Sheet 9: Elementary homotopy theory

Florian Kranhold

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1. Monoid objects

Definition 1.1. Let C be a category with finite products (in particular, a terminal object $*$). A *monoid object* is a triple (X, μ, e) where $X \in \mathrm{ob}(C)$ and $\mu : X \times X \longrightarrow X$ and $e : * \longrightarrow X$ with

$$\begin{tikzcd} X^3 \arrow[r, "1_X \times \mu"] \arrow[d, "\mu \times 1_X"'] & X^2 \arrow[d, "\mu"] \\ X^2 \arrow[r, "\mu"] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \times * \arrow[r, "1_X \times e"] \arrow[dr, equal, "\mathrm{pr}_1"'] & X^2 \arrow[d, "\mu"] & * \times X \arrow[l, "e \times 1_X"'] \arrow[dl, equal, "\mathrm{pr}_2"] \\ & X & \end{tikzcd}$$

A *homomorphism* $\varphi : (X, \mu, e) \longrightarrow (X', \mu', e')$ is a morphism $\varphi : X \longrightarrow X'$ such that $\varphi \circ e = e'$ and $\mu' \circ (\varphi \times \varphi) = \varphi \circ \mu$. This defines a category $\mathrm{Mon}(C)$.

Remark 1.2. Dually, if C has finite coproducts (in particular, an initial object $\varnothing$), a *comonoid object* is (Y, ν, c) with $\nu : Y \longrightarrow Y \sqcup Y$ and $c : Y \longrightarrow \varnothing$. Homomorphisms $\psi : (Y, \nu, c) \longrightarrow (Y', \nu', c')$ are maps $\psi : Y \longrightarrow Y'$ with $c' \circ \psi = c$ and $(\psi + \psi) \circ \nu = \nu' \circ \psi$. This defines a category $\mathrm{Comon}(C)$.

Proposition 1.3.

- *A monoid structure on X is the same as a functor $F : C^{op} \longrightarrow$ **Mon** which lifts the functor $C(-, X) : C^{op} \longrightarrow$ **Set** over $U :$ **Mon** $\longrightarrow$ **Set**.*
- *A comonoid structure on X is the same as a functor $G : C \longrightarrow$ **Mon** which lifts the functor $C(X, -) : C \longrightarrow$ **Set** over $U :$ **Mon** $\longrightarrow$ **Set**.*

Proof. For duality reasons, we only have to prove the first statement:

(i) Let (μ, e) be a monoid structure on X. We define a monoid structure on $C(W, X)$ by

$$f_1 \star f_2 := \mu \circ (f_1, f_2),$$
$$e_{C(W,X)} := e \circ *_W \in C(W, X)$$

where $*_W : W \longrightarrow *$ denotes the terminal morphism. We check everthing which is necessary:

- *Associativity.* We see

$$\begin{aligned} (f_1 \star f_2) \star f_3 &= \mu \circ \big[(\mu \circ (f_1, f_2)), f_3\big] \\ &= (\mu \circ (\mu \times 1_X)) \circ (f_1, f_2, f_3) \\ &= (\mu \circ (1_X \times \mu)) \circ (f_1, f_2, f_3) \\ &= f_1 \star (f_2 \star f_3). \end{aligned}$$

- *Unitality.* We see

$$f \star e_{C(W,X)} = \mu \circ (f, e \circ *_W) = \mu \circ (1_X \times e) \circ (f, *_W) = f.$$

- *For $g : V \longrightarrow W$, the map $g^* : C(W, X) \longrightarrow C(V, X)$ is a homomorphism.* We see

$$g^*(e_{C(W,X)}) = e_{C(W,X)} \circ g = e \circ (*_W \circ g) = e \circ *_V = e_{C(V,X)}$$

by the uniqueness of the terminal arrow $V \longrightarrow *$. Moreover, we get

$$g^*(f_1 \star f_2) = \mu \circ (f_1, f_2) \circ g = \mu \circ (f_1 g, f_2 g) = (g^* f_1) \star (g^* f_2).$$

(II) Conversely, let $C(W,X)$ carry a monoid structure $(\star, e_{C(W,X)})$ for each $W \in \mathrm{ob}(C)$ and let the precompositions be homomorphisms. We define

$$\begin{aligned} e &:= e_{C(*,X)} \in C(*,X), \\ \mu &:= \mathrm{pr}_1 \star \mathrm{pr}_2 \in C(X^2, X). \end{aligned}$$

