

Forcing: How to prove unprovability I

The Continuum Hypothesis

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Motivation

Theorem (Cantor)

There is no surjection from \mathbb{N} onto $\mathcal{P}(\mathbb{N})$.

This lead to the study of *infinite cardinals*.

Definition

- \aleph_0 denotes the cardinality of $\omega = \mathbb{N}$.
- \aleph_1 denotes the least uncountable cardinal.
- $\mathfrak{c} = 2^{\aleph_0}$ denotes the cardinality of $\mathcal{P}(\omega)$.

The Continuum Hypothesis

In 1878, Georg Cantor conjectured the *Continuum Hypothesis*:

Conjecture (The Continuum Hypothesis (CH))

There is no cardinal between \aleph_0 and 2^{\aleph_0} , i.e. $2^{\aleph_0} = \aleph_1$.

The Continuum Hypothesis was the first of Hilbert's famous list of 23 open Problems.

In 1963, Paul Cohen proved that CH is actually *independent* of the axioms of ZFC, i.e. both $\text{ZFC} + \text{CH}$ and $\text{ZFC} + \neg\text{CH}$ are consistent. The method he used (and developed for this result) is *forcing*.

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Forcing notions

Definition

A *forcing (notion)* is a non-atomic partial order $\mathbb{P} = \langle P, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}} \rangle$ with a maximal element $\mathbb{1}_{\mathbb{P}}$, i.e.

- $\leq_{\mathbb{P}}$ is reflexive, transitive and antisymmetric
- for every $p \in \mathbb{P}$, $p \leq_{\mathbb{P}} \mathbb{1}_{\mathbb{P}}$
- for every $p \in \mathbb{P}$ there are $p_0, p_1 \leq_{\mathbb{P}} p$ which are *incompatible*, i.e. there is no $r \in \mathbb{P}$ with $r \leq_{\mathbb{P}} p_0, p_1$.

Forcing notions

Example (Cohen forcing)

Let $\mathbb{P} = \text{Fn}(\omega, 2, \aleph_0)$ be the set of partial functions $p : \text{dom}(p) \rightarrow 2$ with $\text{dom}(p) \subseteq \omega$ finite, ordered by reverse inclusion, i.e.

$$p \leq_{\mathbb{P}} q \iff \text{dom}(p) \supseteq \text{dom}(q) \text{ and } p \upharpoonright \text{dom}(q) = q.$$

Furthermore, let $\mathbb{1}_{\mathbb{P}} = \emptyset$, the empty function.

Generic filters

Definition

Let M be a countable transitive model of ZFC and $\mathbb{P} \in M$ be a forcing notion.

- 1 A subset $D \subseteq \mathbb{P}$ is said to be *dense*, if for every $p \in \mathbb{P}$ there is some $q \leq_{\mathbb{P}} p$ with $q \in D$.
- 2 A subset $G \subseteq \mathbb{P}$ is said to be a \mathbb{P} -*generic filter*, if it has the following properties:
 - If $p \leq_{\mathbb{P}} q$ and $p \in G$, then $q \in G$.
 - If $p, q \in G$ then there is $r \in G$ such that $r \leq_{\mathbb{P}} p, q$.
 - If $D \subseteq \mathbb{P}$ is a dense set which is in M , then $G \cap D \neq \emptyset$.

Example

Consider Cohen forcing $\mathbb{P} = \text{Fn}(\omega, 2, \aleph_0)$. Then for each $n \in \omega$, the set $D_n = \{p \in \mathbb{P} \mid n \in \text{dom}(p)\}$ is dense.

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Names

Definition

Let \mathbb{P} be a forcing notion. A \mathbb{P} -name is a set whose elements are of the form $\langle \sigma, p \rangle$, where σ is a \mathbb{P} -name and $p \in \mathbb{P}$.

This definition requires *transfinite recursion*.

Example

Let $M \models \text{ZFC}$ and $\mathbb{P} \in M$ a forcing notion.

- \emptyset is a \mathbb{P} -name.
- Let $x \in M$. Then there is a *canonical* \mathbb{P} -name for x given by

$$\check{x} = \{ \langle \check{y}, 1_{\mathbb{P}} \rangle \mid y \in x \}.$$

- The set $\dot{G} = \{ \langle \check{p}, p \rangle \mid p \in \mathbb{P} \}$ is a \mathbb{P} -name, the canonical \mathbb{P} -name for a \mathbb{P} -generic filter.

