Sheet 8: Grassmannians and coefficient systems

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1. Orientability of Grassmannians

Reminder 1.1. Let V be an n-dimensional $\mathbb{R}$-vector space. $\mathrm{GL}(k,\mathbb{R})$ acts on $\mathrm{Hom}^{\mathrm{inj}}(\mathbb{R}^k, V)$ by precomposition and we defined the Grassmannian $\mathrm{Gr}_k(V) := \mathrm{Hom}^{\mathrm{inj}}(\mathbb{R}^k, V)/\mathrm{GL}(k,\mathbb{R})$.

Remark 1.2. We define the *oriented Grassmannian*

$$\mathrm{Gr}_k^+(V) := \mathrm{Hom}^{\mathrm{inj}}(\mathbb{R}^k, V)/\mathrm{GL}^+(k,\mathbb{R}).$$

As $[\mathrm{GL}(k,\mathbb{R}) : \mathrm{GL}^+(k,\mathbb{R})] = 2$, the projection $\mathrm{pr} : \mathrm{Gr}_k^+ \longrightarrow \mathrm{Gr}_k$ is a double cover, in particular, a closed smooth manifold. As $\mathrm{Hom}^{\mathrm{inj}}(\mathbb{R}^k, V)$ is connected for $k < n$, the space $\mathrm{Gr}_k^+(V)$ is connected.

Lemma 1.3. *Let $f : P \longrightarrow P$ be linear. Consider $f^\star : \mathrm{Hom}(P,Q) \longrightarrow \mathrm{Hom}(P,Q)$. Then*

$$\det(f^\star) = \det(f)^{\dim_{\mathbb{R}}(Q)}.$$

The same holds for postcomposition $g_\star : \mathrm{Hom}(P,Q) \longrightarrow \mathrm{Hom}(P,Q)$ for $g : Q \longrightarrow Q$.

Proof. Let $(p_1,\dots,p_r)$ a basis for P and $(q_1,\dots,q_s)$ a basis for Q and identify $\mathrm{Hom}_{\mathbb{R}}(P,Q) \cong P^*\otimes Q$. Then $f^\star = f^* \otimes \mathrm{id}_Q$ where $f^* : P^* \longrightarrow P^*$ is just the dual. We know $\det(f^*) = \det(f)$. We have a basis $(q_j^* \otimes p_i)_{i,j}$ for $\mathrm{Hom}_{\mathbb{R}}(Q,P)$ and we see that

$$\begin{aligned}
\bigwedge_j \bigwedge_i f^\star\left(p_i^* \otimes q_j\right) &= \bigwedge_j \bigwedge_i \left(f^*(p_i^*) \otimes q_j\right) \\
&= \bigwedge_j \left(\bigwedge_i f^*(p_i^*)\right) \otimes q_j \\
&= \bigwedge_j \det(f) \cdot \left(\bigwedge_i p_i^*\right) \otimes q_j \\
&= \det(f)^s \cdot \bigwedge_j \bigwedge_i \left(p_i^* \otimes q_j\right).
\end{aligned}$$

□

Proposition 1.4. *Any orientation on V determines an orientation of Gr_k^+.*

Proof. We construct an explicit section $\mathrm{Gr}_k^+ \longrightarrow \mathrm{or}(T\mathrm{Gr}_k^+)$: Let $P^+ \in \mathrm{Gr}_k^+$ and $P := \mathrm{pr}(P^+)$. By construction, we know that $T_{P^+}\mathrm{Gr}_k^+ \cong T_P\mathrm{Gr}_k$ and the latter is by Exercise VII.2 given as follows: Choose a complementary subspace $Q \subseteq V$, we have an identification

$$\mathrm{Hom}(P,Q) \longrightarrow T_P\mathrm{Gr}_k, B \longmapsto \big[t \longmapsto [\mathrm{id}_P + tB]\big].$$

Given a representative $A \in \mathrm{Hom}^{\mathrm{inj}}(\mathbb{R}^k, V)$ of P^+, we have a basis $(p_1,\dots,p_k)$ with $p_i = Ae_i$ for P and choose basis $(q_1,\dots,q_{n-k})$ for Q such that $(p_1,\dots,p_k,q_1,\dots,q_{n-k})$ is positively oriented. This gives as basis $(p_i^*\otimes q_j)$ for $\mathrm{Hom}(P,Q)$ and therefore an orientation. We have to check that this orientation is independent of the choice of A: For another representative A', there is a $S \in \mathrm{GL}^+(k,\mathbb{R})$ such that $A' = AS$. The basis $(p_1',\dots,p_k')$ has the same orientation and each consistent choice of $(q_1',\dots,q_{n-k}')$ also has the same orientation. Call the base changes $S_P : p_i \longmapsto p_i'$ and $S_Q : q_j \longmapsto q_j'$. The map $(p_i^*\otimes q_j) \longmapsto (p_i'^* \otimes q_j')$ is precomposition with S_P^{-1} and postcomposition with S_Q, i. e.

