Consistency strength results about mutual stationarity

Forcing $\text{MS}(\aleph_3, \aleph_5, \aleph_7, \ldots; \omega_1)$

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A model theoretic characterization of stationarity

For $\kappa \geq \aleph_1$ regular and $S \subseteq \kappa$ the following are equivalent:

- $S$ is stationary in $\kappa$
- every first-order structure $\mathfrak{A} \supseteq \kappa$ with countable language has an elementary substructure $X < \mathfrak{A}$ such that $X \cap \kappa \in S$
- every first-order structure $\mathfrak{A} \supseteq \kappa$ with countable language has an elementary substructure $X < \mathfrak{A}$ such that $\sup (X \cap \kappa) \in S$
Mutual stationarity

\((S_i)\) is mutually stationary in \((\kappa_i)\) if every first-order structure \(\mathcal{A} \supseteq \bigcup_i \kappa_i\) with countable language has an elementary substructure \(X \prec \mathcal{A}\) such that \(\forall i \sup (X \cap \kappa_i) \in S_i\).

Obviously: if \((S_i)\) is mutually stationary in \((\kappa_i)\) then \(\forall i S_i\) is stationary in \(\kappa_i\).

The mutual stationarity problem (Foreman, Magidor):
(When) does the converse hold?
Mutual Ramseyness

Consider regular cardinals

\[ \kappa_0 < \kappa_1 < \ldots < \kappa_n < \ldots, \, n < \omega, \, \kappa = \sup \kappa_n \]

\((\kappa_n)\) is mutually Ramsey (coherently Ramsey) if for all \(F: [\kappa]^{<\omega} \to 2\) there are sets \(A_n \subseteq \kappa_n\), \(\text{card}(A_n) = \kappa_n\) such that \((A_n)\) is homogeneous for \(F\):

for all \(x, y \subseteq \bigcup A_n\), \(x, y\) finite, \(\forall n < \omega \ \text{card}(x \cap A_n) = \text{card}(y \cap A_n)\) holds

\[ F(x) = F(y). \]

The sequence \((A_n)\) is mutually indiscernible for a structure coded by \(F\) (all structures are assumed to have built-in Skolem functions).
Mutual stationarity from mutual indiscernibles

**Theorem.** Let \((\kappa_n)\) be mutually Ramsey. Then the mutual stationarity property \(\text{MS}(\kappa_0, \kappa_1, \ldots)\) holds: if \(\forall n < \omega S_n\) is stationary in \(\kappa_n\) then \((S_n)\) is mutually stationary in \((\kappa_n)\).

**Proof.** Let \((A_n)\) be mutually indiscernible for a given structure \(\mathcal{A} \supseteq \kappa\). Let \(\beta_n \in S_n\), \(\sup (A_n \cap \beta_n) = \beta_n\). Let \(X\) be the elementary substructure of \(\mathcal{A}\) generated by \(\bigcup_{n < \omega} (A_n \cap \beta_n)\). Then

\[
\beta_n \leq \sup (X \cap \kappa_n) \leq \beta_n .
\]

Let \(t^A(x) = t^A(x \cap \kappa_n, x \setminus \kappa_n) < \kappa_n\). Let \(t^A(x) = t^A(x \cap \kappa_n, x \setminus \kappa_n) < \xi\), \(\xi \in A_n \cap \kappa_n\). By indiscernibility, \(t^A(x) = t^A(x \cap \kappa_n, x \setminus \kappa_n) < \bar{\xi} < \beta_n\) for some \(\bar{\xi} \in A_n \cap \beta_n\).
\[ \beta_n \in S_n \quad \kappa_n \quad \beta_{n+1} \quad \kappa_{n+1} \]

\[ t(x) < \kappa_n ? \]
Consistency strengths

$\kappa$ measurable

$\Downarrow \text{Prikry forcing}$

Endsegment of a Prikry sequence $(\kappa_n)$ is mutually Ramsey

$\Downarrow$

$\text{MS}(\kappa_0, \kappa_1, \ldots)$ (Cummings, Foreman, Magidor)

$\Downarrow$

$\kappa$ is a singular Jónsson cardinal

$\Downarrow \text{inner models}$

$\kappa$ is measurable in an inner model (Mitchell)
Accessible $\kappa_i$’s

$\text{MS}(\mathcal{N}_1, \mathcal{N}_2, \ldots) \to \mathcal{N}_\omega$ is Jónsson $\to$ ???

