The axiom of choice

How (not) to choose infinitely many socks

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Content

Motivation

The axiom of choice is necessary to select a set from an infinite number of socks but not an infinite number of shoes.

Bertrand Russell

The axiom of choice

The *axiom of choice* AC states the following:

$$\forall x\,[\emptyset \notin x \to \exists f : x \to \bigcup x\,(\forall y \in x(f(y) \in y))].$$

Or, in a less cryptic way,

If $(X_i)_{i\in I}$ is a family of non-empty sets, then there is a family $(y_i)_{i\in I}$ such that $y_i \in X_i$ for every $i \in I$.

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The axiom of choice

The Axiom of Choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn's lemma?

Jerry Bona

Some equivalences of the axiom of choice

Theorem

The following statements are equivalent.

1. *The axiom of choice.*
2. *The well-ordering principle.*
3. *Zorn's lemma.*
4. *Every vector space has a basis.*
5. *Every non-trivial unital ring has a maximal ideal.*
6. *Tychonoff's theorem.*
7. *Every connected graph has a spanning tree.*
8. *Every surjective map has a right inverse.*

Proof.

Left as an excercise. □

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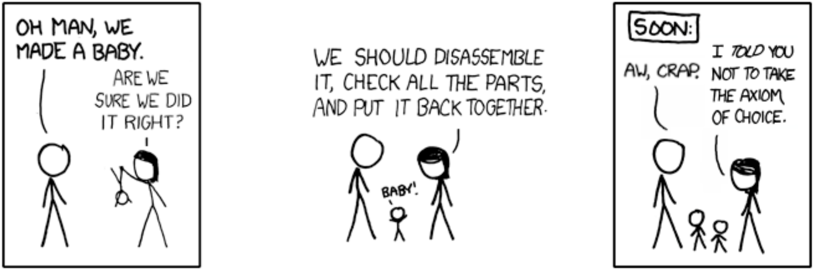

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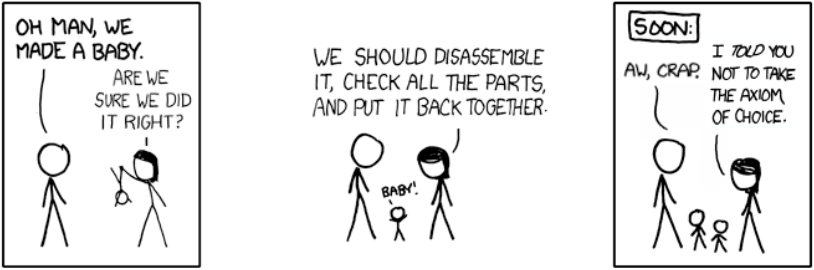

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Weaker forms of choice

The following choice principles are strictly weaker then AC.

- $AC_{\aleph_0}$ Every countable family of non-empty sets has a choice function (*Axiom of countable choice*).
- $AC_{\aleph_0}^{<\aleph_0}$ Every countable family of non-empty finite sets has a choice function.
- $AC_{\aleph_0}^{n}$ Every family of countably many n-element sets has a choice function.
- KL König's lemma
- RPP Ramsey's partition principle

König's lemma

Theorem (Dénes König, 1927)

Every finitely branching tree that contains infinitely many vertices has an infinte branch.

It was shown by Pincus in 1972 that König's lemma is equivalent to the axiom $AC_{\aleph_0}^{<\aleph_0}$.

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Ramsey's partition principle

A **2-coloring** of a set X is a map $f : [X]^2 \to \{0, 1\}$. A subset Y of X is said to be **monochromatic (or homogeneous)** with respect to f, if $f \restriction [Y]^2$ is constant.

Theorem (Ramsey's partition principle)

If $f : [X] \to \{0, 1\}$ is a 2-coloring of an infinite set X, then there is an infinite subset Y of X which is monochromatic.

It holds that $AC_{\aleph_0} \Rightarrow RPP \Rightarrow KL$.

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Stronger forms of choice

In class theory, one often uses the **axiom of global choice**:

There is a class function $F : \mathrm{V} \setminus \{\emptyset\} \to \mathrm{V}$ such that $F(x) \in x$ for every set $x \in \mathrm{V} \setminus \emptyset$.

This is equivalent to postulating that there is a global well-order of the set-theoretic universe V.