Let us introduce $\mathrm{pr}^i : X^3 \longrightarrow X$ for $1 \le i \le 3$ and $\mathrm{pr}^{ij} : X^3 \longrightarrow X^2$ for $1 \le i < j \le 3$. Then $f := 1_X \times (\mathrm{pr}_1 \times \mathrm{pr}_2) : X^3 \longrightarrow X^2$ satisfies $\mathrm{pr}_1 \circ f = \mathrm{pr}^1$ and $\mathrm{pr}_2 \circ f = (\mathrm{pr}_1 \star \mathrm{pr}_2) \circ \mathrm{pr}^{23} = \mathrm{pr}^2 \star \mathrm{pr}^3$. We see that

$$\begin{aligned} \mu \circ (1_X \times \mu) &= \big[\mathrm{pr}_1 \circ (1_X \times (\mathrm{pr}_1 \star \mathrm{pr}_2))\big] \star \big[\mathrm{pr}_2 \circ (1_X \times (\mathrm{pr}_1 \star \mathrm{pr}_2))\big] \\ &= \mathrm{pr}^1 \star \big((\mathrm{pr}_1 \star \mathrm{pr}_2) \circ \mathrm{pr}^{23}\big) \\ &= \mathrm{pr}^1 \star \mathrm{pr}^2 \star \mathrm{pr}^3 \\ &= \mu \circ (\mu \times 1_X). \end{aligned}$$

Now let $\mathrm{pr}^1 : X \times * \longrightarrow X$ and $\mathrm{pr}^2 : X \times * \longrightarrow *$. We get

$$\begin{aligned} \mu \circ (1_X \times e) &= \big[\mathrm{pr}_1 \circ (1_X \times e)\big] \star \big[\mathrm{pr}_2 \circ (1_X \times e)\big] \\ &= \mathrm{pr}^1 \star (e_{C(*,X)} \circ \mathrm{pr}^2) \\ &= \mathrm{pr}^1 \star e_{C(X \times *, X)} \\ &= \mathrm{pr}^1. \end{aligned}$$

□

Proposition 1.4. *If C has finite biproducts, every object admits both a unique monoid and a unique comonoid structure.*

Proof. We know by Exercise IV.1 that there is a unique way to enrich C over **Mon**. This is the same as lifting each $C(-,X)$ and $C(X,-)$ over **Mon** $\longrightarrow$ **Set**. □

Proposition 1.5. *If X is a monoid object and Y is a comonoid object, the two induced monoid structures on $C(Y,X)$ coincide and are abelian.*

Proof. Let $(\star, i)$ be the monoid structure coming from the monoid object (X, μ, e) and $(\cdot, j)$ be the monoid structure coming from the comonoid object (Y, ν, c). Let's write $(f,g) : Y \longrightarrow X \times X$ as before, but $[f, f'] : Y \sqcup Y \longrightarrow X$. First, we see that the satisfy the *interchange law*: By the universal property of product and coproduct, it is clear that

$$\big([f,f'],[g,g']\big) = \big[(f,g),(f',g')\big].$$

Therefore, we get

$$\begin{aligned} (f \star g) \cdot (f' \star g') &= \big[(f \star g), (f' \star g')\big] \circ \nu \\ &= \big[\mu \circ (f,g), \mu \circ (f',g')\big] \circ \nu \\ &= \mu \circ \big[(f,g),(f',g')\big] \circ \nu \\ &= \mu \circ \big([f,f'],[g,g']\big) \circ \nu \\ &= (f \cdot f') \star (g \cdot g') \end{aligned}$$

The rest of the proof is the famous Eckmann–Hilton argument for each two unital structures satisfying the interchange law:

- $j = j \cdot j = (i \star j) \cdot (j \star i) = (i \cdot j) \star (j \cdot i) = i \star i = i$, so just write $i = j = 1$,
- $f \cdot g = (1 \star f) \cdot (g \star 1) = (1 \cdot g) \star (f \cdot 1) = g \star f = (g \cdot 1) \star (1 \cdot f) = (g \star 1) \cdot (1 \star f) = g \cdot f$.

It is even true that we can *conclude* associativity from this without assuming it:

- $(f \cdot g) \star h = (f \cdot g) \cdot (1 \cdot h) = (f \cdot 1) \cdot (g \cdot h) = f \cdot (g \cdot h)$. □

Definition 1.6. An *H-space* is a monoid object in $\mathrm{Ho}(\mathbf{Top}_*)$. Dually, a *co-$H$-space* is a comonoid object in $\mathrm{Ho}(\mathbf{Top}_*)$.