$$\varphi = (S_Q)_\star \circ (S_P^{-1})^\star : \mathrm{Hom}(P,Q) \longrightarrow \mathrm{Hom}(P,Q), B \longmapsto S_Q \circ B \circ S_P^{-1}$$

By the previous Lemma, we get the determinant

$$\det(\varphi) = \det\left((S_Q)_\star \circ (S_P^{-1})^\star\right) = \det(S_P^{-1})^{n-k} \cdot \det(S_Q)^k > 0.$$

□

Proposition 1.5. *If n is even, then this gives an orientation of* Gr_k*. If n is odd and $k < n$, then* Gr_k *is not orientable.*

Proof. If n is even, we construct a section $\mathrm{Gr}_k \longrightarrow \mathrm{or}(T\mathrm{Gr}_k)$ in the same way: Define the orientation of $\mathrm{Hom}(P, Q)$ to be the induced by the basis $(p_1, \ldots, p_k, q_1, \ldots, q_{n-k})$ coming from a representative A: For another representative A', there is now a $T \in \mathrm{GL}(k, \mathbb{R})$ such that $A' = AT$. Let $\varepsilon \in \{\pm 1\}$ be its sign. If n is even, we see that

$$\det(\varphi) = \det\big((T_Q)_\star \circ (T_P^{-1})^\star\big) = \varepsilon^{n-k+k} \cdot |\det(T_P^{-1})|^{n-k} \cdot |\det(T_Q)|^k > 0.$$

Now let n be odd and $k < n$. Then GL_k^+ is connected. If $t : \mathrm{Gr}_k \longrightarrow \mathrm{or}(T\mathrm{Gr}_k)$ is a section, this lifts to a section $\tilde{t} : \mathrm{Gr}_k^+ \longrightarrow \mathrm{or}(T\mathrm{Gr}_k^+)$. If we write $[A]$ for an equivalence class in Gr_k^+ and $[\![A]\!]$ for an equivalence class in Gr_k, we get

$$\tilde{t}[A] = t[\![A]\!] = t[\![-A]\!] = \tilde{t}[-A].$$

On the other hand, the above section s on Gr_k^+ has the property $s[A] = -s[-A]$, so it sometimes coincide with $\tilde{t}$ and sometimes does not. As Gr_k^+ is connected, this is not possible. □

2. Coefficients in the orientation bundle

Proposition 2.1. *Let M be a manifold and $K \subseteq M$ compact. We have a canonical isomorphism*

$$\varphi : H_m(M, M \setminus K; \mathrm{or}(M, A)) \longrightarrow \mathrm{Maps}(K, A).$$

In particular, if M is closed and connected, we have an isomorphism $H_m(M, \mathrm{or}(M, A)) \cong A$.

Proof. Let $\mathcal{O}^{\mathbb{Z}} := H_m(M, M \setminus \cdot\,; \mathbb{Z})$ and $p : E := \mathrm{or}(M, A) \longrightarrow M$ and $\mathcal{O}^E := H_m(M, M \setminus \cdot\,; E)$ and let $\mathscr{A} : \Pi_1(M) \longrightarrow \mathbf{Ab}, x \longmapsto p^{-1}(x)$ the corresponding coefficient system.

(i) We give a version of the universal coefficient theorem with coefficient system: Let $x \in M$ and $U \subseteq M$ a euclidean chart around x. By excision, we get

$$\mathcal{O}^{\mathbb{Z}}_x \cong H_m(\overline{U}, \overline{U} \setminus \{x\}) \qquad \text{and} \qquad \mathcal{O}^E_x \cong H_m(\overline{U}, \overline{U} \setminus \{x\}; \mathscr{A}|_{\overline{U}}).$$

We use that $\mathscr{A}|_{\overline{U}} = \mathscr{A}x$ is constant and we get the map from the UCT

$$\eta_x : \mathcal{O}^{\mathbb{Z}}_x \otimes \mathscr{A}x \cong H_m(\overline{U}, \overline{U} \setminus \{x\}) \otimes \mathscr{A}x \longrightarrow H_m(\overline{U}, \overline{U} \setminus \{x\}; \mathscr{A}x) = \mathcal{O}^E_x.$$

As in Exercise VI.1, we use that $H_{m-1}(\overline{U}, \overline{U} \setminus \{x\}) = 0$ to see that η_x is an isomorphism. Moreover, η_x is independent from the chart U as we do *not* identify $\mathscr{A}|_{\overline{U}}$ with the constant functor A, but with $\mathscr{A}x$, we make no choice of generators so far.

