Sheet 7: Knots, smooth structures and the Hopf invariant

Florian Kranhold

Topology 2, summer term 2019

1. Complements of torus knots

Remark 1.1. We will use the following describe the 3-sphere $\mathbb{S}^3$:

$$\mathbb{S}^3 = \partial\mathbb{D}^4 \cong \partial(\mathbb{D}^2 \times \mathbb{D}^2) = (\mathbb{S}^1 \times \mathbb{D}^2) \cup (\mathbb{D}^2 \times \mathbb{S}^1) =: A \cup B.$$

Their intersection $A \cap B = \mathbb{S}^1 \times \mathbb{S}^1$ is a 2-torus.

Definition 1.2. For two coprime integers $p, q \in \mathbb{Z}$, we define the *torus knot* as the embedding

$$K_{p,q} : \mathbb{S}^1 \xrightarrow{z \longmapsto (z^p, z^q)} \mathbb{S}^1 \times \mathbb{S}^1 \hookrightarrow \mathbb{S}^3.$$

Note that this is injective: There is no $1 \neq z \in \mathbb{S}^1$ with $z^p = 1$ and $z^q = 1$ as there are no $1 \leq k < p$ and $1 \leq l < q$ with $kq + lp = 0$.

Proposition 1.3. $\pi_1(\mathbb{S}^3 \setminus K_{p,q}) = \langle a, b \mid a^p = b^q \rangle$.

Proof. Throughout the proof, let $K := K_{p,q}$.

(I) Let the complement inside the torus be $C := \mathbb{T}^2 \setminus K$. Now let $W := C \times (1 - \varepsilon, 1]$. We can identify $W \cong W_A \subseteq A$ and $W \cong W_B \subseteq B$. We thicken $U := (A \setminus K) \cup W_B$ and $V := (B \setminus K) \cup W_A$. Then $U \simeq A$ and $V \simeq B$. Moreover, $U \cap V = C \times (1 - \varepsilon, 1 + \varepsilon) \simeq C$. Then $U \cup V = \mathbb{S}^3 \setminus K$, so we found an open cover.

(II) We have $\pi_1(U) = \mathbb{Z} = \pi_1(V)$ and we see that $U \cap V$ is just an annulus and its fundamental loop is a perturbation of the removed path $z \longmapsto (z^p, z^q)$. Therefore $\pi_1(U \cap V) = \mathbb{Z}$ and the inclusions $\imath^U : U \cap V \longrightarrow U$ and $\imath^V : U \cap V \longrightarrow V$ induce multiplication with p resp. q. By Seifert–van Kampen, we get

$$\begin{aligned} \pi_1(\mathbb{S}^3 \setminus K) &= \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \\ &= \mathrm{pushout}_{\mathbf{Grp}} \left(\mathbb{Z} \xleftarrow{p} \mathbb{Z} \xrightarrow{q} \mathbb{Z} \right) \\ &= \langle a, b \mid a^p = b^q \rangle. \end{aligned}$$

□

Proposition 1.4. $H_1(\mathbb{S}^3 \setminus K_{p,q}; \mathbb{Z}) = \mathbb{Z}$ *(without Alexander duality).*

Proof. We know $H_1(-; \mathbb{Z}) = \pi_1^{\mathrm{ab}}$ for connected arguments. As the abelianisation preserves colimits,

$$H_1(\mathbb{S}^3 \setminus K_{p,q}; \mathbb{Z}) = \mathrm{pushout}_{\mathbf{Ab}} \left(\mathbb{Z} \xleftarrow{p} \mathbb{Z} \xrightarrow{q} \mathbb{Z} \right) = \mathrm{coker}\tbinom{p}{-q}.$$

To show that the latter is isomorphic to $\mathbb{Z}$, consider the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\binom{p}{-q}} \mathbb{Z}^2 \xrightarrow{(q\ p)} \mathbb{Z} \longrightarrow 0.$$

The map $(q\ p)$ is surjective by Bézout's Lemma, the composite is 0 and if $kq + lp = 0$, then $q \mid l$ and $p \mid k$, so there are $a, b \in \mathbb{Z}$ with $qb = l$ and $pa = k$. Then $pq(a + b) = 0$, so $b = -a$ and $(k, l) = \tbinom{p}{-q} \cdot a$, so the sequence is exact and $\mathbb{Z} = \mathrm{coker}\tbinom{p}{-q}$. □

