Sheet 11: Suspensions, Hopf and Hurewicz

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1. Precomposition and postcomposition

Proposition 1.1. *For $g : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{k-1}$ based, $(\Sigma g)^* : \pi_k(X) \longrightarrow \pi_n(X)$ is a homomorphism.*

Proof. For $l \geq 1$, let $\Phi_l : [\mathbb{S}^l, X]_* \longrightarrow [\mathbb{S}^{l-1}, \Omega X]_*$ be the adjunction. We already checked that this is an isomorphism of groups when considering the multiplication coming from the monoid structure on ΩX on the right hand side. Moreover, we know that

$$\begin{tikzcd}
{[\mathbb{S}^k, X]_*} \arrow[r, "(\Sigma g)^*"] \arrow[d, equal, "\Phi_k"'] & {[\mathbb{S}^n, X]_*} \arrow[d, equal, "\Phi_n"] \\
{[\mathbb{S}^{k-1}, \Omega X]_*} \arrow[r, "g^*"'] & {[\mathbb{S}^{n-1}, \Omega X]_*},
\end{tikzcd}$$

the bottom map is by Exercise IX.1 a homomorphism. Thus, $(\Sigma g)^*$ is also a homomorphism. □

Proposition 1.2. *Let $f : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ be of degree d. Then*

$$\pi_{k-1}(\mathbb{S}^{n-1}) \xrightarrow{\Sigma} \pi_k(\mathbb{S}^n) \overset{f_*}{\underset{\cdot d}{\rightrightarrows}} \pi_k(\mathbb{S}^n).$$

Proof. Let $\Sigma g \in \operatorname{im}(\Sigma : \pi_{k-1}(\mathbb{S}^{n-1}) \longrightarrow \pi_k(\mathbb{S}^n))$. By the first part of the exercise, $(\Sigma g)^*$ is a homomorphism. Thus,

$$f_*[\Sigma g] = (\Sigma g)^*[f] = (\Sigma g)^*(d \cdot [\mathrm{id}]) = d \cdot (\Sigma g)^*[\mathrm{id}] = d \cdot [\Sigma g]. \qquad \square$$

2. The Hopf map

Proposition 2.1. *Consider the conjugations*

$$\begin{aligned}
\kappa^n &: \mathbb{S}^{2n-1} \subseteq \mathbb{C}^n \longrightarrow \mathbb{S}^{2n-1}, (z_1, \dots, z_n) \longmapsto (\overline{z}_1, \dots, \overline{z}_n), \\
\kappa_n &: \mathbb{C}P^n \longrightarrow \mathbb{C}P^n, \qquad [z_0 : \dots : z_n] \longmapsto [\overline{z}_0 : \dots : \overline{z}_n].
\end{aligned}$$

Then $\deg(\kappa^n) = \deg(\kappa_n) = (-1)^n$.

Proof. (i) We see $\mathbb{S}^{2n-1} \subseteq \mathbb{R}^{2n}$ and $\det_{\mathbb{R}}(\kappa^n) = (-1)^n$. Thus, $\deg(\kappa^n) = (-1)^n$.

(ii) Consider the collapsing

$$\begin{aligned}
r : \mathbb{C}P^n &\longrightarrow \mathbb{C}P^n / \mathbb{C}P^{n-1} \cong \mathbb{S}^{2n} \cong (\mathbb{C}^n)^\infty, \\
[z : 1] &\longmapsto z, \\
[z : 0] &\longmapsto \infty.
\end{aligned}$$

Then r induces an isomorphism in H_{2n}. Moreover, we have a self map $l : (\mathbb{C}^n)^\infty \longrightarrow (\mathbb{C}^n)^\infty$ which extends the conjugation $\mathbb{C}^n \longrightarrow \mathbb{C}^n$. Apparently $r \circ \kappa_n = l \circ r$, so $\deg(\kappa_n) = \deg(l)$. We can calculcate $\deg(l)$ locally: All $z \in \mathbb{C}^n$ are regular with $l^{-1}(z) = \{\overline{z}\}$, so

$$\deg(l) = \operatorname{sg}\det(\mathrm{D}l|_{\overline{z}}) = (-1)^n. \qquad \square$$

Proposition 2.2. *For the Hopf map $\eta : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ and the antipode $\alpha : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$, we have $\alpha\eta \simeq \eta$.*