(ii) For each $x \in K$, we choose a generator $e_x \in \mathcal{O}^{\mathbb{Z}}_x$. Recall that $\mathscr{A}x = \{\pm e_x\} \times_{\mathbb{Z}^*} A$. For $u \in \mathcal{O}^E(K)$ and $x \in K$ there is a unique $a_x \in A$ with $\eta_x^{-1}(u_x) = e_x \otimes [e_x, a_x]$. We define

$$\varphi(u) : K \longrightarrow A, x \longmapsto a_x.$$

Note that a_x does not depend on the choice of e_x because in contrast to Exercise VI.1, the generator occurs twice here: If $e'_x = -e_x$, we get

$$e'_x \otimes [e'_x, a_x] = (-e_x) \otimes [-e_x, a_x] = e_x \otimes [e_x, a_x].$$

Moreover, it is clear that $\varphi(u + u') = \varphi(u) + \varphi(u')$. One now checks the continuity of $\varphi(u)$ exactly as in Exercise VI.1.

(iii) Injectivity and surjectivity are again proven exactly as in Exercise VI.1: If $\varphi(u) = 0$, then $u_x = 0$ for all $x \in K$ and Lemma 8.9 gives us $u = 0$. For the surjectivity, we use the Mayer–Vietoris sequence, now with twisted coefficients. □

3. Cellular homology with local coefficients

Construction 3.1. Let X be a CW complex, let E_n be the set of n-cells and for each n-cell e let $\chi_e : \mathbb{D}^n \longrightarrow X_n \subseteq X$ be the characteristic map. Let $\eta_e := \chi_e|_{\mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \longrightarrow X_{n-1}$ be the attaching map and $z_e := \chi_e(0)$ be the center of the cell. For an $(n-1)$-cell e' define

$$M(e,e') := \left\{c \in \pi_0(\eta_e^{-1}(e'^{\circ}));\ z_{e'} \in \eta_e(c)\right\}.$$

Note that $c \subseteq \mathbb{S}^{n-1}$ for each $c \in M(e,e')$. We define

$$f_{e,c} : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}/(\mathbb{S}^{n-1} \setminus c) \xrightarrow{\eta_e} X_{n-1}/(X_{n-1} \setminus e') \cong \mathbb{S}^{n-1}.$$

Now let $\mathscr{A} : \Pi_1(X) \longrightarrow \mathbf{Ab}$ be a coefficient system. We define

$$C_n^{\mathrm{cell}}(X;\mathscr{A}) := \bigoplus_{e \in E_n} \mathscr{A} z_e$$

where we identify a summand e with the generator of $\mathscr{A} z_e$. As differentials we use

$$d_n e = \sum_{e' \in E_{n-1}} \sum_{a \in M(e,e')} \deg(f_{e,c}) \cdot (\mathscr{A}\gamma_c)(e)$$

where $\gamma_c : I \longrightarrow X$ is a path from z_e to $z_{e'}$ arising as follows: Choose a path $\overline{\gamma}_c$ inside $\mathbb{D}^n$ from 0 to a preimage of $z_{e'}$ inside c and let $\gamma_c = \chi_e \circ \overline{\gamma}_c$.

Proposition 3.2. *The construction is well-defined. This means:*

(I) *The sum in $d_n e$ is finite.*
(II) *$\mathscr{A}\gamma_c$ does not depend on the choice involved for $\overline{\gamma}_c$.*

Proof. (I) For $e' \in E_{n-1}$, the space $\eta_e^{-1}(e'^{\circ}) \subseteq \mathbb{S}^{n-1}$ is an open submanifold and each $c \in M(e,e')$ is open as a subspace $c \subseteq \eta_e^{-1}(e'^{\circ}) \subseteq \mathbb{S}^{n-1}$. Moreover, the set $S := \{z_{e'};\ e' \in E_{n-1}\} \subseteq X_{n-1}$ is discrete, in particular closed. Let $U := \mathbb{S}^{n-1} \setminus \eta_e^{-1}(S) \subseteq \mathbb{S}^{n-1}$ open. Then we have an open cover of $\mathbb{S}^{n-1}$ by

$$\{U\} \cup \{c \in M(e,e') \text{ for some } e'\}.$$

As $\mathbb{S}^{n-1}$ is compact, we find a finite subcover. Since U contains no preimage of a center, there is a finite subfamily of (c) covering all preimages of centres. As the cs are pairwise disjoint and each contains a preimage of a center, there are only finitely many c.