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Evaluations of names

Let \mathbb{P} be a forcing notion and G a \mathbb{P} -generic filter. We can *evaluate* a \mathbb{P} -name σ as follows:

$$\sigma^G = \{\tau^G \mid \exists p \in G (\langle \tau, p \rangle \in \sigma)\}.$$

Example

- $\emptyset^G = \emptyset$.
- Let $x \in M$. Then

$$\check{x}^G = \{\check{y}^G \mid y \in x\} = x$$

by transfinite induction.

- $\dot{G}^G = \{\check{p}^G \mid p \in G\} = G$.

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Generic extensions

Let $M \models \text{ZFC}$, $\mathbb{P} \in M$ be a non-trivial forcing notion and G a \mathbb{P} -generic filter. Then we define

$$M[G] = \{\sigma^G \mid \sigma \text{ is a } \mathbb{P}\text{-name}\}.$$

Fact

- 1 $M \cup \{G\} \subseteq M[G]$
- 2 $G \notin M$.
- 3 $M[G] \models \text{ZFC}$.

Proof.

For (2) suppose that $G \in M$. Then the set $D = \mathbb{P} \setminus G$ is in M . Moreover, D is dense: Let $p \in \mathbb{P}$. Since \mathbb{P} is non-atomic, there are $p_0, p_1 \leq_{\mathbb{P}} p$ such that p_0 and p_1 are incompatible. But then at least one of p_0, p_1 is not in G . \dashv

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Where do generic filters live?

Suppose that $V \models \text{ZFC}$ and V contains countable, transitive models $M \in V$ of ZFC. Let $\mathbb{P} \in M$ be a forcing notion. We extend M to $M[G]$, where G is a \mathbb{P} -generic filter contained in V , i.e. G is a filter which intersects all dense subsets of \mathbb{P} which lie in M .

Fact

For every $p \in \mathbb{P}$ there is a \mathbb{P} -generic filter G in V .

Proof.

Since M is countable, M contains only countably many dense subsets of \mathbb{P} . Let $\langle D_n \mid n \in \omega \rangle$ enumerate them (in V). Now we inductively construct a sequence of conditions $\langle p_n \mid n \in \omega \rangle$ by

- $p_0 = p$
- Given p_n , let $p_{n+1} \leq_{\mathbb{P}} p_n$ with $p_{n+1} \in D_n$.

Then $G = \{q \in \mathbb{P} \mid \exists n \in \omega (p_n \leq_{\mathbb{P}} q)\}$ is a \mathbb{P} -generic filter over M . \dashv

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The idea behind names

Every element of $M[G]$ has a \mathbb{P} -name $\sigma \in M$ but M does not know how σ will be evaluated. This is similar to the case of *extensions of fields*:

Consider \mathbb{Q} and an algebraic closure $\bar{\mathbb{Q}}$ of \mathbb{Q} .

- In \mathbb{Q} , the polynomial $X^2 - 2$ *names* the root $\sqrt{2} \in \bar{\mathbb{Q}}$.
- If we extend \mathbb{Q} to $\mathbb{Q}[\sqrt{2}]$, then $\sqrt{2}$ is the *evaluation* of $X^2 - 2$.

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Cohen forcing

Let \mathbb{P} denote $\text{Fn}(\omega, 2, \aleph_0)$ and let G be \mathbb{P} -generic over M . In $M[G]$, consider the function

$$c = \bigcup G. \quad (c \text{ is called a } \textit{Cohen real})$$

Fact

- 1 c is a function $\omega \rightarrow 2$.
- 2 If $f \in M$ is a function $f : \omega \rightarrow 2$ then $f \neq c$.

Proof.

(1) Let $n \in \omega$. If there are $p, q \in G$ such that $n \in \text{dom}(p) \cap \text{dom}(q)$ and $p(n) \neq q(n)$, then p and q are incompatible.

To see that $\text{dom}(c) = \omega$, note that $D_n = \{p \in \mathbb{P} \mid n \in \text{dom}(p)\}$ is dense. Let $p \in G \cap D_n$. Then $n \in \text{dom}(c)$ and $c(n) = p(n)$. \dashv

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(2) Consider $D_f = \{p \in \mathbb{P} \mid \exists n \in \text{dom}(p)(p(n) \neq f(n))\}$. We claim that D_f is a dense subset of \mathbb{P} . Let $p \in \mathbb{P}$. Since $\text{dom}(p)$ is finite, there is $n \in \omega \setminus \text{dom}(p)$. Then $q = p \cup \{\langle n, f(n) + 1 \rangle\} \leq_{\mathbb{P}} p$ and $q \in D_f$.