2. Loop spaces

Reminder 2.1. Let A be locally compact and Y an arbitrary space. The *compact–open topology* is a topology on the set $\langle A, Y\rangle = \mathbf{Top}(A,Y)$ of continous maps such that we have a bijection

$$\begin{aligned}\Phi := \Phi_{X,A,Y} : \mathbf{Top}(X\times A, Y) &\longrightarrow \mathbf{Top}(X, \langle A,Y\rangle),\\ f &\longmapsto \Big(x \longmapsto \big(f_x : a \longmapsto f(x,a)\big)\Big),\\ \Big((x,a)\longmapsto g(x)(a)\Big) &\longleftarrow g\end{aligned}$$

This means that the endofunctor $-\times A : \mathbf{Top} \longrightarrow \mathbf{Top}$ has a right adjoint $\langle A, -\rangle : \mathbf{Top} \longrightarrow \mathbf{Top}$.

Remark 2.2. We see the following:

(i) There is a based version. Let $(X, x_0), (A, a_0), (Y, y_0)$ be based.

- Let $\langle A, Y\rangle_*$ be the subspace of based maps. This is again a based space with basepoint the constant map $c_{y_0} : a \longmapsto y_0$. We have

$$\mathbf{Top}_*(X, \langle A,Y\rangle_*) \subseteq \mathbf{Top}(X, \langle A,Y\rangle_*) \subseteq \mathbf{Top}(X, \langle A,Y\rangle).$$

- We have a projection $\mathrm{pr} : X\times A \longrightarrow X\wedge A$ and $\mathrm{pr}^* : \mathbf{Top}(X\wedge A, Y) \longrightarrow \mathbf{Top}(X\times A, Y)$ is injective as pr is epic. Therefore, we have an inclusion

$$\mathbf{Top}_*(X\wedge A, Y) \subseteq \mathbf{Top}(X\wedge A, Y) \subseteq \mathbf{Top}(X\times A, Y).$$

It is easy to check that Φ restricts to a natural bijection

$$\Psi : \mathbf{Top}_*(X\wedge A, Y) \longrightarrow \mathbf{Top}_*(X, \langle A, Y\rangle_*),$$

so the endofunctor $-\wedge A : \mathbf{Top}_* \longrightarrow \mathbf{Top}_*$ has a right adjoint $\langle A, -\rangle_* : \mathbf{Top}_* \longrightarrow \mathbf{Top}_*$.

(ii) Two maps $f, f' : X\times A \longrightarrow Y$ are homotopic iff $\Phi(f), \Phi(f') =$ are homotopic: The map $H : X\times A\times I = X\times I\times A \longrightarrow Y$ is a homotopy between f and f' iff

$$\Phi_{X\times I, A, Y}(H) : X\times I \longrightarrow \langle A, Y\rangle$$

is a homotopy between $\Phi_{X,A,Y}(f)$ and $\Phi_{X,A,Y}$. Two based maps $f, f' : X\wedge A \longrightarrow Y$ are based homotopic iff $\Psi(f), \Psi(f') : X \longrightarrow \langle A, Y\rangle_*$ are based homotopic.

Proposition 2.3. *The reduced suspension* $\Sigma : \mathrm{Ho}(\mathbf{Top}_*) \longrightarrow \mathrm{Ho}(\mathbf{Top}_*)$ *has a right adjoint* Ω.

Proof. The sphere $\mathbb{S}^1$ is locally compact. Define the *loop space* $\Omega := \langle \mathbb{S}^1, -\rangle_* : \mathbf{Top}_* \longrightarrow \mathbf{Top}_*$. This is right adjoint to $-\wedge \mathbb{S}^1 = \Sigma$. Now as the natural identification respects based homotopies, this adjunction descends to the homotopy category:

$$[\Sigma X, Y]_* = [X\wedge \mathbb{S}^1, Y]_* = [X, \Omega Y]_*. \qquad \square$$

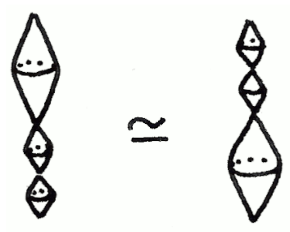
Proposition 2.4. *Let X be a space.*

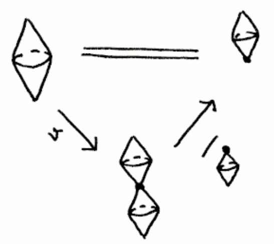
(i) *ΣX has a co-H-space structure,*

(ii) *ΩX has an H-space structure.*

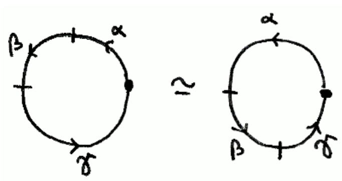
Proof.