(II) Since we are in $\mathbb{D}^n$, all paths starting in 0 and having the same endpoint are homotopic, so when fixing a preimage b of $z_{e'}$ inside c, each path from 0 to b is fine. If c contains two images b and b' of $z_{e'}$, we can connect them by a path $\beta : I \longrightarrow c \subseteq \mathbb{S}^{n-1}$ and $\eta_e \circ \beta : I \longrightarrow X_{n-1}$ lifts over the interior of e' and is therefore a null-homotopic loop. If we have to paths $\overline{\gamma}_a$ resp. $\overline{\gamma}'_c$ ending in b resp. b' Hence, then $\overline{\gamma}'_c \simeq \overline{\gamma}_c \star \beta$, so

$$\gamma'_c = \chi_e \circ \overline{\gamma}'_a \simeq (\chi_e \circ \overline{\gamma}_c) \star \underbrace{(\chi_e \circ \beta)}_{\simeq *} \simeq \gamma_c. \qquad \square$$

Proposition 3.3. *Consider the triple $(X_n; X_{n-1}, X_{n-2})$ and the corresponding boundary map*

$$\partial_n : H_n(X_n, X_{n-1};\mathscr{A}) \xrightarrow{\delta_1} H_{n-1}(X_{n-1};\mathscr{A}) \xrightarrow{p} H_{n-1}(X_{n-1}, X_{n-2};\mathscr{A})$$

Then we have isomorphisms $\varphi_n : C_n^{cell}(X;\mathscr{A}) \longrightarrow H_n(X_n, X_{n-1};\mathscr{A})$ which are chain morphisms with respect to ∂ and d, i. e.

$$\begin{tikzcd}
C_n^{cell}(X;\mathscr{A}) \arrow[r, "d"] \arrow[d, "\varphi_n"'] & C_{n-1}^{cell}(X;\mathscr{A}) \arrow[d, "\varphi_{n-1}"] \\
H_n(X_n, X_{n-1};\mathscr{A}) \arrow[r, "\partial_n"'] & H_{n-1}(X_{n-1}, X_{n-2};\mathscr{A}).
\end{tikzcd}$$

Proof. We follow THEOREM VI.4.1 from Whitehead's *Elements of Homotopy Theory* and LEMMA 3.36 from Lück's book *Homologie und Mannigfaltigkeiten*.

(I) Fix an n-cell e and the corresponding characteristic map $\chi_e : (\mathbb{D}^n, \mathbb{S}^{n-1}) \longrightarrow (X_n, X_{n-1})$. We have an induced coefficient system $\chi_e^*\mathscr{A} : \Pi_1(\mathbb{D}^n) \longrightarrow \mathbf{Ab}$. As $\mathbb{D}^n$ is contractible, $\chi_e^*\mathscr{A}$ is trivial, so $\chi_e^*\mathscr{A}(t) \cong \chi_e^*\mathscr{A}(0) = \mathscr{A}z_e$ and $\chi_e^*\mathscr{A}(\alpha) = \mathrm{id}$ for $\alpha : I \longrightarrow \mathbb{D}^n$. We have

$$(\chi_e)_* : \mathscr{A}z_e \cong H_n(\mathbb{D}^n, \mathbb{S}^{n-1}; \mathscr{A}z_e) \longrightarrow H_n(X_n, X_{n-1}; \mathscr{A}).$$

Their direct sum gives a map

$$\varphi_n := \bigoplus_{e\in E_n} (\chi_e)_* : C_n^{\mathrm{cell}}(X; \mathscr{A}) = \bigoplus_{e\in E_n} \mathscr{A}z_e \longrightarrow H_n(X_n, X_{n-1}; \mathscr{A}).$$