2. Orientation and tangent bundles

Reminder 2.1. Let V be an m-dimensional $\mathbb{R}$-vector space. The set $\mathrm{Fr}(V) := \mathrm{Fr}_m(V)$ of bases $\mathcal{B} = (v_1, \dots, v_m)$ behaves functorially: For each isomorphism $\varphi : V \longrightarrow W$, we have a map

$$\varphi : \mathrm{Fr}(V) \longrightarrow \mathrm{Fr}(W), (v_1, \dots, v_m) \longmapsto (\varphi(v_1), \dots, \varphi(v_m)).$$

We impose on it an equivalence relation: Two bases $\mathcal{B} = (v_1, \dots, v_m)$ and $\mathcal{B}' = (v_1', \dots, v_m')$ are *equally oriented*, if the determinant a of the base change

$$v_1' \wedge \dots \wedge v_m' = a \cdot (v_1 \wedge \dots \wedge v_m)$$

is positive. The quotient consists O_V of two equivalence classes $[\mathcal{B}]$ of bases. A choice of one is called *orientation* of V. The map φ_* respects this equivalence relation and thus gives rise to a bijection $O_\varphi : O_V \longrightarrow O_W$. For a choice of orientation, φ is called *orientation-preserving* if O_φ is based, and *orientation-reversing* if not.

Construction 2.2. Let $V \longrightarrow E \longrightarrow X$ be a real vector bundle, so assume we have charts $\varphi_i : E|_{U_i} \longrightarrow U_i \times V$ such that for all $p \in U_{ij}$, the coordinate changes $\varphi_{ij}(p) : V \longrightarrow V$ are linear automorphisms. We can build the *associated orientation bundle* $\mathbb{Z}^* \longrightarrow \mathrm{or}(E) \longrightarrow X$ by

$$\mathrm{or}(E) := \coprod_{i \in I} U_i \times \mathbb{Z}^* \Big/ \Big((p, [\mathcal{B}]) \sim (p, O_{\varphi_{ij}(p)}[\mathcal{B}]) \text{ for } p \in U_{ij} \Big).$$

This is a $\mathbb{Z}^*$-bundle over X, with trivialisations $\mathrm{or}(E)|_{U_i} \longrightarrow U_i \times \mathbb{Z}^*, [p, [\mathcal{B}]] \longrightarrow (p, [\mathcal{B}])$. We call E *orientable*, if $\mathrm{or}(E) \cong X \times \mathbb{Z}^*$. If each $\varphi_{ij}(p)$ has positive determinant, then $\mathrm{or}(E)$ is orientable.

Proposition 2.3. *All complex vector bundles are orientable (as real vector bundles).*

Proof. Being a complex vector bundle means that the transitions $\varphi_{ij}(p) : V \longrightarrow V$ are $\mathbb{C}$-linear. We show that $\mathbb{C}$-linear maps are always orientation-preserving (as $\mathbb{R}$-linear maps): We choose a $\mathbb{C}$-basis $(v_1, \dots, v_r)$ of V. As an $\mathbb{R}$-vector space, we have

$$V \cong \underbrace{\mathbb{R}\langle v_1, \dots, v_r \rangle}_{=:V_1} \oplus \underbrace{\mathbb{R}\langle iv_1, \dots, iv_r \rangle}_{=:V_2}.$$

Now let $\varphi : V \longrightarrow V$ be a $\mathbb{C}$-linear automorphism. Then there are $\varphi_{ab} : V_a \longrightarrow V_b$ such that

$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix},$$

as $\mathbb{R}$-linear map, so $\det(\varphi) = \varphi(\varphi_{11}) \det(\varphi_{22}) - \det(\varphi_{12}) \det(\varphi_{21})$. Now we use $\mathbb{C}$-linearity to get $\varphi(iv_k) = i\varphi(v_k)$, so $\varphi_{22} = \varphi_{11}$ and $\varphi_{21} = -\varphi_{12}$. Therefore

$$\det(\varphi) = \det(\varphi_{11})^2 + \det(\varphi_{12})^2 > 0. \qquad \square$$

Lemma 2.4. *Fix a generator $o \in H_m(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}; \mathbb{Z})$ and for $x \in \mathbb{R}^m$ let $o_x \in H_m(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}; \mathbb{Z})$ be its translations. Now let $\varphi : V \longrightarrow V'$ be a diffeomorphism. For $x \in V$, we have*

$$\varphi_* o_x = \mathrm{sgn}(\det(\mathrm{D}\varphi|_x)) \cdot o_{\varphi(x)}.$$

Proof.

