Determinacy of Infinite Games

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What are infinite games?

Player I chooses $x_0 \in \omega$, Player II chooses $x_1 \in \omega$ and so on.

Player I: $x_0$ $x_2$ $x_4$ ...

Player II: $x_1$ $x_3$ ...

This yields a sequence of natural numbers $x = \langle x_n \mid n \in \omega \rangle$, i.e. a real.

Alternative: all $x_n \in 2 = \{0, 1\}$. 
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This yields a sequence of natural numbers \( x = \langle x_n \mid n \in \omega \rangle \), i.e. a real.

Alternative: all \( x_n \in 2 = \{0, 1\} \).
How to win an infinite game

- Given a **payoff set** \( A \subseteq \omega \omega \) (resp. \( \omega^2 \)), Player I wins, if \( x \in A \), otherwise Player II wins.

- A winning strategy for Player I is a map

\[
\sigma : \bigcup_{n \in \omega} 2^{n \omega} \rightarrow \omega
\]

such that Player I wins, if he plays as follows:

1. \( x_0 = \sigma(\emptyset) \);
2. Given \( x_0, \ldots, x_{2n+1} \), Player I plays \( x_{2n+2} = \sigma(\langle x_0, \ldots, x_{2n+1} \rangle) \).

Similarly for Player II.
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Examples:

Is there a winning strategy for the games where $A \subseteq \omega \omega$ is given by...

- all eventually periodic sequences
- the set of all sequences where every natural number occurs at least once
- the set of all sequences where every natural number occurs infinitely often
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So let’s play...

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What is a winning strategy?

**Definition**

A *position* according to a strategy \( \sigma \) is a partial play such that the last play was played by the player using \( \sigma \).

**Lemma**

Let \( A \) be a payoff set. Assume that Player I (II) has a winning strategy \( \sigma \). If \( x \notin A \) (\( x \in A \)), then there is a position \( p \subseteq x \) according to \( \sigma \) such that whatever Player II (I) plays next, the consequent position according to \( \sigma \) is not contained in \( x \).
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Let $A$ be a payoff set. Assume that Player I (II) has a winning strategy $\sigma$. If $x \notin A$ ($x \in A$), then there is a position $p \subseteq x$ according to $\sigma$ such that whatever Player II (I) plays next, the consequent position according to $\sigma$ is not contained in $x$. 
Let’s define a topology on $\omega^\omega$, the so-called Baire space.

Basic clopen sets are given by

$$I_s = \{ x \in \omega^\omega | s \subseteq x \}$$

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Are all open sets determined?

**Theorem (Gale-Stewart)**

*All open sets are determined.*

**Proof (Sketch).**

Let $A \subseteq \omega \omega$ be open. Assume Player I has no winning strategy. Then at the position $\langle x_0 \rangle$ there must be $x_1$ such that Player I has no winning strategy at position $\langle x_0, x_1 \rangle$ etc. In general, Player II has a strategy $\tau$ such that at every position $\langle x_0, \cdots, x_{2n-1} \rangle$ Player I has no winning strategy. Then $\tau$ is a winning strategy for Player II: Otherwise let $x \in A$ be a play according to $\tau$. Since $A$ is open, there exists $s \in \omega \omega$ such that $x \in I_s \subseteq A$. But then Player I has a winning strategy at all positions later than $s$. Contradiction.

This uses the axiom of choice!
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This uses the axiom of choice!
Is every set of reals determined?

No...

**Theorem**

There is a set of reals which is not determined.

**Proof.**

Idea: There are more sets of reals than strategies...

Enumerate all strategies \(\langle \sigma_\alpha \mid \alpha < 2^{\aleph_0} \rangle\) for Player I and all strategies \(\langle \tau_\alpha \mid \alpha < 2^{\aleph_0} \rangle\) for Player II. At every step choose reals \(a_\alpha, b_\alpha\) such that \(a_\alpha \notin \{b_\beta \mid \beta < \alpha\}\), \(b_\alpha \notin \{a_\beta \mid \beta < \alpha\}\) and \(a_\alpha\) can be obtained using strategy \(\sigma_\alpha\), \(b_\alpha\) using \(\tau_\alpha\) and starting with 0. Then

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**Axiom of Determinacy (AD)**

Every set of reals is determined.
What is the idea behind AD?

