# Forcing: How to prove unprovability

## The Continuum Hypothesis

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# Content

# Motivation

**Theorem (Cantor)**

*There is no surjection from $\mathbb{N}$ onto $\mathcal{P}(\mathbb{N})$.*

This lead to the study of *infinite cardinals.*

**Definition**

- $\aleph_0$ denotes the cardinality of $\omega = \mathbb{N}$.
- $\aleph_1$ denotes the least uncountable cardinal.
- $\aleph_2$ denotes the second-least uncountable cardinal.
- $\mathfrak{c} = 2^{\aleph_0}$ denotes the cardinality of $\mathcal{P}(\omega)$.

## The Continuum Hypothesis

In 1878, Georg Cantor conjectured the *Continuum Hypothesis*:

**Conjecture (The Continuum Hypothesis (CH))**

*There is no cardinal between* $\aleph_0$ *and* $2^{\aleph_0}$*, i.e.* $2^{\aleph_0} = \aleph_1$*.*

The Continuum Hypothesis was the first of Hilbert's famous list of 23 open Problems.

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In 1963, Paul Cohen proved that CH is actually *independent* of the axioms of ZFC, i.e. both ZFC + CH and ZFC + $\neg$CH are consistent. The method he used (and developed for this result) is *forcing*.

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# Forcing notions

**Definition**

A *forcing (notion)* is a non-atomic partial order $\mathbb{P} = (P, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$ with a maximal element $\mathbb{1}_{\mathbb{P}}$, i.e.

- $\leq_{\mathbb{P}}$ is reflexive, transitive and antisymmetric,
- for every $p \in \mathbb{P}$, $p \leq_{\mathbb{P}} \mathbb{1}_{\mathbb{P}}$,
- for every $p \in \mathbb{P}$ there are $p_0, p_1 \leq_{\mathbb{P}} p$ which are *incompatible*, i.e. there is no $r \in \mathbb{P}$ with $r \leq_{\mathbb{P}} p_0, p_1$.

# Forcing notions

**Example (Cohen forcing)**

Let $\mathbb{P} = \mathrm{Fn}(\omega, 2, \aleph_0)$ be the set of partial functions $p : \mathrm{dom}(p) \to 2$ with $\mathrm{dom}(p) \subseteq \omega$ finite, ordered by reverse inclusion, i.e.

$$p \leq_{\mathbb{P}} q \iff \mathrm{dom}(p) \supseteq \mathrm{dom}(q) \text{ and } p \restriction \mathrm{dom}(q) = q.$$

Furthermore, let $\mathbb{1}_{\mathbb{P}} = \emptyset$, the empty function.

# Generic filters

## Definition

Let $M$ be a countable transitive model of ZFC and $\mathbb{P} \in M$ be a forcing notion.

1. A subset $D \subseteq \mathbb{P}$ is said to be *dense*, if for every $p \in P$ there is some $q \leq_{\mathbb{P}} p$ with $q \in D$.
2. A subset $G \subseteq \mathbb{P}$ is said to be a $\mathbb{P}$-*generic filter*, if it has the following properties:
   - If $p \leq_{\mathbb{P}} q$ and $p \in G$, then $q \in G$.
   - If $p, q \in G$ then there is $r \in G$ such that $r \leq_{\mathbb{P}} p, q$.
   - If $D \subseteq \mathbb{P}$ is a dense set which is in $M$, then $G \cap D \neq \emptyset$.

## Example

Consider Cohen forcing $\mathbb{P} = \mathrm{Fn}(\omega, 2, \aleph_0)$. Then for each $n \in \omega$, the set $D_n = \{p \in \mathbb{P} \mid n \in \mathrm{dom}(p)\}$ is dense.

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# Names

## Definition

Let $\mathbb{P}$ be a forcing notion. A $\mathbb{P}$-name is a set whose elements are of the form $(\sigma, p)$, where $\sigma$ is a $\mathbb{P}$-name and $p \in \mathbb{P}$.

This definition requires *transfinite recursion*.

## Example

Let $M \models \mathrm{ZFC}$ and $\mathbb{P} \in M$ a forcing notion.

