Sheet 10: More homotopy theory

FLORIAN KRANHOLD

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1. π_1-ACTION ON BASED MAPS

Reminder 1.1. Let (X, x) be well-pointed (i. e. $* \hookrightarrow X, * \longmapsto x$ is a cofibration) and Y another space. For each path $\alpha : I \longrightarrow Y$ from y to y' and $f : X \longrightarrow Y$ with $f(x) = y$, we consider the homotopy extension

$$\begin{tikzcd}
* \times 0 \arrow[r] \arrow[d] & X \times 0 \arrow[d] \arrow[ddr, bend left, "f"] & \\
* \times I \arrow[r] \arrow[drr, bend right, "\alpha"'] & X \times I \arrow[dr, dashed, "H"] & \\
& & Y
\end{tikzcd}$$

and define $\alpha_\# f := H(-, 1)$. Then $\alpha_\# f : (X, x) \longrightarrow (Y, \alpha(1))$ is based and the homotopy type $[\alpha_\# f] \in [(X, x), (Y, y')]$ only depends on $[\alpha] \in \Pi_1(Y)$ and $[f] \in [(X, x), (Y, y)]_*$. This gives an action

$$\Pi_1(Y)(y, y') \times [(X, x), (Y, y)]_* \longrightarrow [(X, x), (Y, y')]_*$$

which generalises the Π_1-action on the homotopy groups.

Proposition 1.2. *Let (X, x) again be well-pointed, Y connected and choose $y \in Y$. Then the following relaxation is surjective:*

$$R : [X, Y]_* \longrightarrow [X, Y], [f]_* \longmapsto [f]$$

Proof. Let $f : X \longrightarrow Y$ and define $y' := f(x)$. Choose a path $\alpha : I \longrightarrow Y$ from y to y'. By construction $\alpha^-_\# f$ is a based map $(X, x) \longrightarrow (Y, y)$ and the map $H : X \times I \longrightarrow Y$ from above is a (free!) homotopy from f to $\alpha^-_\# f$, so we get

$$[f] = R\Big([\alpha^-_\# f]_*\Big). \qquad \square$$

Proposition 1.3. *In the same setting, we have a group action of $\pi_1(Y)$ on $[X, Y]_*$ and the relaxation R is precisely the projection to the quotient $[X, Y]_* \longrightarrow [X, Y]_*/\pi_1(Y)$.*

Proof. For two based maps $f, g : (X, x) \longrightarrow (Y, y)$, it is enough to show that $f \simeq g$ iff $[g]_* = [\alpha_\# f]_*$ for some based $\alpha : I \longrightarrow Y$.

($\Rightarrow$) If $[g]_* = [\alpha_\# f]_*$, there is a $H : X \times I \longrightarrow Y$ with $H(-, 0) = f$ and $H(-, 1) \simeq g$, so $f \simeq g$.

($\Leftarrow$) Let $H : X \times I \longrightarrow Y$ be a homotopy from f to g, we consider $\alpha := H(x, -) : I \longrightarrow Y$. Then $\alpha(0) = \alpha(1) = y$ and $[\alpha_\# f]_* = [g]_*$. $\square$

2. $\mathbb{R}P^\infty$ AND A COUNTEREXAMPLE

Proposition 2.1. *$\pi_1(\mathbb{R}P^\infty) = \mathbb{Z}_2$ and $\pi_i(\mathbb{R}P^\infty) = 0$ for $i \geq 2$.*

Proof.
- Consider $\mathbb{S}^\infty := \varinjlim \mathbb{S}^n$. Then $\pi_i(\mathbb{S}^\infty) = 0$: Let $f : \mathbb{S}^i \longrightarrow \mathbb{S}^\infty$ a based map, then by compactness $\mathrm{im}(f) \subseteq \mathbb{S}^n$ for some $n \in \mathbb{N}$. By cellular approximation, f can be contracted inside $\mathbb{S}^{n+1}$, in particular $f \simeq 0$ as a map to $\mathbb{S}^\infty$.
- We know that $\mathbb{R}P^\infty = \mathbb{S}^\infty/\mathbb{Z}^*$ where $\mathbb{Z}^*$ describes the antipodal action. As $\mathbb{S}^\infty$ is weakly contractible, we get $\pi_1(\mathbb{R}P^\infty) = \mathbb{Z}^* = \mathbb{Z}_2$ and $\pi_i(\mathbb{R}P^\infty) = \pi_i(\mathbb{S}^\infty) = 0$ for $i \geq 2$. $\square$