Take $p_0 \in G \cap D_f$ and let $n_0 \in \text{dom}(p_0)$ such that $p_0(n_0) \neq f(n_0)$. Then $c(n_0) = p_0(n_0)$ and so $c \neq f$. \dashv

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A nice application: Cantor's Theorem

Theorem (Cantor)

There is no surjection from ω onto $\mathcal{P}(\omega)$.

Proof.

Suppose the contrary. Observe that $\mathbb{P} = \text{Fn}(\omega, 2, \aleph_0)$ is countable. Then so is $\mathcal{P}(\mathbb{P})$, so we can enumerate (in M) all dense subsets of \mathbb{P} . But then we can construct in M a \mathbb{P} -generic filter over M . \dashv

The Cantor space

We identify \mathbb{R} with ${}^\omega 2$, the space of functions $\omega \rightarrow 2$.

Observation

If \mathbb{P} is a forcing notion and G is \mathbb{P} -generic over M , then \mathbb{R}^M and $\mathbb{R}^{M[G]}$ do not have to be the same.

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Intervals in the Cantor space

We define intervals in the following way: For $s \in {}^{<\omega}2 = \{t : \text{dom}(t) \rightarrow 2 \mid \text{dom}(t) \in \omega\}$ we define

$$I_s = \{x \in \mathbb{R} \mid s \subseteq x\} \subseteq \mathbb{R}$$

and we define $\mu(I_s) = 2^{-\text{dom}(s)}$.

Definition

We say that a set $X \subseteq \mathbb{R}$ has *measure zero*, if for every $\varepsilon > 0$ there exists a sequence $\langle I_n \mid n \in \omega \rangle$ of intervals in \mathbb{R} such that $X \subseteq \bigcup_{n \in \omega} I_n$ and $\sum_{n \in \omega} \mu(I_n) < \varepsilon$.

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Theorem

Let $M[G]$ be a generic extension of M by Cohen forcing, and let $c \in M[G]$ be the canonical Cohen real. Then in $M[G]$, \mathbb{R}^M has measure zero.

Proof.

Let $\varepsilon > 0$, let $k \in \omega$ such that $\varepsilon > 2^{-k}$. For $n \in \omega$ define

$$s_n : k + n + 1 \rightarrow 2, s_n(l) = c(n + l).$$

Now define $I_n = I_{s_n}$. Then

$$\sum_{n \in \omega} \mu(I_n) = \sum_{n \in \omega} 2^{-(k+n+1)} = 2^{-k} < \varepsilon.$$

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Now define $I_n = I_{s_n}$. Then

$$\sum_{n \in \omega} \mu(I_n) = \sum_{n \in \omega} 2^{-(k+n+1)} = 2^{-k} < \varepsilon.$$

+

Theorem

Let $M[G]$ be a generic extension of M by Cohen forcing, and let $c \in M[G]$ be the canonical Cohen real. Then in $M[G]$, \mathbb{R}^M has measure zero.

Proof.

Let $\varepsilon > 0$, let $k \in \omega$ such that $\varepsilon > 2^{-k}$. For $n \in \omega$ define

$$s_n : k + n + 1 \rightarrow 2, s_n(l) = c(n + l).$$

Now define $I_n = I_{s_n}$. Then

$$\sum_{n \in \omega} \mu(I_n) = \sum_{n \in \omega} 2^{-(k+n+1)} = 2^{-k} < \varepsilon.$$

⊥

Proof, continued.

$s_n : k + n + 1 \rightarrow 2$, $s_n(l) = c(n + l)$, $I_n = I_{s_n}$.

It remains to check: $\mathbb{R}^M \subseteq \bigcup_{n \in \omega} I_n$. Let $x \in \mathbb{R}^M$. Then

$$D_x = \{p \in \mathbb{P} \mid \exists n \in \omega \forall l < k + n + 1 (n + l \in \text{dom}(p) \wedge p(n + l) = x(l))\} \in M$$

is a dense subset of Cohen forcing \mathbb{P} . Take $p \in G \cap D_x$ and $n \in \omega$ which witnesses that $p \in D_x$. Then

$$\forall l < k + n + 1 (s_n(l) = c(n + l) = p(n + l) = x(l))$$

and so $s_n \subseteq x$ and $x \in I_n$. ⊣

To be continued...

Continuation: 18.11.2015 by Peter Holy.

Outline:

- Add \aleph_2 -many Cohen reals
- Then $\mathcal{P}(\omega)^{M[G]}$ has cardinality \aleph_2^M .
- But is $(\aleph_2) = (\aleph_2)^{M[G]}$?

To be continued...

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