow$ PA $\vdash \neg\varphi(\ulcorner \alpha \urcorner)$.

Diagonalization Lemma $\Rightarrow$ there is σ with $\sigma \equiv \neg\varphi(\ulcorner \sigma \urcorner)$.

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Definability of truth

Question.

Can truth in $\mathbb{N}$ be defined?

Definition

- A formula φ is **true** in $\mathbb{N}$ if $\mathbb{N} \models \varphi$.
- A formula $T(x)$ is a **truth predicate** for $\mathbb{N}$, if $\mathbb{N} \models \varphi \Leftrightarrow \mathbb{N} \models T(\ulcorner \varphi \urcorner)$ for every formula φ.

Undefinability of truth

Theorem (Tarski)

Truth in $\mathbb{N}$ is not definable.

Beweis.

Assume that $T(x)$ is a truth predicate. Diagonalization Lemma $\Rightarrow$ there is a sentence L such that $\mathrm{PA} \vdash L \leftrightarrow \neg T(\ulcorner L \urcorner)$. Therefore $\mathbb{N} \models L \leftrightarrow \neg T(\ulcorner L \urcorner) \leftrightarrow \neg L$. $\dashv$

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The Second Incompleteness Theorem

What does consistency PA of mean?

Definition

Define Con_{PA} by $\neg\text{provable}(\ulcorner 0 = 1 \urcorner)$.

Question

Does $\text{PA} \vdash \text{Con}_{\text{PA}}$ hold?

Paradoxon

A student asks his theology professor whether God exists and gets the following answer:
"God exists if and only if you will never believe that God exists."
Can the student believe the professor without becoming inconsistent?

The Second Incompleteness Theorem

Theorem (The Second Incompleteness Theorem)

If PA *is consistent then* $\mathrm{PA} \nvdash \mathrm{Con}_{\mathrm{PA}}$.

The Derivability Conditions

We define for an $\mathcal{L}_{\mathsf{PA}}$-formula φ

$$\Box\varphi :\leftrightarrow \mathsf{provable}(\ulcorner\varphi\urcorner).$$

Theorem (Derivability Conditions)

Let φ and ψ be $\mathcal{L}_{\mathsf{PA}}$-formulas. Then:

(D1) $\mathsf{PA} \vdash \varphi \Rightarrow \mathsf{PA} \vdash \Box\varphi$

(D2) $\mathsf{PA} \vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

(D3) $\mathsf{PA} \vdash \Box\varphi \to \Box\Box\varphi$.

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Diagonalization Lemma $\Rightarrow$ there exists a closed $\mathcal{L}_{\mathsf{PA}}$-formula σ such that $\sigma \equiv_{\mathsf{PA}} \neg\Box\sigma$.

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$\mathrm{Con}_{\mathsf{PA}} \equiv_{\mathsf{PA}} \neg\Box(\sigma \wedge \neg\sigma) \equiv_{\mathsf{PA}} \neg\Box(\sigma \wedge \Box\sigma) \equiv_{\mathsf{PA}} \neg(\Box\sigma \wedge \Box\Box\sigma) \equiv_{\mathsf{PA}} \neg\Box\sigma \equiv_{\mathsf{PA}} \sigma$.

If we have $\mathsf{PA} \vdash \sigma$, then $\mathsf{PA} \vdash \Box\sigma$ by (D1), which is a contradiction. $\dashv$

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$\neg(\Box\sigma \wedge \Box\Box\sigma) \equiv_{\mathsf{PA}} \neg\Box\sigma \equiv_{\mathsf{PA}} \sigma$.

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Consequences

Every axiom system which is powerful enough to prove the Incompleteness Theorems is either inconsistent or cannot prove its own consistency.

Heisenberg, Gödel, and Chomsky walk into a bar. Heisenberg looks around the bar and says, "Because there are three of us and because this is a bar, it must be a joke. But the question remains, is it funny or not?" And Gödel thinks for a moment and says, "Well, because we're inside the joke, we can't tell whether it is funny. We'd have to be outside looking at it" And Chomsky looks at both of them and says, "Of course it's funny. You're just telling it wrong."