Proof. The Hopf map is defined by $\mathbb{S}^3 \subseteq \mathbb{C}^2 \longrightarrow \mathbb{C}P^1 = \mathbb{S}^2, (z, w) \longmapsto [z, w]$. Thus, $\eta\kappa^2 = \kappa_1\eta$. Moreover, $\deg(\kappa_1) = -1$, so as $\deg : [\mathbb{S}^2, \mathbb{S}^2] \longrightarrow \mathbb{Z}$ is an isomorphism, we get $\kappa_1 \simeq \alpha$. Likewise, $\deg(\kappa^2) = 1$, so as $\deg : [\mathbb{S}^3, \mathbb{S}^3] \longrightarrow \mathbb{Z}$ is an isomorphism, we get $\kappa^2 \simeq \mathrm{id}$. Thus, we get

$$\eta = \eta \circ \mathrm{id} \simeq \eta \circ \kappa^2 = \kappa_1 \circ \eta \simeq \alpha \circ \eta. \qquad \square$$

Remark 2.3. The last two PROPOSITIONS give counterexamples for the statements in EXERCISE 1 when dropping the suspension condition:

(I) The precomposition $\eta^* : \pi_2(\mathbb{S}^2) \longrightarrow \pi_3(\mathbb{S}^2)$ is not a homomorphism because

$$\eta^* \left(-[\mathrm{id}]\right) = [\alpha\eta] = [\eta] \neq -[\eta] = -\eta^*[\mathrm{id}].$$

(II) The postcomposition $\alpha_* : \pi_3(\mathbb{S}^2) \longrightarrow \pi_3(\mathbb{S}^2)$ is not multiplication with its degree

$$\alpha_*[\eta] = [\alpha\eta] = [\eta] \neq -[\eta] = \deg(\alpha) \cdot [\eta].$$

Proposition 2.4. $2 \cdot [\Sigma\eta] = 0$. *In particular,* $\pi_4(\mathbb{S}^3) \in \{\mathbb{Z}/2, 0\}$.

Proof. We use that Σ is a homotopy functor and that the suspension $\Sigma : \pi_k(X) \longrightarrow \pi_{k+1}(\Sigma X)$ as well as the precomposition $(\Sigma\eta)^*$ are homomorphisms. Thus, we get

$$[\Sigma\eta] = [\Sigma(\alpha\eta)] = (\Sigma\eta)^*\Sigma[\alpha] = (\Sigma\eta)^* \left(-\Sigma[\mathrm{id}]\right) = -(\Sigma\eta)^*\Sigma[\mathrm{id}] = -[\Sigma\eta].$$

By FREUDENTHAL'S SUSPENSION THEOREM, we know that $\Sigma : \pi_3(\mathbb{S}^2) \longrightarrow \pi_4(\mathbb{S}^3)$ is surjective, so $\pi_4(\mathbb{S}^3)$ is generated by $\Sigma\eta$. As the latter has 2-torsion, we get either $\pi_4(\mathbb{S}^3) = \mathbb{Z}/2$ or $\pi_4(\mathbb{S}^3) = 0$. $\square$

3. The relative Hurewicz homomorphism

Reminder 3.1. Fix a generator $[\alpha] \in H_n(\mathbb{D}^n, \mathbb{S}^{n-1}; \mathbb{Z})$ and define the HUREWICZ MAP

$$\mathrm{hur}_{X,A} : \pi_n(X, A, *) \longrightarrow H_n(X, A; \mathbb{Z}), [f] \longmapsto f_*[\alpha].$$

These maps are homorphisms and natural and the absolute Hurewicz map hur_X arises from the case $A = *$. Using naturality and the collapsing $q : (X, A) \longrightarrow (X/A, *)$, we have

$$\begin{array}{ccc} \pi_n(X, A, *) & \xrightarrow{\mathrm{hur}_{X,A}} & H_n(X, A\,; \mathbb{Z}) \\ {\scriptstyle \pi_n(q)}\big\downarrow & & \big\downarrow{\scriptstyle H_n(q)} \\ \pi_n(X/A, *) & \xrightarrow[\mathrm{hur}_{X/A}]{} & H_n(X/A, *\,; \mathbb{Z}). \end{array}$$

Reminder 3.2 (Quotient theorem)**.** Let $A \hookrightarrow X$ be a cofibration and let A be p-connected and (X, A) be q-connected. Then $q : (X, A) \longrightarrow (X/A, *)$ is a $(p + q + 1)$-equivalence, meaning:

- $q_* : \pi_k(X, A) \longrightarrow \pi_k(X/A, *)$ is an isomorphism for $k \leq p + q$,
- $q_* : \pi_{p+q+1}(X, A) \longrightarrow \pi_{p+q+1}(X/A, *)$ is an epimorphism.