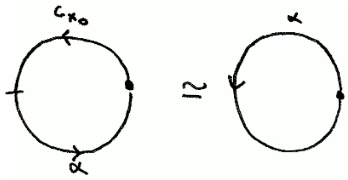
(i) Recall that the coproduct in $\mathrm{Ho}(\mathbf{Top}_*)$ is given by the wedge sum $\vee$. The comultiplication is given by the equatorial squeeze $\nu : \Sigma X \longrightarrow \Sigma X \vee \Sigma X$, the counit is just the terminal morphism $c : \Sigma X \longrightarrow *$. Associativity and unitality can proven pictorially by:





(II) The multiplication is given by the concatenation of paths $\mu : \Omega X \times \Omega X \longrightarrow \Omega X$, the unit is the based map $e : * \longrightarrow \Omega X, * \longmapsto c_{x_0}$. Associativity and unitality are proven in the same way we checked the group axioms for the $\pi_1(X)$:





□

Proposition 2.5.

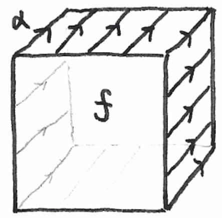
(I) *If X is an H-space, then $\pi_1(X, e)$ is abelian.*
(II) *$\pi_k(X, x_0)$ is abelian for each based (X, x_0) and $k \geq 2$.*

Proof. (I) We see $\pi_1(X, e) = [\Sigma\mathbb{S}^0, X]_*$ and the left side is a comonoid, the right side is a monoid. The Eckmann–Hilton argument from the last exercise yields abelianity.

(II) We see $\pi_k(X, x_0) = [\Sigma^2\mathbb{S}^{k-2}, X]_* = [\Sigma\mathbb{S}^{k-2}, \Omega X]_*$ and again, the left side is a comonoid, the right side is a monoid. □

3. Π_1-action on higher homotopy groups

Reminder 3.1. Recall that $\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)]$. We have a Π_1-action on $\pi_n(X, \cdot)$ as follows: Given a relative map $f : (I^n, \partial I^n) \longrightarrow (X, x_1)$ and a path $\alpha : I \longrightarrow X$ from x_1 to x_2, we just choose *any* map $H : I^n \times I \longrightarrow X$ with:



- $H(-, 0) : (I^n, \partial I^n) \longrightarrow (X, x_1)$ is homotopic to f,
- $H(z, -) : I \longrightarrow X$ is homotopicto α for each $z \in \partial I^n$.

Such an H always exists and we define

$$\alpha_\# f : (I^n, \partial I^n) \longrightarrow (X, x_2), z \longmapsto H(z, 1).$$

The homotopy type of $\alpha_\# f$ only depends on the homotopy type of f and α, *not* on the choice of H. Hence, we get a map

$$\Pi_1(X)(x_1, x_2) \times \pi_n(X, x_1) \longrightarrow \pi_n(X, x_2), ([\alpha], [f]) \longmapsto \alpha_\#[f] = [\alpha_\# f].$$

Proposition 3.2. *Let X be a connected H-space. Then the Π_1-action on π_n is trivial.*

Proof. We have to see that for each two paths $\alpha, \beta : I \longrightarrow X$ from x_1 to x_2, we have $\alpha_\# = \beta_\#$:

(I) Choose a path γ from e to x_1. It is enough to show that $\gamma^-_\# \circ \beta^-_\# \circ \alpha_\# \circ \gamma_\# = \mathrm{id}$, because

$$\begin{aligned}
\alpha_\# &= \beta_\# \circ \gamma_\# \circ \overbrace{\gamma^-_\# \circ \beta^-_\# \circ \alpha_\# \circ \gamma_\#}^{=\mathrm{id}} \circ \gamma^-_\# \\
&= \beta_\# \circ \gamma_\# \circ \gamma^-_\# \\
&= \beta_\#
\end{aligned}$$

This leaves us to show that if $\alpha : I \longrightarrow X$ is a loop based in e, then $\alpha_\# = \mathrm{id}_\#$.