(i) We start with the following general remark: Let $A \in \mathrm{GL}(m, \mathbb{R})$ with $\varepsilon = \mathrm{sgn}(\det(A)) = \pm 1$. As $\mathrm{GL}(m, \mathbb{R}) = \mathrm{GL}(m, \mathbb{R})_+ \sqcup \mathrm{GL}(m, \mathbb{R})_-$ has exactly two path components, there is a path $\gamma : I \longrightarrow \mathrm{GL}(m, \mathbb{R})$ from A to I_ε with $I_\varepsilon(x) = (\varepsilon x_1, x_2, \dots, x_m)$. As $\gamma(t)(x) \neq 0$ for $x \neq 0$, we get $A \simeq I_\varepsilon$ as self maps $\mathbb{R}^m \setminus \{0\} \longrightarrow \mathbb{R}^m \setminus \{0\}$. Therefore $\deg(A) = \varepsilon$.

(ii) Now by translation, we may assume $x = \varphi(x) = 0$. By Taylor approximation, there is a neighbourhood $U \subseteq V$ homeomorphic to $\mathbb{B}^m$ such that

$$\big\|\varphi(x) - \mathrm{D}\varphi|_0(x)\big\| < \big\|\mathrm{D}\varphi|_0(x)\big\|$$

for all $x \in U$, so $0 \notin [\varphi(x), \mathrm{D}\varphi|_0(x)]$. Therefore we have a homotopy

$$H : (U \setminus \{0\}) \times I \longrightarrow \mathbb{R}^m \setminus \{0\}, (x, t) \longmapsto t \cdot \varphi(x) + (1 - t) \cdot \mathrm{D}\varphi|_0(x),$$

so $\varphi|_U \simeq \mathrm{D}\varphi|_0$. We finally get the diagram

$$\begin{tikzcd} o \arrow[d, mapsto] & H_m(U, U\setminus\{0\};\mathbb{Z}) \arrow[r, equal, "\text{excision}"] \arrow[d, shift right, "\varphi_*"'] \arrow[d, shift left, "(\mathrm{D}\varphi|_0)_*"] & H_m(\mathbb{R}^m, \mathbb{R}^m\setminus\{0\};\mathbb{Z}) \arrow[d, "(\mathrm{D}\varphi|_0)_*"] & o \arrow[d, mapsto] \\ \varphi_* o & H_m(\varphi(U), \varphi(U)\setminus\{0\};\mathbb{Z}) \arrow[r, equal, "\text{excision}"'] & H_m(\mathbb{R}^m, \mathbb{R}^m\setminus\{0\};\mathbb{Z}) & \varepsilon\cdot o \end{tikzcd}$$

□

Proposition 2.5. *Let M be a smooth manifold. There is a bundle isomorphism* $\mathrm{or}(TM) \cong \mathrm{or}(M)$.

Proof. We find charts $(U_i, x_i)_{i\in I}$ of M such that U_i is convex. We define $\varepsilon_{ij} := \mathrm{sgn}(\det(\mathrm{D}x_{ij})) \in \mathbb{Z}^*$ (which is independent of the point where we evaluate). Then we can trivialise

$$\Phi_i : U_i \times \mathbb{Z}^* \longrightarrow \mathrm{or}(TM)|_{U_i}, (p, \pm 1) \longmapsto \Big(p, [\pm\tfrac{\partial}{\partial x_i^1}, \tfrac{\partial}{\partial x_i^2}, \dots, \tfrac{\partial}{\partial x_i^m}]\Big),$$
$$\Psi_i : U_i \times \mathbb{Z}^* \longrightarrow \mathrm{or}(M)|_{U_i}, \quad (p, \pm 1) \longmapsto \big(p, \pm (x_i)_*^{-1} o_{x_i(p)}\big)$$

and we define locally

$$f_i := \Psi_i \circ \Phi_i^{-1} : \mathrm{or}(TM)|_{U_i} \longrightarrow \mathrm{or}(M)|_{U_i}$$