Restricting cofinalities

The **mutual stationarity property in cofinality** $\gamma$ (Foreman, Magidor):

$\text{MS}(\kappa_0, \kappa_1, \ldots; \gamma)$: if $\forall n < \omega \ S_n \subseteq \text{cof}_\gamma$ is stationary in $\kappa_n$ then $(S_n)$ is mutually stationary in $(\kappa_n)$. 
Foreman, Magidor:
\[ ZFC \vdash \text{MS}(\kappa_0, \kappa_1, \ldots; \omega) \]

K., Welch:
\[ \text{MS}(\kappa_0, \kappa_1, \ldots; \omega_1) \rightarrow \text{there is an inner model with one measurable cardinal} \]

K., Welch:
\[ \text{MS}(\aleph_2, \aleph_3, \ldots; \omega_1) \rightarrow \text{there is an inner model with many measurable cardinals} \]

No upper consistency bound for \( ZFC + \text{MS}(\aleph_2, \aleph_3, \ldots; \omega_1) \) is known.
Main Theorem (K.)

Let \((\kappa_n)\) be mutually Ramsey with supremum \(\kappa\). Then there is a generic extension \(V[G]\) such that

\[ V[G] \models MS(\aleph_3, \aleph_5, \aleph_7, \ldots; \omega_1). \]

Elements of the Proof. Let

\[ P = \prod_{n<\omega} \text{Col}(\kappa_{n-1}^+, < \kappa_n), \text{ where } \kappa_{-1} = \aleph_1. \]

Every \(p \in P\) is of the form \(p = (p_n | n < \omega)\).

Let \(G\) be \(P\)-generic over \(V\).
\[ V \]

\[ \kappa_{-1} \quad \kappa_0 \quad \kappa_0^+ \quad \kappa_1 \quad \kappa_1^+ \quad \kappa_2 \quad \ldots \quad \kappa \]

\[ \mathcal{N}_1 \quad \mathcal{N}_2 \quad \mathcal{N}_3 \quad \mathcal{N}_4 \quad \mathcal{N}_5 \quad \mathcal{N}_6 \quad \mathcal{N}_7 \quad \ldots \quad \mathcal{N}_\omega \]

\[ V[G] \]
Let \((\kappa, F) \in V[G], F: [\kappa]^\omega \rightarrow \kappa\).
Let \(F = \dot{F}^G\).
Let \(p \in P, p \Vdash \dot{F}: [\kappa]^\omega \rightarrow \kappa\).

**Fixing suprema**
Let \((A_n)\) be “good” mutual indiscernibles for \((V_\theta, \in, \ldots, \dot{F}, p, (\dot{S}_n))\).
Let \(I_n \subseteq A_n, \mathrm{otp}(I_n) = \omega, \sup(I_n \cap \beta_n) = \beta_n\).

Let \([\bigcup I_n]^\omega = \{x_i | i < \omega\}\).
Construct a “generic sequence”
\[p \geq p(x_0) \geq p(x_1) \geq \ldots\]
deciding the terms \(\dot{F}(x_i)\):
\[p(x_i) \Vdash \dot{F}(x_i) = \alpha_i \in \mathrm{Ord}\]
\( \alpha_i = t(x_0, x_1, \ldots, x_{i}) < \kappa_n \rightarrow \alpha_i < \beta_n \)
Let \( q = \bigcup p(x_i) \) be the coordinatewise union of \((p(x_i))\):

\[
q_n = \bigcup_{i < \omega} p_n(x_i).
\]

Let \( X = \{\alpha_i | i < \omega\} \). Then

\[
q \models \check{X} < (\kappa, \check{F}) \land \sup (\check{X} \cap \kappa_n) = \beta_n.
\]
Meeting stationary sets

Let \( V[G] \models S_n = \dot{S}_n^G \subseteq \text{cof}_\omega \) is stationary in \( \kappa_n \). Assume

\[ p \Vdash \dot{S}_n \subseteq \text{cof}_\omega \text{ is stationary in } \kappa_n. \]

Let \( \beta'_n \in S_n \) be a high-level limit of \( A_n \). Let \( r \leq q \) such that

\[ r \Vdash \beta'_n \in \dot{S}_n. \]

Choose \( I'_n \subseteq A_n \), \( \text{otp}(I'_n) = \omega \), \( \sup(I'_n \cap \beta'_n) = \beta'_n \), so that \( I'_n \)

“lies apart” from the condition \( r \):
\[ \text{dom}(r) \]

\[ | \text{some diagram with nodes and edges} | \]

\[ \| \|_1 = A_n \]

\[ \beta'_n \]

\[ I_n' \]
The system \((I'_n)\) is order-isomorphic to \((I_n)\). By this isomorphism let

\[x'_i \cong x_i, \; p(x'_i) \cong p(x_i), \; q' \cong q, \; \alpha'_i \cong \alpha_i, \; X' \cong X\]