(II) Let $f : (I^n, \partial I^n) \longrightarrow (X, e)$. Consider the map

$$H^{n+1} : I^n \times I \longrightarrow X, (z, t) \longmapsto f(z) \cdot \alpha(t).$$

First of all, we see that if $z \in \partial I^n$, then $H(z, 0) = H(z, 1) = \mu(e, e) = e$ because the multiplication $\mu : (X \times X, (e, e)) \longrightarrow (X, e)$ is a based map. Now we check:

- Then $H(-,0) = \mu \circ (f \times e) \circ \mathrm{pr}_1^{-1} = \mu \circ (1_X \times e) \circ (f\mathrm{pr}_1^{-1})$ where $\mathrm{pr}_1 : X \times * \longrightarrow X$. We know that $\mu \circ (1_X \times e)$ is based homotopic to pr_1, so the homotopy $H(-,0) \simeq f$ is a homotopy relative $f^{-1}(e) \supseteq \partial I^n$.
- For $z \in \partial I^n$, we have $H(z,-) = \mu \circ (e \times \alpha) \circ \mathrm{pr}_2^{-1} = \mu \circ (e \times 1_X) \circ (\alpha\mathrm{pr}_2^{-1})$. As above, the homotopy $H(z,-) \simeq \alpha$ is a homotopy relative $\alpha^{-1}(e) \supseteq \{0,1\}$.

Therefore, H is an admissible function as above and we get

$$\alpha_\#[f] = [H(-,1)] = [f]. \qquad \square$$

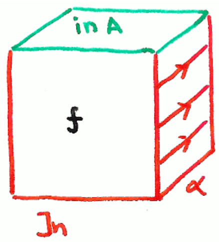
Reminder 3.3. Recall that $\pi_n(X,A,x_0) = [(I^n, \partial I^n, J^n), (X,A,x_0)]$ where

$$J^n := \partial I^n \setminus \big(I^{n-1} \times \{1\}\big) = \big(\partial I^{n-1} \times I\big) \cup \big(I^{n-1} \times \{0\}\big).$$

We have a connecting homomorpism

$$\begin{aligned} \delta_n : \pi_n(X,A,x_0) = [(I^n, \partial I^n, J^n), (X,A,x_0)] &\longrightarrow [(I^{n-1}, \partial I^{n-1}), (X,A)] = \pi_{n-1}, \\ [f] &\longmapsto \big[f|_{I^{n-1}\times\{1\}}\big]. \end{aligned}$$

The extra assumption that $f|_{J_n} \equiv x_0$ ensures that $(\delta_n[f])|_{\partial I^{n-1}} \equiv x_0$. We have a $\pi_1(A,x_0)$-action on $\pi_n(X,A,x_0)$ as follows: Given a relative map $f : (I^n, \partial I^n, J^n) \longrightarrow (X,A,x_0)$ and a path $\alpha : I \longrightarrow X$, choose any map $H : I^n \times I \longrightarrow X$ with:



- $H(-,0) : (I^n, \partial I^n, J^n) \longrightarrow (X,A,x_0)$ is homotopic to f,
- $H(z,t) \in A$ for $z \in \partial I^n$,
- $H(z,-) : I \longrightarrow A$ is homotopic to α for $z \in J^n$.

Again, such an H always exists and the relative homotopy type of

$$H(-,1) : (I^n, \partial I^n, J^n) \longrightarrow (X,A,x_0)$$

does only depend on the homotopy type of f and α.

Proposition 3.4. *δ_n is equivariant, i. e. we have $\delta_n(\alpha_\#[f]) = \alpha_\#(\delta_n[f])$.*

Proof. Fix α and f and choose H as above. Consider the restriction to the green face

$$G(-,-) := H(-,1,-) : I^{n-1} \times I \longrightarrow A.$$

By construction, $G(-,0) \simeq f|_{I^{n-1}\times\{1\}} = \delta_n f$ and for $z \in \partial I^{n-1}$, we have $(z,1) \in J^n$, so we get

$$G(z,-) = H(z,1,-) \simeq \alpha.$$

Thus, G is an admissible map for constructing $\alpha_\#\delta_n[f]$ and we get

$$\delta_n(\alpha_\# f) = [H(-,1,1)] = [G(-,1)] = \alpha_\#(\delta_n[f]). \qquad \square$$

4. The Hopf invariant, Part 2

Proposition 4.1. *Let M be a closed and oriented m-manifold. Then the mapping degree as a map $\deg : \pi_m(M) \longrightarrow \mathbb{Z}$ is a homomorphism.*