We have to see that $f_j|_{U_{ij}} = f_i|_{U_{ij}}$: First, we see for $p \in U_{ij}$ by the LEMMA that

$$(x_j)_*^{-1} o_{x_j(p)} = \varepsilon_{ij} \cdot (x_j)_*^{-1}(x_{ij})_* o_{x_i(p)} = \varepsilon_{ij} \cdot (x_i)^{-1} o_{x_i(p)}.$$

Therefore, we get

$$\begin{aligned} f_j\Big(p, [\tfrac{\partial}{\partial x_i^1}, \dots, \tfrac{\partial}{\partial x_i^m}]\Big) &= f_j\Big(p, [\varepsilon_{ij}\cdot\tfrac{\partial}{\partial x_j^1}, \dots, \tfrac{\partial}{\partial x_j^m}]\Big) \\ &= \Psi_j(p, \varepsilon_{ij}) \\ &= (p, \varepsilon_{ij}\cdot (x_j)_*^{-1} o_{x_j(p)}) \\ &= (p, (x_i)_*^{-1} o_{x_i(p)}) \\ &= f_i\Big(p, [\tfrac{\partial}{\partial x_i^1}, \dots, \tfrac{\partial}{\partial x_i^m}]\Big). \end{aligned}$$

□

Corollary 2.6. *All complex manifolds are orientable.*

Proof. For complex manifolds, the chart changes are biholomorphic, so the differentials are $\mathbb{C}$-linear. Hence the tangent bundle can be enhanced to a complex vector bundle, which is orientable by PROPOSTION 2.3. □

3. TANGENT SPACE OF THE GRASSMANNIANS

Reminder 3.1. Let M be a smooth m-manifold and $p \in M$. Choose a chart (U, x) and define

$$T_pM := \{\alpha : (-\varepsilon, \varepsilon) \longrightarrow M \text{ smooth}; \alpha(0) = p\}/\sim$$

where $\alpha \sim \beta$ iff $(x\circ\alpha)'(0) = (x\circ\beta)'(0)$. We have a canonical bijection

$$T_pM \longrightarrow \mathbb{R}^m, [\alpha] \longmapsto (x\circ\alpha)'(0).$$

Remark 3.2 (Coordinate free identification)**.** For a based chart $x : (U, p) \longrightarrow (W, 0)$ with $W \subseteq V$ in some $\mathbb{R}$-vector space open, one has an isomorphism

$$V \longrightarrow T_pM, v \longmapsto [t \longmapsto x^{-1}(tv)] = \mathrm{D}x^{-1}|_0(v).$$

Proposition 3.3. *Let $P \in \mathrm{Gr}_k(\mathbb{R}^n)$. Choose a complementary $Q \subseteq \mathbb{R}^n$. We have an isomorphism*

$$\Phi : \mathrm{Hom}(P, Q) \longrightarrow T_P\mathrm{Gr}_k(\mathbb{R}^n), B \longmapsto [t \longmapsto \mathrm{im}(\mathrm{id}_P + tB)].$$

Proof. This is just an application of EXERCISE V.2 where we found charts

$$\varphi^{-1} : \mathrm{Hom}(P, Q) \longrightarrow U_P \subseteq \mathrm{Gr}_k(\mathbb{R}^n), B \longmapsto \mathrm{im}(\mathrm{id}_P + B).$$

□

4. The Hopf invariant

Reminder 4.1. Let $n > 1$ and $f : \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^n$ be a map. We consider the pushout $X_f := e^n \sqcup_f \mathbb{S}^n$. It has one 0-, one n- and one $2n$-cell and since all differentials vanish, we get $H^n(X_f) = \mathbb{Z}\langle\alpha\rangle$ and $H^{2n}(X_f) = \mathbb{Z}\langle\beta\rangle$. There is a $h_f \in \mathbb{Z}$ such that $\alpha^2 = h_f \cdot \beta$. We call h_f *Hopf invariant*. It does not depend on the homotopy type of f, whence we get a map

$$h : [\mathbb{S}^{2n-1}, \mathbb{S}^n] \longrightarrow \mathbb{Z}.$$

Proposition 4.2. *If n is odd, then $h_f = 0$ for all $f : \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^n$.*

Proof. We have $2\alpha^2 = 0$ by graded commutativity. As $H^{2n}(X_f)$ is free, we get $\alpha^2 = 0 = 0 \cdot \beta$. □