- $\emptyset$ is a $\mathbb{P}$-name.
- Let $x \in M$. Then there is a *canonical* $\mathbb{P}$*-name* for $x$ given by

$$\check{x} = \{(\check{y}, \mathbb{1}_{\mathbb{P}}) \mid y \in x\}.$$

- The set $\dot{G} = \{(\check{p}, p) \mid p \in \mathbb{P}\}$ is a $\mathbb{P}$-name, the canonical $\mathbb{P}$-name for a $\mathbb{P}$-generic filter.

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# Evaluations of names

Let $\mathbb{P}$ be a forcing notion and $G$ a $\mathbb{P}$-generic filter. We can *evaluate* a $\mathbb{P}$-name $\sigma$ as follows:

$$\sigma^G = \{\tau^G \mid \exists p \in G : (\tau, p) \in \sigma\}.$$

## Example

- $\emptyset^G = \emptyset$.
- Let $x \in M$. Then
$$\check{x}^G = \{\check{y}^G \mid y \in x\} = x$$
by transfinite induction.
- $\dot{G}^G = \{\check{p}^G \mid p \in G\} = G$.

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# Generic extensions

Let $M \models \text{ZFC}$, $\mathbb{P} \in M$ be a non-trivial forcing notion and $G$ a $\mathbb{P}$-generic filter. Then we define

$$M[G] = \{\sigma^G \mid \sigma \text{ is a } \mathbb{P}\text{-name}\}.$$

**Fact**

1. $M \cup \{G\} \subseteq M[G]$
2. $G \notin M$.
3. $M[G] \models \text{ZFC}$.

**Proof.**

For (2) suppose that $G \in M$. Then the set $D = \mathbb{P} \setminus G$ is in $M$. Moreover, $D$ is dense: Let $p \in \mathbb{P}$. Since $\mathbb{P}$ is non-atomic, there are $p_0, p_1 \leq_{\mathbb{P}} p$ such that $p_0$ and $p_1$ are incompatible. But then at least one of $p_0, p_1$ is not in $G$. □

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# Where do generic filters live?

Suppose that $V \models$ ZFC and V contains countable, transitive models $M \in V$ of ZFC. Let $\mathbb{P} \in M$ be a forcing notion. We extend $M$ to $M[G]$, where $G$ is a $\mathbb{P}$-generic filter contained in V, i.e. $G$ is a filter which intersects all dense subsets of $\mathbb{P}$ which lie in $M$.

**Fact**

*For every $p \in \mathbb{P}$ there is a $\mathbb{P}$-generic filter $G$ in V with $p \in G$.*

**Proof.**

Since $M$ is countable, $M$ contains only countably many dense subsets of $\mathbb{P}$. Let $(D_n \mid n \in \omega)$ enumerate them (in V). Now we inductively construct a sequence of conditions $(p_n \mid n \in \omega)$ by

- $p_0 = p$
- Given $p_n$, let $p_{n+1} \leq_{\mathbb{P}} p_n$ with $p_{n+1} \in D_n$.

Then $G = \{q \in \mathbb{P} \mid \exists n \in \omega (p_n \leq_{\mathbb{P}} q)\}$ is a $\mathbb{P}$-generic filter over $M$. □

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# The idea behind names

Every element of $M[G]$ has a $\mathbb{P}$-name $\sigma \in M$ but $M$ does not know how $\sigma$ will be evaluated. This is similar to the case of *extensions of fields:*

Consider $\mathbb{Q}$ and an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$.

- In $\mathbb{Q}$, the polynomial $X^2 - 2$ *names* the root $\sqrt{2} \in \bar{\mathbb{Q}}$.
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# Cohen forcing

Let $\mathbb{P}$ denote $\mathrm{Fn}(\omega, 2, \aleph_0)$ and let $G$ be $\mathbb{P}$-generic over $M$. In $M[G]$, consider the function

$$c = \bigcup G. \qquad (c \text{ is called a } \textit{Cohen real})$$

**Fact**

1. $c$ *is a function* $\omega \to 2$.
2. *If* $f \in M$ *is a function* $f : \omega \to 2$ *then* $f \neq c$.