Proof. Let $f, g : \mathbb{S}^m \longrightarrow M$. We know that $[f] + [g]$ has the homotopy type of $(f \vee g) \circ \nu$ where $\nu : \mathbb{S}^m \longrightarrow \mathbb{S}^m \vee \mathbb{S}^m$ is the equatorial squeeze. If we identify $H_m(\mathbb{S}^m) = \mathbb{Z}\langle[\mathbb{S}^m]\rangle$ and $H_m(\mathbb{S}^m \vee \mathbb{S}^m) = \mathbb{Z}\langle[\mathbb{S}^m]_1, [\mathbb{S}^m]_2\rangle$, we have a sequence

$$\begin{tikzcd} H_m(\mathbb{S}^m) \arrow[r, "\nu_*"] \arrow[rr, bend left, "(f+g)_*"] \arrow[d, equal] & H_m(\mathbb{S}^m \vee \mathbb{S}^m) \arrow[r, "(f\vee g)_*"] \arrow[d, equal] & H_m(M) \arrow[d, equal] \\ \mathbb{Z}\langle[\mathbb{S}^m]\rangle \arrow[r, "\binom{1}{1}"'] & \mathbb{Z}\langle[\mathbb{S}^m]_1, [\mathbb{S}^m]_2\rangle \arrow[r, "f_*+g_*"'] & \mathbb{Z}\langle[M]\rangle\,. \end{tikzcd}$$

Therefore, we get

$$\big((f \vee g) \circ \nu\big)[\mathbb{S}^m] = (f \vee g)_* \left([\mathbb{S}^m]_1 + [\mathbb{S}^m]_2\right) = f_*[\mathbb{S}^m] + g_*[\mathbb{S}^m] = \big(\deg(f) + \deg(g)\big) \cdot [M]. \qquad \square$$

Proposition 4.2. *The Hopf invariant $h : \pi_{4n-1}(\mathbb{S}^{2n}) \longrightarrow \mathbb{Z}$ is a homomorphism.*

Proof. Let $f, g : \mathbb{S}^{4n-1} \longrightarrow \mathbb{S}^{2n}$. Consider the space $X_{f+g} := e^{4n} \sqcup_{f+g} \mathbb{S}^{2n}$ with $4n$-cell β_{f+g} and $2n$-cell α_{f+g}, and also $X_{f\vee g} = (e^{4n} \vee e^{4n}) \sqcup_{f\vee g} \mathbb{S}^{2n}$ with $4n$-cells β'_f, β'_g and $2n$-cell $\alpha_{f\vee g}$.

(i) We have canonical inclusions $i_f : X_f \hookrightarrow X_{f\vee g}$ and $i_g : X_g \hookrightarrow X_{f\vee g}$ sending the $2n$-cells α_f, α_g to $\alpha_{f\vee g}$ and β_f to β'_f and β_g to β'_g. Therefore, we get $i_f^*\alpha_{f\vee g} = \alpha_f$ and $i_g^*\alpha_{f\vee g} = \alpha_g$ as well as $i_f^*\beta'_f = \beta_f$ and $i_g^*\beta'_g = \beta_g$. Note that $i_f^* + i_g^* : H^{4n}(X_{f\vee g}) \longrightarrow H^{4n}(X_f) \oplus H^{4n}(g)$ is injective. We see

$$(i_f^* + i_g^*)\alpha_{f\vee g}^2 = \alpha_f^2 + \alpha_g^2 = h_f\beta_f + h_g\beta_g = (i_f^* + i_g^*)\left(h_f \cdot \beta'_f + h_g\beta'_g\right),$$

so we conclude $\alpha_{f\vee g}^2 = h_f\beta'_f + h_g\beta'_g$.

(ii) By collapsing $e^{4n-1} \times \{0\} \subseteq e^{4n} \hookrightarrow X_{f+g}$, we get a projection

$$\mathrm{pr} : X_{f+g} \longrightarrow X_{f\vee g}.$$

Apparently $\mathrm{pr}_*\beta_{f+g} = \beta'_f + \beta'_g$, so $\mathrm{pr}^*\beta'_f = \mathrm{pr}^*\beta'_g = \beta_{f+g}$ in cohomology. As pr is a homeomorphism on the $2n$-cells, we have $\mathrm{pr}^*\alpha' = \alpha$. We conclude

$$\alpha_{f+g}^2 = \mathrm{pr}^*\alpha_{f\vee g} = \mathrm{pr}^*(h_f\beta'_f + h_g\beta'_g) = (h_f + h_g) \cdot \beta_{f+g}. \qquad \square$$