Determinacy of Infinite Games

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Contents

What are infinite games?

Player I chooses $x_0 \in \omega$, Player II chooses $x_1 \in \omega$ and so on.

Player I:	x_0		x_2		x_4	
						$\dots$
Player II:		x_1		x_3		

This yields a sequence of natural numbers $x = \langle x_n \mid n \in \omega \rangle$, i.e. a real.
Alternative: all $x_n \in 2 = \{0, 1\}$.

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How to win an infinite game

- Given a **payoff set** $A \subseteq {}^{\omega}\omega$ (resp. ${}^{\omega}2$), Player I wins, if $x \in A$, otherwise Player II wins.
- A **winning strategy** for Player I is a map

$$\sigma : \bigcup_{n \in \omega} {}^{2n}\omega \to \omega$$

such that Player I wins, if he plays as follows:

1. $x_0 = \sigma(\emptyset)$;
2. Given $x_0, \dots, x_{2n+1}$, Player I plays $x_{2n+2} = \sigma(\langle x_0, \dots, x_{2n+1} \rangle)$.

Similarly for Player II.

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So let's play...

Examples:

Is there a winning strategy for the games where $A \subseteq {}^{\omega}\omega$ is given by...

- all eventually periodic sequences
- the set of all sequences where every natural number occurs at least once
- the set of all sequences where every natural number occurs infinitely often
- a countable set?

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What is a winning strategy?

Definition

A **position** according to a strategy σ is a partial play such that the last play was played by the player using σ.

Lemma

Let A be a payoff set. Assume that Player I (II) has a winning strategy σ. If $x \notin A$ ($x \in A$), then there is a position $p \subseteq x$ according to σ such that whatever Player II (I) plays next, the consequent position according to σ is not contained in x.

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Baire space

Let's define a topology on ${}^{\omega}\omega$, the so-called **Baire space**.

Basic clopen sets are given by

$$I_s = \{x \in {}^{\omega}\omega \mid s \subseteq x\}$$

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Are all open sets determined?

Theorem (Gale-Stewart)

All open sets are determined.

Proof (Sketch).

Let $A \subseteq {}^{\omega}\omega$ be open. Assume Player I has no winning strategy. Then at the position $\langle x_0 \rangle$ there must be x_1 such that Player I has no winning strategy at position $\langle x_0, x_1 \rangle$ etc. In general, Player II has a strategy τ such that at every position $\langle x_0, \cdots, x_{2n-1} \rangle$ Player I has no winning strategy. Then τ is a winning strategy for Player II: Otherwise let $x \in A$ be a play according to τ. Since A is open, there exists $s \in {}^{<\omega}\omega$ such that $x \in I_s \subseteq A$. But then Player I has a winning strategy at all positions later than s. Contradiction. $\dashv$

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Theorem

There is a set of reals which is not determined.

Proof.

Idea: There are more sets of reals than strategies...

Enumerate all strategies $\langle \sigma_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ for Player I and all strategies $\langle \tau_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ for Player II. At every step choose reals a_α, b_α such that $a_\alpha \notin \{b_\beta \mid \beta < \alpha\}, b_\alpha \notin \{a_\beta \mid \beta < \alpha\}$ and a_α can be obtained using strategy σ_α, b_α using τ_α and starting with 0. Then

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What if don't have AC?

... and replace it by

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The Axiom of Determinacy

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- Player I has a winning strategy means: $\exists x_0 \forall x_1 \exists x_2 \ldots (x \in A)$
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AD states:

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ZF+AD has some very nice consequences:

- Every set of reals is Lebesgue measurable
- Every set of reals has the Baire property
- Every set of reals has the perfect subset property
- $AC_\omega(\mathbb{R})$ (every countable set of nonempty sets of reals has a choice function).

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Definition

A set of reals is said to be **perfect**, if it is closed, non-empty and has no isolated points.

In order to prove CH, Cantor postulated

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The Perfect Subset Property

Theorem (Cantor-Bendixson)

Every closed set of reals satisfies the PSP.

Sketch of the proof.

For $C \subseteq {}^{\omega}\omega$, let C' be the set of limit points of C. Define iteratively,

- $C_0 = C$.
- $C_{\alpha+1} = C'_{\alpha}$.
- $C_{\alpha} = \bigcap_{\beta<\alpha} C_{\beta}$, if α is a limit ordinal.

Then $\bigcup_{\alpha<\omega_1} C_{\alpha} \setminus C_{\alpha+1}$ is countable, so the sequence $\langle C_{\alpha} \mid \alpha < \omega_1 \rangle$ stabilizes after countably many steps. Then $P = \bigcap_{\alpha<\omega_1} C_{\alpha}$ is perfect. ⊣

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Does every set of reals satisfy the PSP?

No...

Bernstein (1908) proved under AC

Theorem

There exists a set of reals A of cardinality $2^{\aleph_0}$ such that for every perfect set of reals P, $P \cap A \neq \emptyset$ and $P \setminus A \neq \emptyset$.

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The perfect set game

Player I chooses $s_0 \in {}^{<\omega}2$, Player II chooses $k_1 \in 2$ and so on.

Player I:	s_0		s_2		s_4	
						$\dots$
Player II:		k_1		k_3		

For $A \subseteq {}^{\omega}2$, Player I wins iff

$$s_0 ^\frown \langle k_1 \rangle ^\frown s_2 ^\frown \langle k_3 \rangle \ldots \in A.$$

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The perfect set game

Theorem (Davis, 1964)

For any set $A \subseteq {}^{\omega}2$,

1. *A is countable iff Player II has a winning strategy.*
2. *A has a perfect subset iff Player I has a winning strategy.*

Proof (Sketch).

1. Assume τ is a winning strategy for II. Then for any $x \in A$ there is a position p_x according to τ such that every consequent position is not contained in x. Then for any $x \neq y$ in A, $p_x \neq p_y$. Since there are only countably many positions, A is countable.

$\dashv$

The perfect set game

Theorem (Davis, 1964)

For any set $A \subseteq {}^{\omega}2$,

1. *A is countable iff Player II has a winning strategy.*
2. *A has a perfect subset iff Player I has a winning strategy.*

Proof (Sketch).

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The perfect set game

Proof (continued).

2. Let $P \subseteq A$ be perfect. Let

$$T = \{x \restriction n \mid x \in P \wedge n \in \omega\}.$$

Then define a strategy σ by

- $\sigma(\emptyset) = s \in T$ such that $s^\frown\langle 0\rangle, s^\frown\langle 1\rangle \in T$.
- Given a position p according to σ and $q = p^\frown\langle k\rangle$ where Player II plays k, choose $\sigma(q)$ to be some s such that $q^\frown s^\frown\langle 0\rangle, q^\frown s^\frown\langle 1\rangle \in T$.

Conversely, if σ is a winning strategy, then the set of all reals resulting from plays according to σ is perfect.

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The perfect set game

Proof (continued).

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Corollary

Under AD, every set of reals satisfies the PSP.

Theorem (Cantor, 1884)

Every perfect set of reals has cardinality $2^{\aleph_0}$.

AD implies a weak form of CH:

Every uncountable set of reals has the cardinality of the continuum.

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