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Set theory with atoms (ZFA)

The language of ZFA is given by $\{\in, A\}$. The axioms are given by the axioms of ZF with a modified version of the **axiom of the empty set** and the **extensionality axiom**

$$\exists x\,[x \notin A \wedge \forall y(y \notin x)]$$
$$\forall x, y\,[(x, y \notin A) \rightarrow \forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y]$$

and the additional **axiom of atoms** given by

$$\forall x\,[x \in A \leftrightarrow (x \neq \emptyset \wedge \neg\exists y(y \in x))].$$

Models of set theory with atoms

For a set S we define a hierarchy by

$$\begin{aligned} \mathcal{P}^0(S) &= S \\ \mathcal{P}^{\alpha+1}(S) &= \mathcal{P}(\mathcal{P}^\alpha(S)) \\ \mathcal{P}^\alpha(S) &= \bigcup_{\beta<\alpha} \mathcal{P}^\beta(S), \ \beta \text{ limit} \\ \mathcal{P}^\infty(S) &= \bigcup_{\alpha\in\mathrm{Ord}} \mathcal{P}^\alpha(S). \end{aligned}$$

Then $\mathrm{V} = \mathcal{P}^\infty(A)$ is a model of ZFA, $\hat{\mathrm{V}} = \mathcal{P}^\infty(\emptyset)$ is a model of ZF.

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Normal filters of permutation groups

Let G be a group of permutations of A.

Definition

Let $\mathcal{F}$ be a set of subgroups of G. Then $\mathcal{F}$ is said to be a **normal filter** on G, if for all subgroups H, K of G the following statements hold.

(A) $G \in \mathcal{F}$

(B) $H \in \mathcal{F}$ and $H \subseteq K \Rightarrow K \in \mathcal{F}$

(C) $H, K \in \mathcal{F} \Rightarrow H \cap K \in \mathcal{F}$

(D) $\pi \in G$ and $H \in \mathcal{F} \Rightarrow \pi H \pi^{-1} \in \mathcal{F}$

(E) for all $a \in A$, $\{\pi \in G \mid \pi a = a\} \in \mathcal{F}$.

Normal filters of permutation groups

Definition

For a subset $E \subseteq A$ we define

$$\mathrm{fix}_G(E) = \{\pi \in G \mid \pi a = a \text{ for all } a \in E\}.$$

Then the filter $\mathcal{F}$ generated by $\{\mathrm{fix}_G(E) \mid E \subseteq A \text{ finite}\}$, i.e.

$$H \in \mathcal{F} \iff \exists E \subseteq A \text{ finite such that } \mathrm{fix}_G(E) \subseteq H$$

is a normal filter on G, denoted $\mathcal{F}_{\mathrm{fin}}$.

From now on, let $\mathcal{F} = \mathcal{F}_{\mathrm{fin}}$.

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From now on, let $\mathcal{F} = \mathcal{F}_{\mathrm{fin}}$.

By transfinite induction we define for every set x and for every $\pi \in G$ the set πx by

$$\pi x = \begin{cases} \emptyset & \text{if } x = \emptyset, \\ \pi x & \text{if } x \in A, \\ \{\pi y \mid y \in x\} & \text{otherwise.} \end{cases}$$

For every set x we define

$$\mathrm{sym}_G(x) = \{\pi \in G \mid \pi x = x\}.$$

Definition

Let $\mathcal{F}$ be a normal filter on G. A set x is said to be

- **symmetric**, if $\mathrm{sym}_G(x) \in \mathcal{F}$.
- **hereditarily symmetric**, if x is symmetric and every element of x is hereditarily symmetric.

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Symmetric sets

Lemma

The following statements hold.

1. *Every atom $a \in A$ is symmetric.*
2. *A set x is hereditarily symmetric if and only if for every $\pi \in G$, πx is hereditarily symmetric.*
3. *For every set $x \in \hat{V}$ and for every $\pi \in G$, $\pi x = x$.*

Symmetric sets

Lemma

A set x is symmetric if and only if there is a finite set $E \subseteq A$ such that $\mathrm{fix}_G(E) \subseteq \mathrm{sym}_G(x)$. Such a set E is said to be a ***support*** *of x.*

Note that if E is a support of x then every finite set F with $E \subseteq F \subseteq A$ is also a support of x.

Permutation models

Definition

Let $\mathcal{F}$ be a normal filter on a group G of permutations of A. Then the class $\mathrm{V}_{\mathcal{F}}$ of all hereditarily symmetric sets in $\mathrm{V} = \mathcal{P}^{\infty}(A)$ is a model of ZFA called a **permutation model**.

We have $A \in \mathrm{V}_{\mathcal{F}}$ and $\hat{\mathrm{V}} \subseteq \mathrm{V}_{\mathcal{F}}$.

Theorem (Jech-Sochor)

Permutation models can always be embedded into models of ZF.

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Theorem (Jech-Sochor)

Permutation models can always be embedded into models of ZF.

The second Fraenkel model

We now construct a specific permutation model with atoms given by

$$A = \bigcup_{n \in \omega} P_n,$$

where $P_n = \{a_n, b_n\}$ consists of two elements for all $n \in \omega$ and $P_n \cap P_m = \emptyset$ for $n \neq m$.

Let G be the group of all permutations π of A such that $\pi P_n = P_n$ for all $n \in \omega$, and let $\mathcal{F} = \mathcal{F}_{\text{fin}}$. We call the corresponding permutation model the **second Fraenkel model** V_{F_2}.

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The second Fraenkel model

Lemma

The following statements hold in V_{F_2}.