(II) We show that φ_n is an isomorphism: Consider $U := X_n \setminus \{z_e;\, e \in E_n\}$. Then $X_{n-1} \hookrightarrow U$ is a deformation retract. By the 5-lemma applied to the long exact sequence with coefficients, we have an isomorphism

$$i : H_n(X_n, X_{n-1}; \mathscr{A}) \longrightarrow H_n(X_n, U; \mathscr{A}).$$

Now let $V := X_n \setminus X_{n-1}$ and $W := V \cap U$. By excision, we have an isomorphism

$$j : H_n(V, W; \mathscr{A}) \longrightarrow H_n(X_n, U; \mathscr{A}).$$

We have $(V, W) \cong \coprod_{e\in E_n} (\mathbb{B}^n, \mathbb{B}^n \setminus \{0\})$, so we $f : \coprod_{e\in E_n} (\mathbb{D}^n, \mathbb{S}^{n-1}) \longrightarrow (V, W)$ by scaling with $1/2$ and then applying χ_e, which induces by additivity an isomorphism

$$f_* : \bigoplus_{e\in E_n} H_n(\mathbb{D}^n, \mathbb{S}^{n-1}; \mathscr{A}z_e) = H_n\left(\coprod_{e\in E_n} (\mathbb{D}^n, \mathbb{S}^{n-1}); \mathscr{A}\right) \longrightarrow H_n(V, W; \mathscr{A}).$$

Finally, we see that $\varphi_n = i^{-1} \circ j \circ f_*$, so indeed φ_n is an isomorphism.

(III) We show that φ_* is a chain map: For each $e \in E_n$ and $e' \in E_{n-1}$, the map

$$\begin{tikzcd}
H_{n-1}(\mathbb{S}^{n-1}; \mathscr{A}z_e) \arrow[rr, dashed, "\psi_{e,e'}"] \arrow[d, "\eta_e"'] & & H_{n-1}(\mathbb{S}^{n-1}; \mathscr{A}z_{e'}) \\
H_{n-1}(X_{n-1}; \mathscr{A}) \arrow[r, "p"'] & H_{n-1}(X_{n-1}, X_{n-2}; \mathscr{A}) \arrow[r, "\mathrm{pr}_{e'} \circ \varphi_{n-1}^{-1}"'] & H_{n-1}(\mathbb{D}^{n-1}, \mathbb{S}^{n-2}; \mathscr{A}z_{e'}) \arrow[u, "/\mathbb{S}^{n-2}"']
\end{tikzcd}$$

is by construction the morphism $\sum_{c\in M(e,e')} \deg(f_{e,c}) \cdot (A\gamma_{e,c})$, so for

$$h_{e,e'} := (/\mathbb{S}^{n-2})^{-1} \circ \psi_{e,e'} \circ \delta_2 : H_n(\mathbb{D}^n, \mathbb{S}^{n-1}; \mathscr{A}z_e) \longrightarrow H_{n-1}(\mathbb{D}^{n-1}, \mathbb{S}^{n-2}; \mathscr{A}z_{e'}),$$

we get $\bigoplus_{e,e'} h_{e,e'} = d_n$. Therefore, we get the diagram

$$\begin{tikzcd}
\bigoplus_{e\in E_n} H_n(\mathbb{D}^n, \mathbb{S}^{n-1}; \mathscr{A}z_e) \arrow[r, "\varphi_n"] \arrow[d, "\bigoplus_e \delta_2"] \arrow[ddd, bend right=60, "d_n"'] & H_n(X_n, X_{n-1}; \mathscr{A}) \arrow[d, "\delta_1"] \arrow[ddd, bend left=60, "\partial_n"] \\
\bigoplus_{e\in E_n} H_{n-1}(\mathbb{S}^{n-1}; \mathscr{A}z_e) \arrow[r, "\bigoplus_e (\eta_e)_*"] \arrow[d, "\bigoplus_{e,e'} \psi_{e,e'}"] & H_{n-1}(X_{n-1}; \mathscr{A}) \arrow[dd, "p"] \\
\bigoplus_{e'\in E_{n-1}} H_{n-1}(\mathbb{S}^{n-1}; \mathscr{A}z_{e'}) \arrow[d, "\bigoplus_{e'} (/\mathbb{S}^{n-2})^{-1}"] & \\
\bigoplus_{e'\in E_{n-1}} H_{n-1}(\mathbb{D}^{n-1}, \mathbb{S}^{n-2}; \mathscr{A}z_{e'}) \arrow[r, "\varphi_{n-1}"] & H_{n-1}(X_{n-1}, X_{n-2}; \mathscr{A})
\end{tikzcd}$$