Proposition 4.3. *Consider the Whitehead square $\omega : \mathbb{S}^{2n-1} \xrightarrow{f} \mathbb{S}^n \vee \mathbb{S}^n \xrightarrow{\nabla} \mathbb{S}^n$ where the first map is the attaching map of the $2n$-cell of $\mathbb{S}^n \times \mathbb{S}^n$. Then $h_\omega = 1 + (-1)^n$.*

Proof. By definition, $\mathbb{S}^n \times \mathbb{S}^n = e^{2n} \sqcup_f (\mathbb{S}^n \vee \mathbb{S}^n)$, so by the universal property, there is a map $g : \mathbb{S}^n \times \mathbb{S}^n \longrightarrow X$ such that

$$\begin{tikzcd}
\mathbb{S}^{2n-1} \arrow[r, "f"] \arrow[d, hook] & \mathbb{S}^n \vee \mathbb{S}^n \arrow[r, "\nabla"] \arrow[d, "i"] & \mathbb{S}^n \arrow[d, "j"] \\
e^n \arrow[r] & \mathbb{S}^n \times \mathbb{S}^n \arrow[r, "g"'] & X_\omega.
\end{tikzcd}$$

As the outer and the left square are pushouts, also the right square is a pushout. As i is a neighbourhood deformation retract, this gives us a Mayer–Vietoris sequence

$$\cdots \longrightarrow H^{k-1}(\mathbb{S}^n \vee \mathbb{S}^n) \xrightarrow{\delta^*} H^k(X_\omega) \xrightarrow{j^*+g^*} H^k(\mathbb{S}^n) \oplus H^k(\mathbb{S}^n \times \mathbb{S}^n) \xrightarrow{i^*-\nabla^*} H^k(\mathbb{S}^n \vee \mathbb{S}^n) \xrightarrow{\delta^*} \cdots$$

Hence, $g^* : H^{2n}(X_\omega) \longrightarrow H^{2n}(\mathbb{S}^n \times \mathbb{S}^n)$ is an iso, so we may assume $g^*\beta = [\mathbb{S}^n] \times [\mathbb{S}^n]$. We see

$$h_\omega \cdot ([\mathbb{S}^n] \times [\mathbb{S}^n]) = h_\omega \cdot g^*\beta = (g^*\alpha)^2,$$

so we are left to see what $g^*\alpha \in H^n(\mathbb{S}^n \times \mathbb{S}^n)$ is. To do this, we determine $j^* + g^*$ by using exactness in the following situation:

$$\begin{tikzcd}
H^n(X_\omega) \arrow[r, "j^*+g^*"] \arrow[d, equal] & H^n(\mathbb{S}^n) \oplus H^n(\mathbb{S}^n \times \mathbb{S}^n) \arrow[r, "\nabla^*-i^*"] \arrow[d, equal] & H^n(\mathbb{S}^n \vee \mathbb{S}^n) \arrow[d, equal] \\
\langle\alpha\rangle \arrow[r, "\left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right)"'] & \langle [\mathbb{S}^n], 1 \times [\mathbb{S}^n], [\mathbb{S}^n] \times 1 \rangle \arrow[r, "\left(\begin{smallmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{smallmatrix}\right)"'] & \langle [\mathbb{S}^n]_1, [\mathbb{S}^n]_2 \rangle .
\end{tikzcd}$$

Then $(\nabla^* - i^*)(j^* + g^*) = 0$ immediately gives us $a - b = 0$ and $a - c = 0$, so $b = c$. On the other hand, as $\ker(\nabla^* - i^*) \subseteq \mathrm{im}(j^* + g^*)$, we get $b, c = \pm 1$. Therefore

$$g^*\alpha = b \cdot (1 \times [\mathbb{S}^n]) + c \cdot ([\mathbb{S}^n] \times 1) = \pm \left((1 \times [\mathbb{S}^n]) + ([\mathbb{S}^n] \times 1) \right).$$

and $(g^*\alpha)^2 = (1 \times [\mathbb{S}^n])([\mathbb{S}^n] \times 1) + ([\mathbb{S}^n] \times 1)(1 \times [\mathbb{S}^n]) = (1 + (-1)^n) \cdot ([\mathbb{S}^n] \times [\mathbb{S}^n])$. □

Corollary 4.4. $[\mathbb{S}^{2n-1}, \mathbb{S}^n] \neq 0$ *for n even.*