**Proof.**

(1) Let $n \in \omega$. If there are $p, q \in G$ such that $n \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$ and $p(n) \neq q(n)$, then $p$ and $q$ are incompatible.
To see that $\mathrm{dom}(c) = \omega$, note that $D_n = \{p \in \mathbb{P} \mid n \in \mathrm{dom}(p)\}$ is dense.
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Take $p_0 \in G \cap D_f$ and let $n_0 \in \mathrm{dom}(p_0)$ such that $p_0(n_0) \neq f(n_0)$. Then $c(n_0) = p_0(n_0)$ and so $c \neq f$. □

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# A nice application: Cantor's Theorem

**Theorem (Cantor)**

*There is no surjection from $\omega$ onto $\mathcal{P}(\omega)$.*

**Proof.**

Suppose the contrary. Observe that $\mathbb{P} = \mathrm{Fn}(\omega, 2, \aleph_0)$ is countable. Then so is $\mathcal{P}(\mathbb{P})$, so we can enumerate (in $M$) all dense subsets of $\mathbb{P}$. But then we can construct in $M$ a $\mathbb{P}$-generic filter over $M$. □

# The Cantor space

We identify $\mathbb{R}$ with ${}^{\omega}2$, the space of functions $\omega \to 2$.

**Observation**

If $\mathbb{P}$ is a forcing notion and $G$ is $\mathbb{P}$-generic over $M$, then $\mathbb{R}^M$ and $\mathbb{R}^{M[G]}$ do not have to be the same.

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# Intervals in the Cantor space

We define intervals in the following way: For $s \in {}^{<\omega}2 = \{t : \mathrm{dom}(t) \to 2 \mid \mathrm{dom}(t) \in \omega\}$ we define

$$I_s = \{x \in \mathbb{R} \mid s \subseteq x\} \subseteq \mathbb{R}$$

and we define $\mu(I_s) = 2^{-\mathrm{dom}(s)}$.

**Definition**

We say that a set $X \subseteq \mathbb{R}$ has *measure zero*, if for every $\varepsilon > 0$ there exists a sequence $(I_n \mid n \in \omega)$ of intervals in $\mathbb{R}$ such that $X \subseteq \bigcup_{n\in\omega} I_n$ and $\sum_{n\in\omega} \mu(I_s) < \varepsilon$.

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and we define $\mu(I_s) = 2^{-\mathrm{dom}(s)}$.

**Definition**

We say that a set $X \subseteq \mathbb{R}$ has *measure zero*, if for every $\varepsilon > 0$ there exists a sequence $(I_n \mid n \in \omega)$ of intervals in $\mathbb{R}$ such that $X \subseteq \bigcup_{n\in\omega} I_n$ and $\sum_{n\in\omega} \mu(I_s) < \varepsilon$.

## Theorem

*Let $M[G]$ be a generic extension of $M$ by Cohen forcing, and let $c \in M[G]$ be the canonical Cohen real. Then in $M[G]$, $\mathbb{R}^M$ has measure zero.*

## Proof.

Let $\varepsilon > 0$, let $k \in \omega$ such that $\varepsilon > 2^{-k}$. For $n \in \omega$ define

$$s_n : k + n + 1 \to 2,\ s_n(l) = c(n + l).$$

Now define $I_n = I_{s_n}$. Then

$$\sum_{n \in \omega} \mu(I_n) = \sum_{n \in \omega} 2^{-(k+n+1)} = 2^{-k} < \varepsilon.$$

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It remains to check: $\mathbb{R}^M \subseteq \bigcup_{n\in\omega} I_n$. Let $x \in \mathbb{R}^M$. Then

$$D_x = \{p \in \mathbb{P} \mid \exists n \in \omega \forall l < k+n+1 : n+l \in \mathrm{dom}(p) \\ \wedge p(n+l) = x(l)\} \in M$$

is a dense subset of Cohen forcing $\mathbb{P}$. Take $p \in G \cap D_x$ and $n \in \omega$ which witnesses that $p \in D_x$. Then

$$\forall l < k+n+1 : s_n(l) = c(n+l) = p(n+l) = x(l)$$

and so $s_n \subseteq x$ and $x \in I_n$. □

# Forcing ¬CH

We have used Cohen forcing to add one real $c$ which is not an element of the ground model $M$. What happens if we add $\aleph_2$-many Cohen reals in this way?

**Definition**

Let $\mathrm{Fn}(\aleph_2^M \times \omega, 2, \aleph_0)$ denote the set of finite partial functions from $\aleph_2^M \times \omega \to 2$, ordered by reverse inclusion.

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Let $\mathbb{P}$ denote $\mathrm{Fn}(\aleph_2^M \times \omega, 2, \aleph_0)$ and let $G$ be $\mathbb{P}$-generic over $M$.