Example 2.2. Consider the spaces $X := \mathbb{R}P^2$ and $Y := \mathbb{R}P^\infty \times \mathbb{S}^2$.

- It is not true that $X \simeq Y$: We see by KÜNNETH $H_2(Y; \mathbb{Z}_2) = H_2(\mathbb{R}P^\infty; \mathbb{Z}_2) \oplus H_2(\mathbb{S}^2; \mathbb{Z}_2) = \mathbb{Z}_2^2 \neq \mathbb{Z}_2 = H_2(X; \mathbb{Z}_2)$.
- We have $\pi_i(Y) = \pi_i(\mathbb{R}P^\infty) \times \pi_i(\mathbb{S}^2) = \pi_i(X)$ for all $i \in \mathbb{N}$.

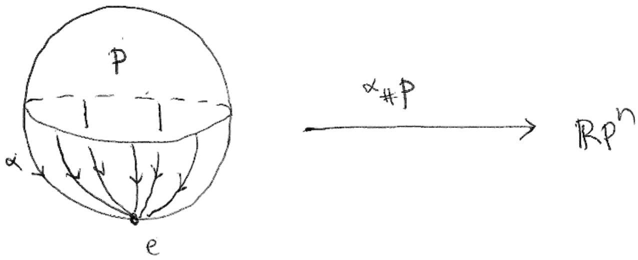
3. π_1-ACTION ON $\pi_n(\mathbb{R}P^n)$

Remark 3.1. We know $\mathbb{R}P^1 \cong \mathbb{S}^1$ and $\pi_1(\mathbb{R}P^1) = \mathbb{Z}$ acts on itself by conjugation. As $\pi_1(\mathbb{R}P^1)$ is abelian, the action is trivial.

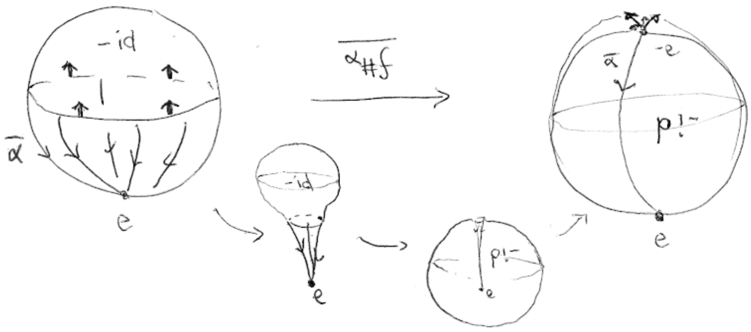
Proposition 3.2. *Let $n \geq 2$. Then $\pi_1(\mathbb{R}P^n) = \mathbb{Z}^*$ acts on $\pi_n(\mathbb{R}P^n)$ by $(\pm 1)^{n+1}$.*

Proof. For technical reasons, let $e = (0, \dots, 0, -1) \in \mathbb{S}^n$ and $\mathbb{S}^n_\pm := \{\pm x_n \geq 1\} \subseteq \mathbb{S}^n$.