$\square$

4. $H_*(\mathbb{R}P^m; \mathbb{S}^m \times_{\mathbb{Z}^*} A)$

Proposition 4.1. *Consider the standard CW decomposition for $\mathbb{R}P^m$ and the bundle $\mathbb{S}^m \longrightarrow \mathbb{R}P^m$. Then the cellular chain complex with coefficients in $\mathbb{S}^m \times_{\mathbb{Z}^*} A$ is of the form*

$$C_*(\mathbb{R}P^m; \mathbb{S}^m \times_{\mathbb{Z}^*} A) = A \xleftarrow{2} A \xleftarrow{0} A \xleftarrow{2} A \longleftarrow \cdots,$$

in particular, we get

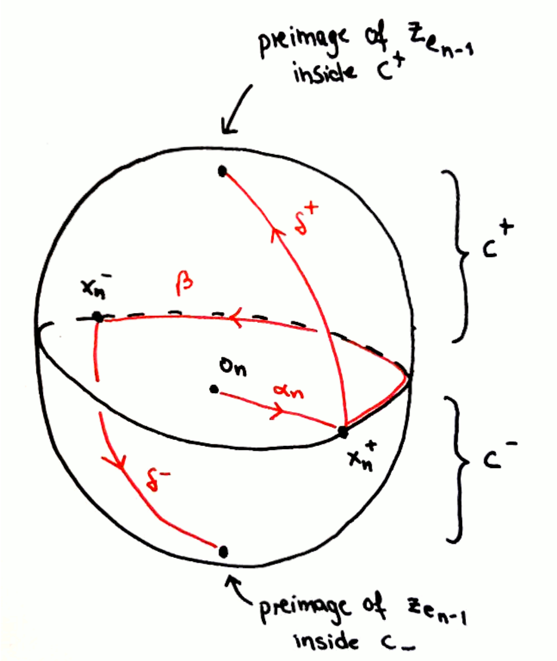
$$H_k(\mathbb{R}P^m; \mathbb{S}^m \times_{\mathbb{Z}^*} A) = \begin{cases} A/2A & \text{for } k \text{ even, } k < m \\ A & \text{for } k = m \text{ even,} \\ \mathrm{tors}_2(A) & \text{for } k \leq m \text{ odd,} \\ 0 & \text{else.} \end{cases}$$

Proof. Call the coefficient system $\mathscr{A}$, so we have $\mathscr{A}(x) = A$ for each $x \in \mathbb{R}P^m$.

- We have one cell e_n in each dimension. For each $n \geq 1$, let the two preimages of e_{n-1}° be $c_+, c_- \subseteq \mathbb{S}^{n-1} \subseteq \mathbb{D}^n$. Let $x_n^\pm := (\pm 1, 0, \ldots, 0) \in \mathbb{D}^n$ be the two preimages of the 0-cell.
- For each n, we consider $\alpha : I \longrightarrow \mathbb{D}^n, t \longmapsto t x_n^+$ from 0_n to x_n^+. Moreover, the half circle $\beta : I \longrightarrow \mathbb{D}^n, t \longmapsto (e^{\pi i \cdot t}, 0, \ldots, 0)$ has the two properties $\chi_e \circ \beta = e_1$ (with orientation) and $\alpha \star \beta$ is a path from 0 to x_n^-. Finally, we define the paths

  $$\delta^\pm(t) := \pm\left(1 - t, 0, \ldots, 0, \sqrt{2t - t^2}\right).$$

  $\delta^\pm$ is a path from $x_n^\pm$ to the preimage of $z_{e_{n-1}}$ inside $c^\pm$ and $\chi_e \circ \delta^+ = \chi_e \circ \delta^- =: \varepsilon$. Pictorially,



- Now we choose our paths γ_{c_+} and γ_{c_-} by setting

  $$\overline{\gamma}_{c_+} := \alpha \star \delta^+$$
  $$\overline{\gamma}_{c_-} := \alpha \star \beta \star \delta^-.$$

  Then $\gamma_{c_+} = (\chi_e \alpha) \star \varepsilon$ and $\gamma_{e_n, c_-} = (\chi_e \alpha) \star e_1 \star \varepsilon$. Hence, we get

  $$\mathscr{A}\gamma_{c_+} = -\mathscr{A}\gamma_{c_-}.$$

  Recall that $\deg(f_{e_n, c_\pm}) = (\pm 1)^n$, whence the boundary operator has the form $d^{\text{old}} e_n = (1 + (-1)^n) \cdot e_{n-1}$. Now we have one extra sign, so we get $d e_n = (1 + (-1)^{n+1}) \cdot e_{n-1}$. □