1. *For all* $n \in \omega$, $P_n \in V_{F_2}$.
2. $\{P_n \mid n \in \omega\} \in V_{F_2}$.

Proof.

1. By definition we have $\pi P_n = P_n$ for all $\pi \in G$, so P_n is symmetric. Since $\mathrm{sym}_G(a_n), \mathrm{sym}_G(b_n) \in \mathcal{F}$, P_n is also hereditarily symmetric.
2. For every $\pi \in G$ we have

$$\pi\{P_n \mid n \in \omega\} = \{\pi P_n \mid n \in \omega\} = \{P_n \mid n \in \omega\},$$

so $\{P_n \mid n \in \omega\}$ is symmetric and by (1) also hereditarily symmetric.

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A failure of $\mathrm{AC}^2_{\aleph_0}$

Theorem

In V_{F_2} the axiom $\mathrm{AC}^2_{\aleph_0}$ fails.

Proof.

We show that there is no choice function on $\{P_n \mid n \in \omega\}$. Suppose for a contradiction that $f : \omega \to \bigcup_{n\in\omega} P_n$ is a choice function, i.e. $f(n) \in P_n$ for all $n \in \omega$. Let E_f be a support of f. WLOG we may assume that E is of the form $E = \{a_0, b_0, \ldots, a_k, b_k\}$ for some $k \in \omega$. Let $\pi \in \mathrm{fix}_G(E)$ with $\pi a_{k+1} = b_{k+1}$. Since $\pi \in \mathrm{fix}_G(E) \subseteq \mathrm{sym}_G(f)$ we have $\pi f = f$ and hence

$$\pi\langle k+1, f(k+1)\rangle = \langle \pi(k+1), \pi f(k+1)\rangle = \langle k+1, \pi f(k+1)\rangle \in f,$$

so $f(k+1) = \pi f(k+1)$, a contradiction. □

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A failure of König's lemma

Theorem

In V_{F_2} there is an infinite binary tree which does not have an infinite branch. In particular, in V_{F_2} König's lemma fails.

Proof.

For every $n \in \omega$ consider

$$V_n = \{s : \{0, \ldots, n-1\} \to A \mid \forall i < n\, [s(i) \in P_i]\}.$$

The vertices are given by $V = \bigcup_{n \in \omega} V_n$. Furthermore, we define $s \prec t$ iff there is $n \in \omega$ such that $s \in V_n, t \in V_{n+1}$ and t extends s. Then $T = \langle V, \prec \rangle$ is an infinite binary tree with the property that if $s \in V$, then $s \cup \{\langle n, a_n \rangle\}, s \cup \{\langle n, b_n \rangle\} \in V$.

But every infinite branch through T would give a choice function on $\{P_n \mid n \in \omega\}$ contradicting our previous theorem. □

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A failure of the partition principle

Theorem

In V_{F_2} there is a 2-coloring of $[A]^2$ such that no infinite subset of A is homogeneous.

Proof.

Consider $f : [A]^2 \to \{0, 1\}$ given by

$$f(\{a,b\}) = \begin{cases} 1 & \text{if } \{a,b\} = P_n \text{ for some } n \in \omega, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $B \subseteq A$ is an infinite homogeneous set. Clearly, $f \restriction [B]^2 \equiv 0$. Let E be a support of B and let $k \in \omega$ be such that $E \cap P_k = \emptyset$ and $B \cap P_k \neq \emptyset$. Then there is $\pi \in \mathrm{fix}_G(E)$ such that $\pi a_k = b_k$. But then $\pi \notin \mathrm{sym}_G(B)$. □

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Consider $f : [A]^2 \to \{0, 1\}$ given by

$$f(\{a, b\}) = \begin{cases} 1 & \text{if } \{a, b\} = P_n \text{ for some } n \in \omega, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $B \subseteq A$ is an infinite homogeneous set. Clearly, $f \restriction [B]^2 \equiv 0$. Let E be a support of B and let $k \in \omega$ be such that $E \cap P_k = \emptyset$ and $B \cap P_k \neq \emptyset$. Then there is $\pi \in \mathrm{fix}_G(E)$ such that $\pi a_k = b_k$. But then $\pi \notin \mathrm{sym}_G(B)$. □

Further statements that are consistent in the absence of AC

- All sets of reals are Lebesgue measurable.
- There is a countable union of countable sets which is not countable.
- The reals cannot be well-ordered.
- The Baire category theorem fails.

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Homework

Infinitely many dwarves are standing in a straight line. Every dwarf wears a hat of color either red or blue and sees the color of the hats of all the dwarves standing in front of him. There is explicitly a first dwarf, who has to start guessing the color of his hat and then the guessing proceeds with the next one in the line.

If a dwarf guessed correctly, it is freed; if he guessed wrong, it is fried. Every dwarf can hear the voice of all other dwarves without a problem. Everybody is only allowed to speak out either the color red or blue, but no further information.

Is there a possibility for (almost) all dwarves to be freed?