Proposition 3.3. *Let* $n \geq 2$ *and* (X, A) *be* $(n-1)$*-connected and* A *simply connected. Then the relative Hurewicz map* $\pi_{n+1}(X, A, *) \longrightarrow H_{n+1}(X, A; \mathbb{Z})$ *is surjective.*

Proof. (I) By naturality of the Hurewicz map, we can replace $\imath : A \hookrightarrow X$ by the cofibration $A \hookrightarrow \mathrm{cyl}(\imath)$: By the 5-lemma, the inclusion $(X, A) \hookrightarrow (\mathrm{cyl}(\imath), A)$ induces isomorphisms in both homology and homotopy. Thus, we can wlog assume that $A \hookrightarrow X$ is a cofibration.

(II) We know that in the above square (with $n + 1$ instead of n), $H_{n+1}(q)$ is an isomorphism. Moreover, X/A is $(n-1)$-connected, so the absolute Hurewicz map $\mathrm{hur}_{X/A} : \pi_{n+1}(X/A, *) \longrightarrow H_{n+1}(X/A, *\,; \mathbb{Z})$ is surjective. By the QUOTIENT THEOREM, the map $\pi_{n+1}(q)$ is also surjective. Hence, also $\mathrm{hur}_{X,A}$ is surjective. $\square$

Example 3.4. Let $(X, A) = (\mathbb{C}P^\infty \times \mathbb{S}^1, * \times \mathbb{S}^1)$:

- A is only 0-connected. The map $\pi_k(A) \longrightarrow \pi_k(X)$ is an isomorphism for $k \in \{0, 1\}$, so $\pi_1(X, A) = 0$, so (X, A) is 1-connected.
- We have $\pi_3(X) = \pi_3(\mathbb{C}P^\infty) \times \pi_3(\mathbb{S}^1) = 0$ and $\pi_2(A) = \pi_2(\mathbb{S}^1) = 0$, so $\pi_3(X, A) = 0$ by the long exact sequence.
- We have $H_3(A) = 0$, so $H_3(X) \subseteq H_3(X, A)$. Moreover, we have $H_3(X) = \mathbb{Z}\big\langle[\mathbb{C}P^2] \times [\mathbb{S}^1]\big\rangle$, in particular, $H_3(X, A) \neq 0$.

We conclude that $\mathrm{hur}_{X,A} : \pi_3(X, A) \longrightarrow H_3(X, A)$ is not surjective.

4. π_1-action on $\pi_2(X, A, *)$

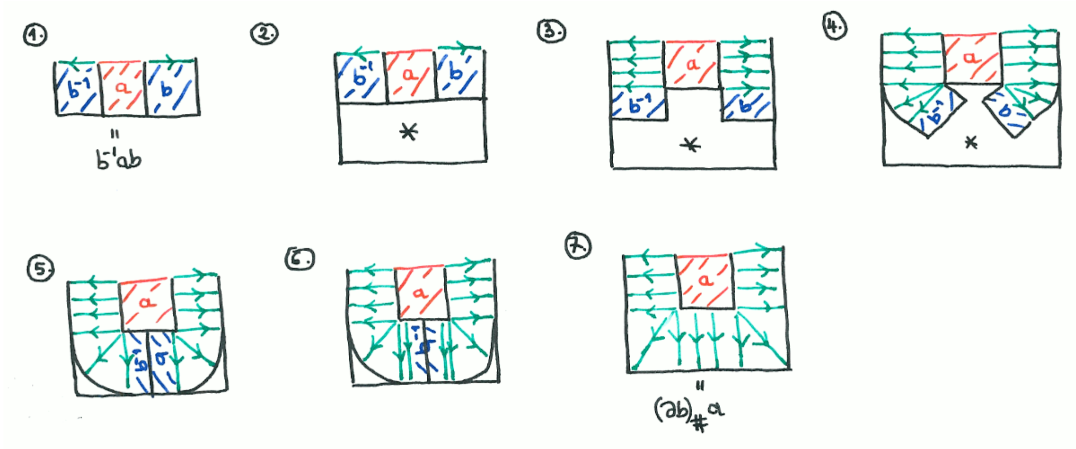
Reminder 4.1. The relative homotopy group $\pi_k(X, A)$ is . . . :

- undefined for $k = 0$,
- just a set for $k = 1$,
- a group for $k = 2$, write the operation multiplicatively,
- an abelian group for $k = 3$, write the operation additively.

Proposition 4.2. *Let $a, b \in \pi_2(X, A)$ and denote the boundary by $\partial : \pi_2(X, A) \longrightarrow \pi_1(A)$. Then*

$$(\partial b)_\# a = b^{-1}ab.$$

Proof. The homotopy is given by the following "movie", where the green arrow is the loop ∂b:



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