- Player I has a winning strategy means: $\exists x_0 \forall x_1 \exists x_2 \ldots (x \in A)$
- Player II has a winning strategy means: $\forall x_0 \exists x_1 \forall x_2 \ldots (x \notin A)$

AD states:

$$\neg (\exists x_0 \forall x_1 \exists x_2 \ldots (x \in A)) \iff \forall x_0 \exists x_1 \forall x_2 \ldots (x \notin A).$$
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Consequences of AD

$\text{ZF}+\text{AD}$ has some very nice consequences:

- Every set of reals is Lebesgue measurable
- Every set of reals has the Baire property
- Every set of reals has the perfect subset property
- $\text{AC}_\omega(\mathbb{R})$ (every countable set of nonempty sets of reals has a choice function).
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1878, Georg Cantor formulated the Continuum Hypothesis (CH)

\[ \aleph_1 = 2^{\aleph_0}, \text{ i.e. every uncountable set has cardinality at least } 2^{\aleph_0}. \]

- \( \aleph_0 = \) cardinality of \( \omega \)
- \( \aleph_1 = \) the least uncountable cardinal number
- \( 2^{\aleph_0} = \) cardinality of \( \mathbb{R} = \mathcal{P}(\omega) \) (the continuum)
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The Perfect Subset Property

Definition

A set of reals is said to be **perfect**, if it is closed, non-empty and has no isolated points.

In order to prove CH, Cantor postulated

**The Perfect Subset Property (PSP)**

Every set of reals is either countable or contains a perfect subset.
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**The Perfect Subset Property**

**Theorem (Cantor-Bendixson)**

*Every closed set of reals satisfies the PSP.*

**Sketch of the proof.**

For $C \subseteq \omega^\omega$, let $C'$ be the set of limit points of $C$. Define iteratively,

- $C_0 = C$.
- $C_{\alpha+1} = C'$.
- $C_\alpha = \bigcap_{\beta < \alpha} C_\beta$, if $\alpha$ is a limit ordinal.

Then $\bigcup_{\alpha < \omega_1} C_\alpha \setminus C_{\alpha+1}$ is countable, so the sequence $\langle C_\alpha | \alpha < \omega_1 \rangle$ stabilizes after countably many steps. Then $P = \bigcap_{\alpha < \omega_1} C_\alpha$ is perfect.
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\[\]
Does every set of reals satisfy the PSP?

No...

Bernstein (1908) proved under AC

Theorem

There exists a set of reals $A$ of cardinality $2^{\aleph_0}$ such that for every perfect set of reals $P$, $P \cap A \neq \emptyset$ and $P \setminus A \neq \emptyset$. 
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Bernstein (1908) proved under AC

**Theorem**

There exists a set of reals $A$ of cardinality $2^{\aleph_0}$ such that for every perfect set of reals $P$, $P \cap A \neq \emptyset$ and $P \setminus A \neq \emptyset$. 
The perfect set game

Player I chooses $s_0 \in <\omega^2$, Player II chooses $k_1 \in 2$ and so on.

For $A \subseteq \omega^2$, Player I wins iff

$$s_0 \sim \langle k_1 \rangle \sim s_2 \sim \langle k_3 \rangle \ldots \in A.$$
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Player I: $s_0 \quad s_2 \quad s_4 \quad \ldots$

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The perfect set game

Theorem (Davis, 1964)

For any set $A \subseteq \omega^2$,

1. $A$ is countable iff Player II has a winning strategy.
2. $A$ has a perfect subset iff Player I has a winning strategy.

Proof (Sketch).

1. Assume $\tau$ is a winning strategy for II. Then for any $x \in A$ there is a position $p_x$ according to $\tau$ such that every consequent position is not contained in $x$. Then for any $x \neq y$ in $A$, $p_x \neq p_y$. Since there are only countably many positions, $A$ is countable.
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Proof (continued).

Let \( P \subseteq A \) be perfect. Let

\[
T = \{ x \mid n \mid x \in P \land n \in \omega \}.
\]

Then define a strategy \( \sigma \) by

- \( \sigma(\emptyset) = s \in T \) such that \( s \upharpoonright \langle 0 \rangle, s \upharpoonright \langle 1 \rangle \in T \).
- Given a position \( p \) according to \( \sigma \) and \( q = p \upharpoonright \langle k \rangle \) where Player II plays \( k \), choose \( \sigma(q) \) to be some \( s \) such that
  \( q \upharpoonright s \upharpoonright \langle 0 \rangle, q \upharpoonright s \upharpoonright \langle 1 \rangle \in T \).

Conversely, if \( \sigma \) is a winning strategy, then the set of all reals resulting from plays according to \( \sigma \) is perfect.

\( \sqcup \)
Proof (continued).

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1. $\sigma(\emptyset) = s \in T$ such that $s \upharpoonright \langle 0 \rangle, s \upharpoonright \langle 1 \rangle \in T$.
2. Given a position $p$ according to $\sigma$ and $q = p \upharpoonright \langle k \rangle$ where Player II plays $k$, choose $\sigma(q)$ to be some $s$ such that $q \upharpoonright s \upharpoonright \langle 0 \rangle, q \upharpoonright s \upharpoonright \langle 1 \rangle \in T$.

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Conversely, if $\sigma$ is a winning strategy, then the set of all reals resulting from plays according to $\sigma$ is perfect.
Corollary

Under AD, every set of reals satisfies the PSP.

Theorem (Cantor, 1884)

Every perfect set of reals has cardinality $2^\aleph_0$.

AD implies a weak form of CH:

Every uncountable set of reals has the cardinality of the continuum.
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