- Consider $p : (\mathbb{S}^n, e) \longrightarrow (\mathbb{R}P^n, *)$ and recall that $p_* : \pi_1(\mathbb{S}^n, e) \longrightarrow \pi_n(\mathbb{R}P^n, *)$ is an isomorphism with the following inverse: Given $f : (\mathbb{S}^n, e) \longrightarrow (\mathbb{R}P^n, *)$, we have a unique lift $\overline{f} : (\mathbb{S}^n, e) \longrightarrow (\mathbb{S}^n, e)$. We have isomorphism $\deg : \pi_n(\mathbb{R}P^n, *) = \pi_n(\mathbb{S}^n, e) = [\mathbb{S}^n, \mathbb{S}^n] \longrightarrow \mathbb{Z}$.
- Consider the generator $\alpha : I \longrightarrow \mathbb{R}P^n$. As $\deg([p]) = 1$, it is enough to show $\deg([\alpha_\# p]) = (-1)^{n+1}$. First, we see that α^- lifts to $\overline{\alpha}^- : I \longrightarrow \mathbb{S}^n$ from e to $-e$. By definition, $\alpha_\# p : (\mathbb{S}^n, e) \longrightarrow (\mathbb{R}P^n, *)$ is (up to relative homotopy) of the following form: On $\mathbb{S}^n_-$ we have $(x_0, \dots, x_n) \longmapsto \alpha(-x_n)$, the arcwise realisation of α^-, and on $\mathbb{S}^n_+$, we have the rescaled p.



- The lift $\overline{\alpha_\# p}$ of $\alpha_\# p$ is of the following form: On $\mathbb{S}^n_-$, we have the arcwise realisation of $\overline{\alpha}^-$, so we go from e to $-e$, and $\mathbb{S}^n_+$, we have the (rescaled) lift of p, now at the *different* point $-e$ at the fibre, which is given by $-\mathrm{id}$.



- As the degree factors over the unbsed homotopy classes, we can contract the path and get that $\overline{\alpha_\# p} \simeq -\mathrm{id}$ and as desired $\deg[\alpha_\# p] = \deg(-\mathrm{id}) = (-1)^{n+1}$. $\square$.

4. FREUDENTHAL

Reminder 4.1. (I) A based space (X, x) is called *n-connected*, if $\pi_i(X, x) = 0$ for $i \leq n$.

(II) A based map $f : (X, x) \longrightarrow (Y, y)$ is called *n-equivalence* if $\pi_i(f)$ is an isomorphism for $i \leq n$ and $\pi_{n+1}(f)$ is an epimorphism.

Proposition 4.2. *Let Y be well-based and n-connected. Then the unit $\eta : Y \longrightarrow \Omega\Sigma Y$ is a $2n$-equivalence.*

Proof. Using the $\Omega\Sigma$-adjunction, we get a diagram

$$\begin{tikzcd}
{[f]} \arrow[r, mapsto] & {[\Sigma f]} \\
\pi_k(Y) \arrow[r, "\Sigma"'] \arrow[dr, "\eta_*"'] & \pi_{k+1}(\Sigma Y) \arrow[d, equal, "\Phi"] \\
& \pi_k(\Omega\Sigma X),
\end{tikzcd}$$

so it is enough to show that $\Sigma : \pi_k(Y) \longrightarrow \pi_{k+1}(\Sigma Y)$ is an isomorphism for $k \leq 2n$ and an epimorphism for $k = 2n + 1$, which is exactly the FREUDENTHAL SUSPENSION THEOREM. □

Corollary 4.3. *Let (X, x) be a based cell-complex and Y as before. Consider*

$$\Sigma : [X, Y]_* \longrightarrow [\Sigma X, \Sigma Y]_*$$

If $\dim(X) \leq 2n$, then Σ is an isomorphism, and if $\dim(X) \leq 2n + 1$, Σ is epic.

Proof. This is an immediate consequence from WHITEHEAD'S THEOREM in the first version:

> *Let $e : Y \longrightarrow Z$ be an m-equivalence. If $\dim(X) \leq m$, then $e_* : [X, Y]_* \longrightarrow [X, Z]_*$ is an isomorphism and if $\dim(X) \leq m + 1$, then e_* is epic.*

Applying this to $m = 2n$ and $e = \eta$, we get that η_* has the desired properties, so we use the adjunction $[X, \Omega\Sigma Y]_* = [\Sigma X, \Sigma Y]_*$ once more to get the statement for $\Sigma = \Phi^{-1} \circ \eta_*$. □