By indiscernibility,

\[q' \models \exists \tilde{X}' \prec (\kappa, \dot{F}) \wedge \sup (\tilde{X}' \cap \kappa_n) = \beta'_n.\]

By the choice of \((I'_n)\), \(q'\) is compatible with \(r\). Hence

\[q' \cup r \models \exists \tilde{X}' \prec (\kappa, \dot{F}) \wedge \sup (\tilde{X}' \cap \kappa_n) \in \dot{S}_n.\]

This is a forcing construction for the Foreman-Magidor ZFC-result:

\[V[G] \models \text{MS}(\kappa_3, \kappa_5, \kappa_7, \ldots; \omega).\]
From cof$_\omega$ to cof$_{\omega_1}$

Fixing a substructure of size $\omega \equiv$ Rasiowa-Sikorski construction of a generic filter for countably many dense sets.

Fixing a substructure of size $\omega_1 \equiv$ getting a generic filter for $\omega_1$ dense sets via Martin’s axiom MA$_{\omega_1}$ like in Silver’s forcing construction of Chang’s conjecture.

Assume $V \models$ MA$_{\omega_1}$ (by small forcing).
Let $I_n \subseteq A_n$, otp$(I_n) = \omega_1$, sup$(I_n \cap \beta_n) = \beta_n$.

Let $[\bigcup I_n]^{<\omega} = \{x_i | i < \omega_1\}$.

**Dense sets**

\[ D_i = \{ s | \exists \alpha s \models \dot{F}(x_i) = \check{\alpha} \}. \]

**But:** $P$ does not have the countable chain condition (ccc).

**Constructing a suitable ccc $Q \subseteq P$**

Silver: Let $Z \prec (V_\theta, \in, \ldots, \dot{F}, p, (\dot{S}_n))$ be generated by $\bigcup I_n$, and let $Q = Z \cap P$. 
\[ \Delta\text{-system of } x_i \text{'s} \]

\[ \kappa_{n-1} \]

\[ p_{n-1}(x_i) = p_{n-1}(x_j) \]

\[ p_n(x_i) \quad p_n(x_j) \]

compatible in \( P \)

\[ \kappa_n \]

\[ p_{n+1}(x_i) \subseteq p_{n+1}(x_j) \]
For the ccc-argument, consider some $\Delta$-system of $x_i$’s in the interval $(\kappa_{n-1}, \kappa_n)$:

- for $m < n$, $p_m(x_i) = p_m(x_j)$ by indiscernibility;

- for $m = n$, $p_n(x_i)$ is compatible with $p_n(x_j)$ by a standard ccc-argument;

- for $m > n$, $p_m(x_i) \subseteq p_m(x_j)$ by some “growth condition”.

Construct the suborder $Q = \{ p(x_i) | i < \omega_1 \}$ such that $D_i \cap Q$ is dense in $Q$. 
By MA_{\omega_1} let $H$ be $Q$-generic over $\{D_i \cap Q \mid i < \omega_1\}$.
Let $q = \bigcup H$ (coordinatewise).
Let $X = \{\alpha \mid \exists i < \omega_1 \ q \models \vec{F}(x_i) = \vec{\alpha}\}$. Then

$$q \models \sup (\bar{X} \cap \kappa_n) = \beta_n.$$  

As before, we can also choose the $I_n$ sufficiently apart from a condition $r$ which fixes $\beta_n \in S_n$. Then

$$q \cup r \models \sup (\bar{X} \cap \kappa_n) \in \dot{S}_n.$$  

Hence

$$V[G] \models \text{MS}(\aleph_3, \aleph_5, \aleph_7, \ldots; \omega_1).$$
Variants

- $\text{MS}(\mathcal{N}_{n(0)}, \mathcal{N}_{n(1)}, \mathcal{N}_{n(2)}, \ldots; \omega_1)$, where

$$\exists i_0 < \omega \forall i \geq i_0 \ n(i + 1) \geq n(i) + 2.$$

- $\text{MS}(\mathcal{N}_{n(0)}, \mathcal{N}_{n(1)}, \mathcal{N}_{n(2)}, \ldots; \omega/\omega_1)$, where

$$\exists i_0 < \omega \forall i \geq i_0 \ n(i + 1) \geq n(i) + 2.$$

The forcing method does not go above cofinality $\omega_1$:

$\text{MS}(\mathcal{N}_3, \mathcal{N}_5, \mathcal{N}_7, \ldots; \omega_2) \rightarrow$ there is an inner model with many measurable cardinals.
Conjecture

The consistency strength of

$$\text{MS}(\aleph_1, \aleph_2, \aleph_3, \aleph_4, \ldots ; \omega, \omega_1, \omega, \omega_1, \ldots)$$

is the existence of 1 measurable cardinal.