## Lemma

*The following statements hold:*

1. $F = \bigcup G$ *is a function* $F : \aleph_2^M \times \omega \to 2$.
2. *For each* $\alpha < \aleph_2^M$, *let* $c_\alpha(n) = F(\alpha, n)$. *Then for all* $\alpha < \beta < \aleph_2^M$, $c_\alpha \neq c_\beta$.

## Proof.

(1) Follows from the density of the sets

$$D_{\alpha,n} = \{p \in \mathbb{P} \mid (\alpha, n) \in \mathrm{dom}(p)\}$$

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(2) For all $\alpha < \beta < \aleph_2^M$ consider

$$D_{\alpha\beta} = \{p \in \mathbb{P} \mid \exists n \in \omega : (\alpha, n), (\beta, n) \in \mathrm{dom}(p) \text{ and } p(\alpha, n) \neq p(\beta, n)\}.$$

Then $D_{\alpha\beta}$ is dense in $\mathbb{P}$. By genericity, take $p \in G \cap D_{\alpha\beta}$ and $n \in \omega$ such that $(\alpha, n), (\beta, n) \in \mathrm{dom} p$ with $p(\alpha, n) \neq p(\beta, n)$. Then

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The previous lemma shows that there are at least $\aleph_2^M$ many reals in $M[G]$, so if we can show that $\aleph_2^M = \aleph_2^{M[G]}$ then in $M[G]$

$$(2^{\aleph_0})^{M[G]} = |\mathbb{R}^{M[G]}| \geq \aleph_2^M = \aleph_2^{M[G]}.$$

**Definition**

Let $\mathbb{P}$ be a forcing notion.

1. An *antichain* in $\mathbb{P}$ is a subset $A \subseteq \mathbb{P}$ with the property that all elements of $A$ are incompatible, i.e. for all $p, q \in A$, there is no $r \in \mathbb{P}$ with $r \leq_{\mathbb{P}} p, q$.
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*If $\mathbb{P}$ satisfies the ccc then $\mathbb{P}$ preserves all cardinals, i.e. $|\kappa|^{M[G]} = \kappa$ in every $\mathbb{P}$-generic extension $M[G]$. In particular, $\aleph_2^{M[G]} = \aleph_2^M$.*

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**Example**

Consider the forcing notion

$$\mathrm{Col}(\omega, \aleph_2^M) = \{p : \mathrm{dom}(p) \to \omega_2^M \mid \mathrm{dom}(p) \subseteq \omega \text{ finite}\}$$

ordered by reverse inclusion. If $G$ is $\mathrm{Col}(\omega, \aleph_2^M)$-generic, then $F = \bigcup G$ is a function $\omega \to \aleph_2^M$. Then $F$ is surjective because for each $\alpha < \aleph_2^M$ the set

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**Lemma**

$\mathbb{P} = \mathrm{Fn}(\aleph_2^M \times \omega, 2, \aleph_0)$ *has the ccc.*

**Proof.**

Suppose that $(p_i \mid i < \aleph_1^M)$ enumerates an uncountable antichain. Then there is a set $X \subseteq \aleph_1^M$ of size $\aleph_1^M$ and $n \in \omega$ such that

$$\forall i \in X \, |\mathrm{dom}(p_i)| = n.$$

Now let $m \leq n$ be maximal such that there is a set $b$ with $|b| = m$ and $Y \subseteq X$ of size $\aleph_1^M$ such that for all $i \in Y$, $b \subseteq \mathrm{dom}(p_i)$. We claim that for all $x \notin b$ there is $i(x) < \aleph_1^M$ such that for all $i > i(x)$ in $Y$, $x \notin \mathrm{dom}(p_i)$. This holds because otherwise $b \cup \{x\}$ would contradict the maximality of $m$. □

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## Proof, continued.

- $Y \subseteq \aleph_1$ of size $\aleph_1^M$ with $b \subseteq \mathrm{dom}(p_i)$ for all $i \in Y$.
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We construct an increasing sequence $(i_\xi \mid \xi < \aleph_1)$ with $i_\xi \in Y$ by recursion: Suppose that $(i_\xi \mid \xi < \zeta)$ has already been defined. Then let $i_\zeta$ be the minimal ordinal $i \in Y$ such that

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